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DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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THE FALL MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The seventeenth regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Washington and Jefferson College, Washington, Pennsylvania, on Saturday, October 25, 1941. Professor J. S. Taylor, chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was fifty-two, including the following twenty-six members of the Association: O. F. H. Bert, J. O. Blumberg, W. G. Brady, A. M. Bryson, W. E. Buker, A. B. Cunningham, H. A. Davis, L. L. Dines, H. L. Dorwart, Mary B. Ferguson, H. C. Hicks, R. P. Johnson, M. L. Manning, David Moskovitz, L. T. Moston, F. W. Owens, Helen B. Owens, C. N. Reynolds, J. B. Rosenbach, H. C. Shaub, J. S. Taylor, R. W. Thomas, C. H. Vehse, M. L. Vest, W. J. Wagner, E. A. Whitman.

The following officers were elected for the coming year: Chairman, R. G. Sturm, Aluminum Research Laboratories, New Kensington, Pennsylvania; Secretary, H. L. Dorwart, Washington and Jefferson College; member of Executive Committee, E. D. Wells, Erie Center, University of Pittsburgh. Professor C. H. Vehse, West Virginia University, continues in office for the second year of his term as the additional member of the Executive Committee.

At the close of the afternoon session, tea was served for the guests. A vote of thanks was extended to the staff of the College for their generous hospitality.

After an address of welcome by Professor O. F. H. Bert of Washington and Jefferson College, the following six papers were read:

1. "Beyond quadratics" by Professor H. L. Dorwart, Washington and Jefferson College.

2. "History of the cycloid" by Professor E. A. Whitman, Carnegie Institute of Technology.

3. "A report on the aims and activities of the committee on the teaching of science in the secondary schools" by Professor W. H. Michener, Carnegie Institute of Technology, introduced by the Secretary.

4. "A non-involutorial space transformation associated with a $Q_{1,n}$ congruence" by M. L. Vest, West Virginia University.

5. "Fermat's last theorem" by E. C. Johnson, Somerset, Pennsylvania, introduced by Professor Taylor.

6. "A report on the summer program in mechanics at Brown University" by A. M. Bryson, University of Pittsburgh.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor Dorwart gave an elementary account of the origin and nature of modern abstract algebra. The paper is to be published in the *National Mathematics Magazine*.

2. Professor Whitman traced briefly the history of the cycloid from the time of Galileo's attempts (1599) to find the area under one arch to the proof by Jean Bernoulli (1696) that this curve is also a brachistochrone. Roberval's method of finding the area was shown in detail, and comments were made on the discovery of three ways of finding tangents and also on Huygens's method of finding the length of an arch. Professor Whitman stated that he found this history was always interesting to students in the calculus and also stimulating in that it showed the power of the calculus as compared to the methods it displaced.

4. Mr. Vest showed that the transformation, of order $m+m'+n$, is associated with the congruence of lines on a plane curve r of order n having an $(n-1)$-point and a secant s through the multiple point. Two projective pencils of surfaces $|F|$ and $|F'|$ of order m and m' contain the secant as $(m-1)$-fold and $(m'-1)$-fold line, respectively. Through a generic point $P(y)$ there passes a single surface F of $|F|$. The unique line t through $P(y)$, s and r meets the associated F' of $|F'|$ in one residual point $P'(x)$, the image of $P(y)$ under the transformation thus defined.

5. Mr. Johnson, an amateur mathematician, explained how he has developed an original approach to the solution of Fermat's last theorem, which requires a proof that $x^n+y^n=z^n$ cannot be satisfied by whole numbers when n is a whole number greater than 2. Through a comprehensive study of the theory of the algebraic form a^2+ab+b^2, which he has published in book form, he has demonstrated the propriety of Euler's assumptions. In addition to his published work,

Mr. Johnson has among his unpublished work a proof that $x^{p-1}+y^{p-1}=z^{p-1}$ cannot be satisfied by whole numbers when p is any prime number greater than 3. This, he believes, is the nearest approach which has been made to a complete solution of Fermat's last theorem.

6. Mr. Bryson gave a survey of the courses of lectures which he attended at Brown University during the summer of 1941.

DAVID MOSKOVITZ, *Secretary*

HEAT FLOW AND NON-EUCLIDEAN GEOMETRY

E. W. BARANKIN, University of California

In *La Science et l'Hypothèse* (Paris, Ernest Flammarion) Poincaré has well established a very fundamental property of physical space—its *passivity*; that is, the applicability thereto of an arbitrary geometry. There, on pp. 84 to 87, he illustrates the thesis with the example of a 3-dimensional world consisting of the interior of a sphere of radius R, each concentric sphere of radius $r<R$ being held at a constant absolute temperature $T \propto R^2-r^2$, and objects in this world being subject, under displacement, to expansion and contraction accordingly. We propose to develop here a similar example of a 2-dimensional world, at once treating rather thoroughly its purely mathematical side and emphasizing its physical import. Concerning the latter, the significant principle which will be brought out is that of the functional relationship between the geometry of physical space and physical law.

Let us imagine that we one day find it necessary to deal in the laboratory with a rather large sheet of copper. The sheet is not, however, of uniform overall temperature; on the contrary, if we take one edge of the sheet to be the x-axis, and erect perpendicular to it a y-axis, then the temperature, in degrees Centigrade, is the following function of the Euclidean coördinates:

$$T = b\left(y - \frac{1}{pb}\right),$$

where b and p are constants. For the present, let there be constraints on the sheet to prevent it from expanding or buckling. Finally, we are in possession of an infinitesimal measuring rod of iron, whereof we ideally take the thermal coefficient of expansion to be p for all T; and we must meet the problem of establishing geometric relations on the sheet.

We should at first be very reluctant to measure distances on the sheet with the iron rod, knowing that it would alter in length as we moved it from place to place.* Rather would we ask for a rod of invar, a metal with a coefficient of

* For we shall agree to make measurements in a region only after the rod has come to thermal equilibrium with that region.

expansion of zero. But in the event that the iron rod were the only one available, we should have to resign ourselves to its use. And the question arises: how can we be consistent in our measurements with such a rod?

There are, in fact, two simple ways:

1. At 0° (the standard temperature at which we take all true or Euclidean length-measures) let the rod be of true length ds; its reading then at all positions on the sheet is ds. But the true length at the point (x, y) is:

$$ds' = [1 + pT]ds = pbyds.$$

Hence, we may preserve Euclidean measure, and therefore Euclidean geometry, for the surface if we carry the temperature correction factor (pby) over the surface. In this case the temperature correction factor is the physical law pertaining to the matter in the space, and the corresponding geometry is Euclidean.

2. *Perform measurements without regard to the changes in the rod.* This suggestion is at first glance rather shocking; and it is not immediately obvious that such a course is a consistent one. Here we are taking as the governing physical law: matter in the space undergoes no changes. Clearly this law is simpler than that of case 1; it remains to discover whether or not there corresponds a permissible geometry. We occupy the following pages with a determination of this geometry and a demonstration of its consistency.

In the first place, it is seen that the x-axis is the line at infinity; for if the length of a line segment perpendicular to the x-axis and having its foot on this axis is measured with a rod, the rod will become shorter and shorter; and in the neighborhood of the axis the rod must repeat itself a great number of times to measure a very small true-length. In other words, this segment will, when measured toward the x-axis, appear infinite in length.

We now turn to the problem, foremost in importance, of discovering in what ways we may move a physical object, such as a triangle* made up of rods, about in the space so that the object remains congruent to itself. Such a movement we call a *displacement*. Obviously all displacements are sense-preserving. But the notion of anticongruence may well figure in our space in the form of a sense-reversing transformation. We therefore employ the broader term *motion* for any congruence transformation. We may then set ourselves the task of determining the set of motions of our space; the set of displacements will be just the collection of sense-preserving motions.

This problem is readily attacked with infinitesimal considerations; but first we must derive the metric of the space. At a point (x, y) we shall have, if the Euclidean projections of the rod on the x- and y-axes are, respectively, dx and dy, the following:

$$pbyds = [dx^2 + dy^2]^{1/2},$$

* In general, a triangle is defined as a closed, three-sided figure, the edges being segments of geodesics.

whence immediately we have our metric:

$$ds^2 = \frac{dx^2 + dy^2}{(pb)^2 y^2}.$$

We may choose our units in such a way that pb is unity, and write:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{1}$$

We emphasize here the fact that the Euclidean coördinates which we have called up constitute merely a frame for analytical treatment, and do not participate intrinsically in the geometry to be developed.

In terms of the metric, a motion is a transformation,

$$x' = x'(x, y), \qquad y' = y'(x, y), \tag{2}$$

to be thought of as a point transformation referred to the above invariable cartesian axes—such that:

$$\frac{dx'^2 + dy'^2}{y'^2} = \frac{dx^2 + dy^2}{y^2}. \tag{3}$$

That this is so follows from a brief consideration. First, equation (3) obviously guarantees the preservation of distance. It also implies the preservation of angles (conformality), in giving:

$$\begin{gathered}
\frac{1}{y'^2}\left[\left(\frac{\partial x'}{\partial x}\right)^2 + \left(\frac{\partial y'}{\partial x}\right)^2\right] = \frac{1}{y^2},\\
\left[\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y} + \frac{\partial y'}{\partial x}\frac{\partial y'}{\partial y}\right] = 0,\\
\frac{1}{y'^2}\left[\left(\frac{\partial x'}{\partial y}\right)^2 + \left(\frac{\partial y'}{\partial y}\right)^2\right] = \frac{1}{y^2}.
\end{gathered} \tag{4}$$

For, let (d_1x, d_1y) and (d_2x, d_2y) be two infinitesimal segments, drawn from the point (x, y) and there making an angle θ. Further, let (d_1x', d_1y') and (d_2x', d_2y') be their respective transforms under (2). θ is given by:

$$\begin{aligned}
\cos\theta &= \frac{d_1xd_2x + d_1yd_2y}{[(d_1x^2 + d_1y^2)(d_2x^2 + d_2y^2)]^{1/2}}\\
&= \frac{\frac{1}{y^2}(d_1xd_2x + d_1yd_2y)}{\left[\left(\frac{d_1x^2 + d_1y^2}{y^2}\right)\left(\frac{d_2x^2 + d_2y^2}{y^2}\right)\right]^{1/2}}.
\end{aligned} \tag{5}$$

Likewise, the angle θ' formed by the transformed segments at (x', y') is given by:

$$\cos\theta' = \frac{\dfrac{1}{y'^2}(d_1x'd_2x' + d_1y'd_2y')}{\left[\left(\dfrac{d_1x'^2 + d_1y'^2}{y'^2}\right)\left(\dfrac{d_2x'^2 + d_2y'^2}{y'^2}\right)\right]^{1/2}}. \tag{6}$$

Under (3) the denominators of (5) and (6) are equal and we have the condition for conformality:

$$\frac{1}{y'^2}(d_1x'd_2x' + d_1y'd_2y') = \frac{1}{y^2}(d_1xd_2x + d_1yd_2y). \tag{7}$$

If we apply to the left-hand member of (7) the induced transformation,

$$dx' = \frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy, \qquad dy' = \frac{\partial y'}{\partial x}dx + \frac{\partial y}{\partial y}dy,$$

and bring to bear the relations (4), we find that the left-hand member reduces indentically to the right-hand member, wherefore the validity of our analytic characterization of a motion is fully established.

Take the infinitesimal line segment of Euclidean length δs (Fig. 1) and any point P_0 on the x-axis; describe rays from P_0 through the end-points of δs. Then

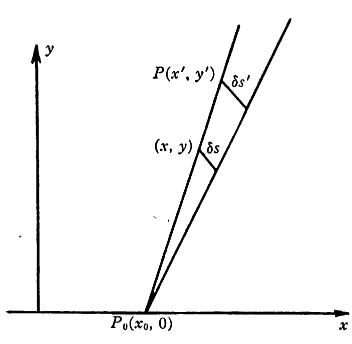

FIG. 1

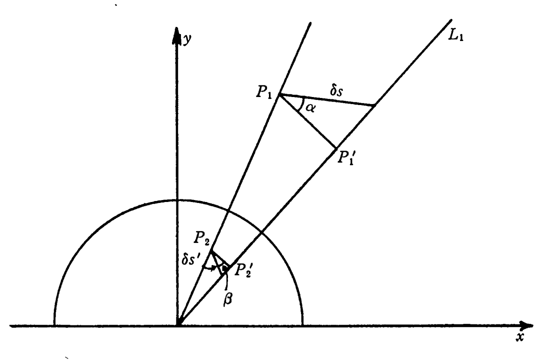

FIG. 2

at any arbitrary point P on one of the rays draw the segment of Euclidean length $\delta s'$ parallel to δs and ending on the second ray. The segment $\delta s'$ is equal to the segment δs under our non-Euclidean metric. For, by similar triangles,

$$\frac{\delta s}{y} = \frac{\delta s'}{y'}.$$

But the non-Euclidean length of the segment δs is given by (1) as

$$ds = \frac{\delta s}{y}$$

and that of the segment $\delta s'$ as:

$$ds' = \frac{\delta s'}{y'}.$$

Hence:

$$ds' = ds.$$

This equation is identifiable with (3), the pertinent transformation being:

$$x' = ax + (1 - a)x_0, \qquad y' = ay, \qquad (a \geqq 0), \tag{8}$$

where a is the factor of magnification.

A second motion makes itself evident when we observe the form of the metric closely. Since only the differential of the variable x occurs, and since the differential is invariant under the addition of a constant to the variable, it is immediately clear that we obtain a motion by augmenting x by a constant and leaving y unchanged. This amounts to the translation parallel to the x-axis:

$$x' = x + t, \qquad y' = y, \tag{9}$$

where t is the parameter of translation.

We know that if, in a two-dimensional space, there definitely exist two independent motions, then a third independent motion must exist.* We therefore seek the third motion; and in fact, we discover this to be an inversion. For, consider again the infinitesimal segment of Euclidean length δs at P_1, and the unit circle (Fig. 2). Inversion in this circle carries δs into $\delta s'$. Construct the segments $\overline{P_1P_1'}$ and $\overline{P_2P_2'}$ perpendicular to the ray L_1. Then, as proved above, segment $\overline{P_1P_1'}$ is equal to segment $\overline{P_2P_2'}$ under the metric (1). Moreover, inversion being a conformal transformation, we have $\alpha=\beta$. It follows therefore, since

$$\cos\alpha = \frac{\overline{P_1P_1'}}{\delta s} = \cos\beta = \frac{\overline{P_2P_2'}}{\delta s'},$$

that segment δs is equal to segment $\delta s'$ under (1). Analytically this inversion is:

$$x' = \frac{x}{x^2 + y^2}, \qquad y' = \frac{y}{x^2 + y^2}. \tag{10}$$

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* See Eisenhart, Riemannian Geometry, p. 241, par. 74.

A similar derivation could have been given for any circle centered on the x-axis, for example, that of radius r with center at $(x_0, 0)$. The resultant inversion, $I_{x_0,r}$,

$$x' = x_0 + \frac{r^2(x - x_0)}{(x - x_0)^2 + y^2}, \qquad y' = \frac{r^2 y}{(x - x_0)^2 + y^2},$$

were then a product of the above simple motions in the following order (Fig. 3): First a translation T_1 of δs:

$$x' = x - x_0, \qquad y' = y,$$

then an inversion $I_{0,1}$ in the unit circle:

$$x' = \frac{x}{x^2 + y^2}, \qquad y' = \frac{y}{x^2 + y^2},$$

a magnification, M, through the origin:

$$x' = r^2 x, \qquad y' = r^2 y,$$

and finally a translation T_2:

$$x' = x + x_0, \qquad y' = y.$$

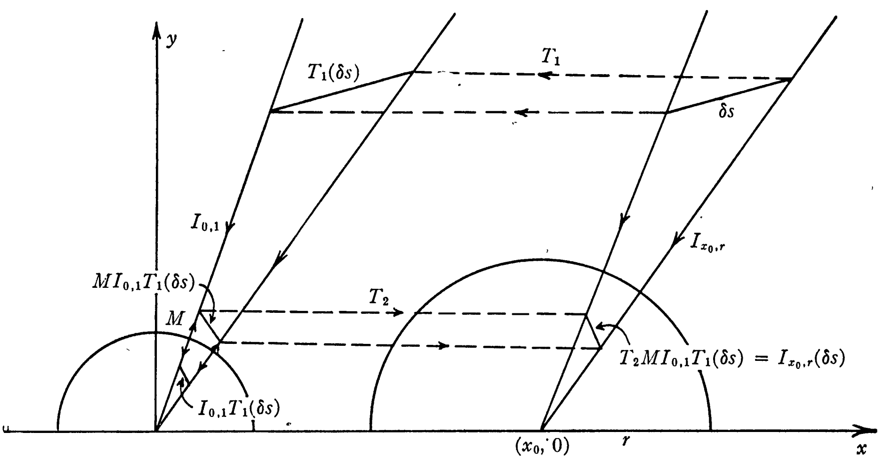

FIG. 3

Concerning the determination of these component transformations, we may remark that the translation T_1 is first performed to bring the domain of the inversion $I_{x_0,r}$ into radial coincidence with that of $I_{0,1}$, the magnification, to account for the different radii of the two circles of inversion, and the translation $T_2 = T_1^{-1}$, to return the domain of $I_{x_0,r}$ to its original position.

The motion of inversion introduces the notion of anti-congruence into our space. In this connection it is of interest to note that Euclid, in order to prove

completely his propositions on the congruence of triangles, had to make use of superposition, an exaction on the third dimension. This difficulty arises when the three independent motions of the Euclidean plane are taken as two translations and a rotation, in the stead of two translations and a reflection. Of course, a rotation may be obtained by the multiplication of two reflections.

We now have the complete* set of three basic independent motions, a translation parallel to the x-axis, a magnification through the origin, and an inversion in the unit circle. These motions forms a group G† which is called the congruence group of the space. What further information we require concerning this group can be most easily obtained by expressing the basic motions in terms of the complex variables $z=x+iy$ and $w=x'+iy'$. We have:

$$\begin{aligned} &\text{Translation,} \quad T: \quad w = z + t, && (t, \text{ real}), \\ &\text{Magnification,} \quad M: \quad w = az, && (a \geqq 0), \\ &\text{Inversion,} \quad I: \quad w = 1/\bar{z}. && \end{aligned} \tag{11}$$

The presence of *three* independent motions in the group is the abstract equivalent of the fact that a two dimensional configuration in the space may be shifted arbitrarily and, so to speak, turned inside-out and still remain congruent, or in the latter case, anticongruent, to itself. This complete freedom of movement is precisely what we hoped to discover for the case of a physical object, and we shall therefore establish the fact explicitly for sense-preserving motions, that is, displacements.

The set of displacements forms a subgroup $\overline{G}$ of G; and $\overline{G}$ is generated by the three independent, sense-preserving motions:

$$\begin{aligned} &T: \quad w = z + t, && \\ &M: \quad w = az, && \\ &U: \quad w = \frac{z}{fz + 1}, && (f, \text{ real}). \end{aligned} \tag{12}$$

The latter claim requires a detailed proof: A displacement is any product of the fundamental motions (11) involving an even number of inversions. It is easily shown that such a general product takes the form D:

$$w = \frac{kz + l}{mz + n}$$

where k, l, m, n are real numbers and $kn-ml \geqq 0$. But this general displacement

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* In an N-dimensional space there are at most $N(N+1)/2$ independent motions.

† This fact is easily proved; we shall not perform the work here.

is obtained by the multiplication of the following displacements of the forms (12):

$$
(13)\qquad
\begin{aligned}
U_1:&\quad w = \frac{z}{\dfrac{m}{n}z + 1},\\
M_1:&\quad w = \left(\frac{kn - ml}{n^2}\right)z,\\
T_1:&\quad w = z + \frac{l}{n}\cdot
\end{aligned}
$$

In fact, $D = T_1M_1U_1$.

The displacement U is the product of an inversion in the unit circle, a translation by amount f, and another inversion. Our method of replacing the inversion of the basis (11) by U, to obtain a basis for the group $\overline{G}$, is equivalent to the substitution, in the case of the Euclidean plane, of a rotation (the product of two reflections) for the basic motion of reflection to obtain a basis for the group of Euclidean displacements.

Now, an arbitrary shifting of a physical object in our space may be characterized in this manner: the allocation of a particular point of the object with an arbitrarily chosen point of the space, and the alignment of a particular direction fixed in the object with an arbitrarily chosen direction in the space. This process clearly requires only *three* parameters for its completion, and the group $\overline{G}$ is a three-parameter group. It is therefore proved that an object in the space enjoys free mobility, a fact denoted by calling the space a *congruence space* and its geometry, a *congruence geometry*.

We now inquire after the geodesics of our space. For the distance between two points P_1 and P_2 we have

$$
(14)\qquad s = \int_{P_1}^{P_2} \frac{1}{y}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.
$$

If $F \equiv 1/y\sqrt{1+(dy/dx)^2}$, the Euler equation for extremality of this integral is

$$
\frac{d}{dx}\,\frac{\partial F}{\partial\left(\dfrac{dy}{dx}\right)} = \frac{\partial F}{\partial y}\cdot
$$

We are led, by the substitution for F, to the differential equation of the family of geodesics,

$$
\frac{d^2y}{dx^2} + \frac{1}{1+y}\left[1 + \left(\frac{dy}{dx}\right)^2\right] = 0.
$$

The solution of this equation gives the family of circles:

$$(x - h)^2 + y^2 = c^2, \tag{15}$$

where h and c are constants of integration. All these circles are centered on the x-axis, and the upper half-circumferences are the geodesics of our space. The geodesic through the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ with different ordinates is seen to degenerate into a straight line parallel to the y-axis when x_2 approaches x_1 in value.

In the Euclidean plane there can be drawn through a given point one and only one straight line parallel to a given straight line. Or, in general terms, there can be drawn through a given point one and only one geodesic parallel to a given geodesic, that is, not intersecting it in any finite point. This is not the case in the heated plane. For, let C_1 be a given geodesic; and let P be a point not on C_1. Describe the geodesics C_2 and C_3 tangent to C_1 at the points where it meets the x-axis. Then both C_2 and C_3 are parallel, in the sense defined above, to C_1. Moreover any geodesic whose slope at P lies between those of C_2 and C_3 is also parallel to C_1.

The student of the non-Euclidean geometries will have immediately recalled the name Lobatschewsky when he read the above result concerning parallels. For it is so in the Lobatschewskian geometry that to a given straight line (geodesic) can be drawn a limited infinity of parallels through a neighboring point.* And in fact our heated-plane geometry is exactly that of Lobatschewsky. A little exercise with triangles in this plane will soon convince the reader that the sum of the angles in a triangle is always less than two right angles.

Two further computations bear results of interest from the projective point of view. First we compute the geodesic distance s_g between two points. Let the two points be $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$; then, substituting $\overset{+}{\sqrt{c^2-(x-h)^2}}$ for y in formula (14), we have:

$$s_g = \int_{x_1}^{x_2} \frac{c}{c^2 - (x - h)^2}\, dx = \tfrac{1}{2} \log_e \left\{ \frac{x_2 - (h - c)}{(h + c) - x_2} \cdot \frac{(h + c) - x_1}{x_1 - (h - c)} \right\}.$$

But, if x_3 and x_4 ($x_4 > x_3$) denote respectively the x-coördinates of the two points in which the geodesic meets the x-axis, we find by (15) that

$$x_3 = h - c, \qquad x_4 = h + c.$$

Hence:

$$s_g = \tfrac{1}{2} \log_e \left\{ \frac{x_2 - x_3}{x_4 - x_2} \cdot \frac{x_4 - x_1}{x_1 - x_3} \right\},$$

* Some mathematicians choose to say that there are only two parallels to a given geodesic through a point, in this geometry, corresponding to C_2 and C_3 in the above description; they employ the term *skew* for all the others that we have called parallel. But this is purely a matter of definition.

or, in terms of the cross-ratio

$$R_1 = \frac{(x_1 - x_4)(x_2 - x_3)}{(x_2 - x_4)(x_1 - x_3)}$$

of the four ordered points $(x_1, 0)$, $(x_2, 0)$, $(x_4, 0)$, $(x_3, 0)$,

$$s_g = \tfrac{1}{2} \log_e R_1.$$

Secondly, we derive the formula for the angle between two geodesics. Let C_1 and C_2 be two geodesics, intersecting at the point (x_0, y_0), and there making an angle θ. From equation (15) we derive the respective slopes of the two geodesics:

$$\mu_1 = -\frac{x - h_1}{y}, \qquad \mu_2 = -\frac{x - h_2}{y}.$$

Hence,

$$\tan\theta = \left(\frac{\mu_2 - \mu_1}{1 + \mu_2\mu_1}\right)_{(x_0, y_0)} = \frac{\dfrac{h_2 - h_1}{y_0}}{1 + \dfrac{(x_0 - h_1)(x_0 - h_2)}{y_0^2}}.$$

If, for $i = 1, 2$, we let the geodesic, C_i intersect the x-axis in the points $(x_{i1}, 0)$, and $(x_{i2}, 0)$ with $x_{i1} > x_{i2}$, we have

$$y_0^2 = (x_0 - x_{11})(x_{12} - x_0) = (x_0 - x_{21})(x_{22} - x_0).$$

The last two members give:

$$x_0 = \frac{x_{21}x_{22} - x_{11}x_{12}}{(x_{21} + x_{22}) - (x_{11} + x_{12})}$$

which in turn gives:

$$y_0 = \frac{\sqrt{(x_{22} - x_{11})(x_{22} - x_{12})(x_{21} - x_{11})(x_{12} - x_{21})}}{[(x_{21} + x_{22}) - (x_{11} + x_{12})]}.$$

Substituting these values and the values

$$h_j = \frac{x_{j1} + x_{j2}}{2}, \qquad (j = 1, 2),$$

into the expression for $\tan\theta$, we obtain, after rearrangement,

$$\tan\theta = \frac{2\sqrt{(x_{22} - x_{11})(x_{22} - x_{12})(x_{21} - x_{11})(x_{12} - x_{21})}}{(x_{12} - x_{21})(x_{22} - x_{11}) - (x_{22} - x_{12})(x_{21} - x_{11})}.$$

In terms of the cross-ratio

$$R_2 = \frac{(x_{21} - x_{11})(x_{22} - x_{12})}{(x_{22} - x_{11})(x_{21} - x_{12})}$$

of the four ordered points $(x_{21}, 0)$, $(x_{22}, 0)$, $(x_{11}, 0)$ and $(x_{12}, 0)$, this formula reduces to:

$$\tan^2 \frac{\theta}{2} = - R_2.$$

We have now quite sufficiently established the validity of our second suggested method of treatment of the copper sheet. For, above all, the prime question is settled: the geometry generated by the simpler physical law there laid down is a congruence geometry. And without doubt we should unanimously choose this second method for dealing with the heated sheet, for the new geometry exactly equals that of Euclid in completeness, and we gain in simplicity of physical law.

If the constraints, which we applied in the beginning, were removed from the copper sheet, it would expand in accordance with the varying temperature and buckle into a surface with the metric,

$$ds^2 = y^2(dx^2 + dy^2). \tag{16}$$

To see this, consider, at the point (x, y) on the constrained sheet, a small scratch with components of length dx and dy. When the sheet is allowed to buckle, these components assume new lengths $dx' = dx(1+pT) = ydx$ and $dy' = dy(1+pT) = ydy$, respectively. The expanded length ds of the scratch is then given by

$$ds^2 = dx'^2 + dy'^2 = y^2(dx^2 + dy^2).$$

For working with this surface we should choose a rod of copper in preference to an invar rod. For now the copper rod would expand at a point of the surface in the same way that the sheet itself has expanded at that point, and under the law that the rod suffers no change we should obtain Euclidean geometry for the surface. The invar rod would, under the same simple law, give us the intrinsic geometry involved in the metric (16). But this geometry is *not* a congruence geometry; objects on this surface must, if they are to remain self-congruent throughout the movement, be moved only along the curves $y = \text{constant}$.

FOUR FINITE GEOMETRIES*

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1. Introduction. A finite geometry is a geometry based on a set of postulates, undefined terms, and undefined relations which limits the set of all points and lines to a finite number. This is usually accomplished by a postulate limiting the number of points on a line. In the first three of the finite geometries considered in this paper there is the following postulate: "No line contains more than three points." In the Desargues finite geometry this is a theorem which follows from the postulates. The set of postulates should fulfill the three requirements of consistency, independence, and categoricalness.

Finite geometries were brought into prominence by the publication of the Veblen and Young *Projective Geometry* (Ginn and Co., Vol. I, 1910; Vol. II, 1918) and by Young's *Lectures on Fundamental Concepts of Algebra and Geometry* (The Macmillan Co., 1911).

The simplest example is the finite geometry of 7 points and 7 lines given in Volume I, Chapter I of the *Projective Geometry* of Veblen and Young. This finite geometry was first considered by Fano in 1892 in 3 dimensions where there are 15 points and 35 lines, but in each plane there are 7 points and 7 lines.

The notion of a class of objects is fundamental in logic. The objects which make up a class are called the elements of the class. The notion of a class and the relation "belonging to a class" will be undefined. Given a set S with elements $A_1, A_2, A_3, \cdots$, let S have certain undefined sub-classes any one of which will be called an m-class, —or in particular, given a set of points $A_1, A_2, A_3, \cdots$, let certain sets of points be associated in an undefined way in sets called lines.

2. The seven point finite geometry. The postulates for the 7 point finite geometry may be stated as follows:

(1′). If A_1 and A_2 are distinct points (elements of S), there is at least one line (m-class) containing A_1 and A_2.

(2′). If A_1 and A_2 are distinct points (elements of S), there is not more than one line (m-class) containing A_1 and A_2.

(3′). Any two lines (m-classes) have at least one point (element of S) in common.

(4′). There exists at least one line (m-class).

(5′). Every line (m-class) contains at least three points (elements of S).

(6′). All the points (elements of S) do not belong to the same line (m-class).

(7′). No line (m-class) contains more than three points (elements of S).

Symbolic diagram where the vertical columns represent lines (m-classes):

$$\begin{matrix} A_1A_2A_3A_4A_5A_6A_7 \\ A_2A_3A_4A_5A_6A_7A_1 \\ A_4A_5A_6A_7A_1A_2A_3 \end{matrix}$$

It is, however, unfortunate that it is necessary to assume that the three points

* Presented at the organization meeting of the Metropolitan New York Section of the Mathematical Association of America at Queens College of the City of New York on April 19, 1941.

A_1, A_3, and A_7, *i.e.*, the diagonal points of the complete quadrangle, are collinear as indicated by the dotted line.

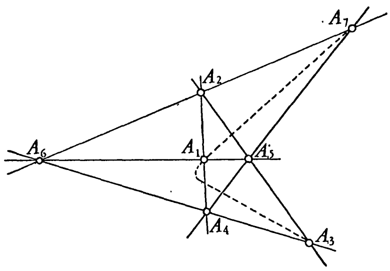


FIG. 1. Geometric diagram for 7 point finite geometry.

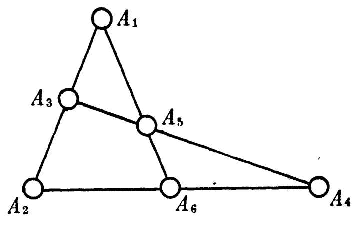
To prove that a postulate of a set of postulates is independent of the rest, it is sufficient to give an example which violates that postulate and fulfills all the rest. Since there are seven postulates in this set, it is necessary to give seven examples to complete the independence proof. This is not always easy to do, and in the case of Hilbert's postulates twenty-one examples would be necessary to complete the independence proof.

If the word "three" were changed to "two" in postulates 5 and 7, the entire geometry would consist of a single triangle which might well be considered as the simplest non-trivial finite geometry; so that seven independent postulates in this case define a geometry consisting of just one triangle.

The independence of the postulates for this finite geometry is shown by the following examples, whose numbers correspond to the number of the postulate which does not hold in the example.

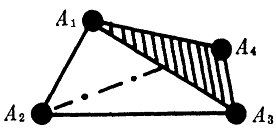
(1′). A complete quadrilateral.

$$\begin{array}{cccc} A_1 & A_1 & A_2 & A_3 \\ A_2 & A_5 & A_4 & A_4 \\ A_3 & A_6 & A_6 & A_5 \end{array}$$



(2′). A tetrahedron $A_1A_2A_3A_4$, where the faces represent lines.

$$\begin{array}{cccc} A_1 & A_1 & A_1 & A_2 \\ A_2 & A_2 & A_3 & A_3 \\ A_3 & A_4 & A_4 & A_4 \end{array}$$



(3′).

$$\begin{array}{cccccccccccc} A_1 & A_1 & A_1 & A_1 & A_2 & A_2 & A_2 & A_3 & A_3 & A_3 & A_4 & A_7 \\ A_2 & A_4 & A_5 & A_6 & A_5 & A_4 & A_6 & A_4 & A_5 & A_6 & A_5 & A_8 \\ A_3 & A_7 & A_9 & A_8 & A_8 & A_9 & A_7 & A_8 & A_7 & A_9 & A_6 & A_9 \end{array}$$

(This is the Young finite geometry of 9 points and 12 lines, section 3.)

(4′). A single point (the remaining postulates are fulfilled vacuously).

(5′). A triangle with A_1, A_2, A_3 as vertices.

(6′). A single line containing 3 points A_1, A_2, A_3.

(7′). Projective geometry.

This seven point finite geometry has been generalized to give a finite geometry of thirteen points and thirteen lines if four is substituted for three in postulates 5 and 7; and to n^2+n+1 points and lines, if $n+1$ is substituted for three in postulates 5 and 7.

Finite geometries of this type have been treated extensively from the standpoint of algebra and finite groups.*

The question now arises as to what theorems there are in this seven point finite geometry. In the first place, the duals of the postulates may be proved as theorems and the geometry will then have duality. Postulates 1 and 3 are duals.

THEOREM 1. (Dual of Postulate 2). *Two distinct lines have only one point in common.*

THEOREM 2. (Dual of Postulate 4). *There exists at least one point.*

THEOREM 3. (Dual of Postulate 5). *At least three lines pass through every point.*

THEOREM 4. (Dual of Postulate 6). *All lines do not pass through the same point.*

THEOREM 5. (Dual of Postulate 7). *Not more than three lines pass through every point.*

These theorems are all easy to prove, and they show that the geometry has duality. Two other theorems suggest themselves.

THEOREM 6. *The geometry contains precisely seven points.*

THEOREM 7. (Dual of Theorem 6). *The geometry contains precisely seven lines.*

The entire body of theorems of this finite geometry of seven points and seven lines consists primarily of these seven theorems. The finite geometry has the characteristics of a projective geometry and might be considered as the simplest type of a projective geometry.

3. A finite geometry of nine points and twelve lines. If in the postulates of section 2, the word "three" is changed to "four" in postulates 5 and 7, the postulates are satisfied by a finite geometry of 13 points and 13 lines. But, if in this geometry one line of four points is omitted, we obtain a geometry of 9 points and 12 lines, which is equivalent to projecting one line to infinity and converting the projective geometry of 13 points and 13 lines without parallel lines into a euclidean geometry of 9 points and 12 lines with parallel lines.

* Veblen and Bussey, Finite projective geometries, Transactions of the American Mathematical Society, vol. 7, 1906, pp. 241–259.

This is in some ways an advantage and a simplification, because in general there is a preference—for historical reasons—for a euclidean geometry in which the parallel postulate is true. This geometry has been used as an example of a complete logical system by Cohen and Nagel in their book, *An Introduction to Logic and Scientific Method* (Harcourt, Brace and Co., 1934).

It is remarkable that the 9 inflection points of a general plane cubic, as far as collinearity properties are concerned, fulfill all of the postulates of this finite geometry.

The following eight postulates define this finite geometry:

(1). If A_1 and A_2 are distinct points (elements of S), there exists one line (m-class) containing A_1 and A_2.

(2). If A_1 and A_2 are distinct points (elements of S), there exists not more than one line (m-class) containing A_1 and A_2.

(3). Given a line a (m-class a) not containing a point A (given element A of S), there exists one line (m-class) containing A and not containing any point of a (element of S belonging to m-class a).

(4). Given a line a (m-class a) not containing a point A (given element of S), there exists not more than one line (m-class) containing A and not containing any point of a (element of S belonging to m-class a).

(5). Every line (m-class) contains at least three points (elements of S).

(6). Not all points (elements of S) are contained by the same line (m-class).

(7). There exists at least one line (m-class).

(8). No line (m-class) contains more than three points (elements of S).

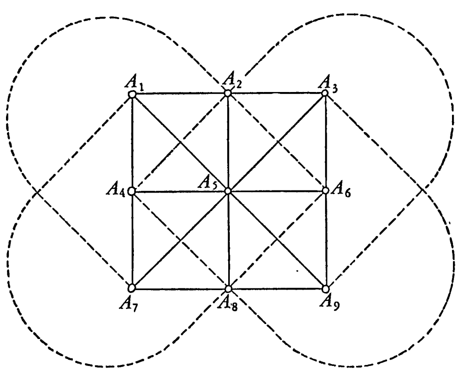


FIG. 2. Geometric diagram for the Young 9 point finite geometry.

Symbolic diagram where the vertical columns represent m-classes (lines):

$$
\begin{array}{cccccccccccc}
A_1 & A_1 & A_1 & A_1 & A_2 & A_2 & A_2 & A_3 & A_3 & A_3 & A_4 & A_7 \\
A_2 & A_4 & A_5 & A_6 & A_5 & A_4 & A_6 & A_4 & A_5 & A_6 & A_5 & A_8 \\
A_3 & A_7 & A_9 & A_8 & A_8 & A_9 & A_7 & A_8 & A_7 & A_9 & A_6 & A_9
\end{array}
$$

The independence of the postulates is shown by the following examples,*

* See article by A. Barshop, Brooklyn College Mathematics Mirror, Issue no. VII, 1939, p. 14.

where parentheses represent lines or m-classes.

(1′). Two lines $(A_1A_2A_3)$, $(A_4A_5A_6)$.

(2′). Six points (elements) A_1, A_2, A_3, A_4, A_5, A_6 taken three at a time to form twenty lines (m-classes).

$$\begin{array}{lllll}
(A_1A_2A_3), & (A_1A_2A_4), & (A_1A_2A_5), & (A_1A_2A_6), & (A_1A_3A_4), \\
(A_1A_3A_5), & (A_1A_3A_6), & (A_1A_4A_5), & (A_1A_4A_6), & (A_1A_5A_6), \\
(A_2A_3A_4), & (A_2A_3A_5), & (A_2A_3A_6), & (A_2A_4A_5), & (A_2A_4A_6), \\
(A_2A_5A_6), & (A_3A_4A_5), & (A_3A_4A_6), & (A_3A_5A_6), & (A_4A_5A_6).
\end{array}$$

(This is a complete 6-point in space, where the planes represent lines.)

(3′). $(A_1A_2A_4)$, $(A_2A_3A_5)$, $(A_3A_4A_6)$, $(A_4A_5A_7)$, $(A_5A_6A_1)$, $(A_6A_7A_2)$, $(A_7A_1A_3)$.

(This is the seven point finite geometry of section 2.)

(4′).
$$\begin{array}{lllll}
(A_1A_4A_5), & (A_2A_5A_6), & (A_3A_6A_7), & (A_4A_7A_8), & (A_5A_8A_9), \\
(A_6A_9A_{10}), & (A_7A_{10}A_{11}), & (A_8A_{11}A_{12}), & (A_9A_{12}A_{13}), & (A_{10}A_{13}A_{14}), \\
(A_{11}A_{14}A_{15}), & (A_{12}A_{15}A_1), & (A_{13}A_1A_2), & (A_{14}A_2A_3), & (A_{15}A_3A_4), \\
(A_1A_3A_9), & (A_2A_4A_{10}), & (A_3A_5A_{11}), & (A_4A_6A_{12}), & (A_5A_7A_{13}), \\
(A_6A_8A_{14}), & (A_7A_9A_{15}), & (A_1A_8A_{10}), & (A_2A_9A_{11}), & (A_3A_{10}A_{12}), \\
(A_4A_{11}A_{13}), & (A_5A_{12}A_{14}), & (A_6A_{13}A_{15}), & (A_1A_7A_{14}), & (A_2A_8A_{15}), \\
(A_1A_6A_{11}), & (A_2A_7A_{12}), & (A_3A_8A_{13}), & (A_4A_9A_{14}), & (A_5A_{10}A_{15}).
\end{array}$$

(5′). A complete quadrilateral.

$$(A_1A_2),\ (A_1A_3),\ (A_1A_4),\ (A_2A_3),\ (A_2A_4),\ (A_3A_4).$$

(6′). A single line of three points $(A_1A_2A_3)$.

(7′). A single point A_1—no lines.

(8′). Plane euclidean geometry.

This finite geometry is euclidean in the sense that through any point not on a line there is one and only one line parallel to that line. The geometry does not have the property of duality because any two distinct points determine one line, but any two distinct lines do not determine a point since they may be parallel.

Several theorems suggest themselves, such as the following:

THEOREM 1. *There exist exactly nine points.*

THEOREM 2. *There exist exactly twelve lines.*

THEOREM 3. *Every line has precisely two lines parallel to it.*

THEOREM 4. *Two lines parallel to a third line are parallel to each other.*

THEOREM 5. *The six points on two parallel lines determine a hexagon such that the intersection points of opposite sides are collinear.* (Pappus-Pascal theorem).

4. The Pappus finite geometry. The postulates of the Pappus finite geometry may be stated as follows:

(1). There exists at least one line (m-class).

(2). Not all points (elements of S) belong to the same line (m-class).

(3). Not more than one line (m-class) contains any two points (elements of S).

(4). Every line (m-class) contains at least 3 points (elements of S).

(5). No line (m-class) contains more than 3 points (elements of S).

(6). Given a line (m-class) and a point (element of S) not on it, there exists a line (m-class) containing the given point (element of S) which has no point (element of S) in common with the first line (m-class).

(7). Given a line (m-class) and a point (element of S) not on it, there exists not more than one line (m-class) containing the given point (element of S) which has no point (element of S) in common with the first line (m-class).

(8). Given a point (element of S) and a line (m-class) not containing it, there exists a point (element of S) contained in the given line (m-class) which is not on any line (m-class) with the first point (element of S).

(9). Given a point (element of S) and a line (m-class) not containing it, there exists not more than one point (element of S) contained in the given line (m-class) which is not on any line (m-class) with the first point (element of S).

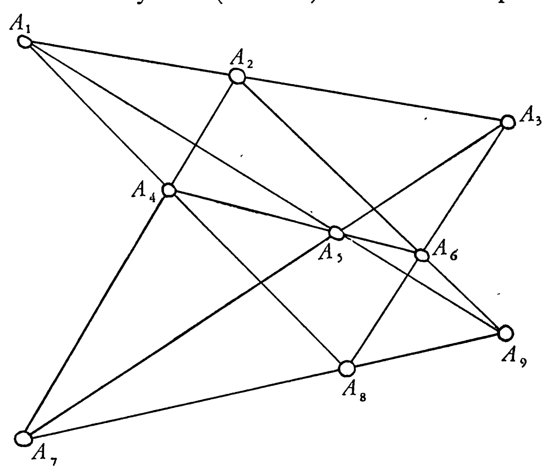
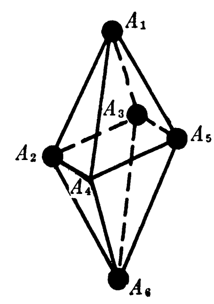


FIG. 3. Geometric diagram for the Pappus finite geometry.

Symbolic diagram where the vertical columns represent lines (m-classes)

$$\begin{matrix} A_1 & A_1 & A_1 & A_2 & A_2 & A_3 & A_3 & A_4 & A_7 \\ A_2 & A_4 & A_5 & A_4 & A_6 & A_5 & A_6 & A_5 & A_8 \\ A_3 & A_8 & A_9 & A_7 & A_9 & A_7 & A_8 & A_6 & A_9 \end{matrix}$$



The independence of the postulates is shown by the following examples.

(1′). A single point A_1.

(2′). A single line containing points A_1, A_2, A_3.

(3′). The faces of an octahedron, where the faces represent lines.

$(A_1A_2A_3)$, $(A_1A_2A_4)$, $(A_1A_4A_5)$, $(A_1A_3A_5)$,
$(A_6A_2A_3)$, $(A_6A_2A_4)$, $(A_6A_5A_4)$, $(A_6A_3A_5)$.

(4′). A simple quadrilateral (A_1A_2), (A_2A_3), (A_3A_4), (A_4A_1).

(5′). $(A_1A_5A_9A_{13})$, $(A_1A_6A_{10}A_{14})$, $(A_1A_7A_{11}A_{15})$, $(A_1A_8A_{12}A_{16})$,
$(A_2A_5A_{10}A_{16})$, $(A_2A_6A_9A_{15})$, $(A_2A_7A_{12}A_{14})$, $(A_2A_8A_{11}A_{13})$,
$(A_3A_5A_{11}A_{14})$, $(A_3A_6A_{12}A_{13})$, $(A_3A_7A_9A_{16})$, $(A_3A_8A_{10}A_{15})$,
$(A_4A_5A_{12}A_{15})$, $(A_4A_6A_{11}A_{16})$, $(A_4A_7A_{10}A_{13})$, $(A_4A_8A_9A_{14})$.

(6′). $(A_1A_2A_3)$, $(A_1A_5A_6)$, $(A_2A_4A_6)$, $(A_3A_4A_5)$.

(7′). $(A_1A_5A_9)$, $(A_1A_6A_{10})$, $(A_1A_7A_{11})$, $(A_1A_8A_{12})$,
$(A_2A_5A_{10})$, $(A_2A_6A_9)$, $(A_2A_7A_{12})$, $(A_2A_8A_{11})$,
$(A_3A_5A_{11})$, $(A_3A_6A_{12})$, $(A_3A_7A_9)$, $(A_3A_8A_{10})$,
$(A_4A_5A_{12})$, $(A_4A_6A_{11})$, $(A_4A_7A_{10})$, $(A_4A_8A_9)$.

(8′). The finite geometry of section 3.
(9′). Two non-intersecting straight lines $(A_1A_2A_3)$, $(A_4A_5A_6)$.

The Pappus finite geometry is treated in the book *Fundamentals of Mathematics* by Moses Richardson (Macmillan, 1941).

The duals of the postulates can be proved, showing that the geometry has duality. The geometry has the euclidean property of parallelism of lines. It also has the dual property of parallelism of points.

DEFINITION. *Two points which are not connected by any line will be called parallel points.*

The most important theorems are the following:

THEOREM 1. *If the six points of two (parallel) lines are connected to form a hexagon, the opposite sides intersect in three collinear points.* (Pappus-Pascal theorem).

THEOREM 2. *There are precisely nine points in the geometry.*

THEOREM 3. (Dual of Theorem 2). *There are precisely nine lines in the geometry.*

The Pappus geometry contains no artificial lines and is associated with one of the simplest non-trivial configurations in geometry. The Master's thesis of John E. Darraugh (Brooklyn College, 1940) lists thirty-five theorems for the Pappus finite geometry.

5. The Desargues finite geometry. The postulates of this geometry may be stated as follows:

(1). There exists a point (element of S).

(2). Two distinct points (elements of S) are contained by at most one line (m-class).

DEFINITION. *Line p is called a polar line of point P if no point of p is connected to P by a line. Point P is called a pole of line p if no line through P contains a point of p.*

(3). For every line p (m-class), there is at most one pole P.

(4). There are at least three distinct points (elements of S) on a line (m-class).

(5). For every point (element of S), there exists a polar line (m-class).

(6). If a line p (m-class) does not contain a given point Q (element of S), the polar line q of point Q has a point in common with line p.

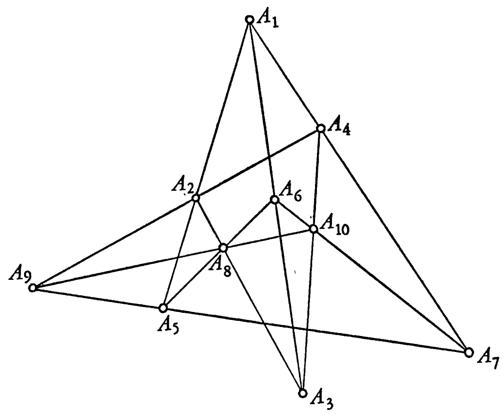


FIG. 4. Geometric diagram for the Desargues finite geometry.

Symbolic diagram where the vertical columns represent lines (m-classes):

$$\begin{array}{cccccccccc} A_1 & A_1 & A_1 & A_2 & A_2 & A_3 & A_5 & A_5 & A_6 & A_8 \\ A_2 & A_3 & A_4 & A_3 & A_4 & A_4 & A_6 & A_7 & A_7 & A_9 \\ A_5 & A_6 & A_7 & A_8 & A_9 & A_{10} & A_8 & A_9 & A_{10} & A_{10} \end{array}$$

The independence of the postulates is shown by the following examples.

(1′). The null set—a geometry without points or lines.

(2′). $(A_1A_2A_3)$, $(A_1A_2A_7)$, $(A_1A_3A_8)$, $(A_2A_5A_7)$, $(A_3A_6A_8)$, $(A_4A_5A_6)$, $(A_4A_5A_7)$, $(A_4A_6A_8)$.

(3′). Two lines $(A_1A_2A_3)$, $(A_4A_5A_6)$.

(4′). A simple hexagon $A_1A_2A_3A_4A_5A_6$.

$$(A_1A_2),\ (A_2A_3),\ (A_3A_4),\ (A_4A_5),\ (A_5A_6),\ (A_6A_1).$$

(5′). A single point A_1.

(6′). $(A_1A_2A_6)$, $(A_1A_3A_5)$, $(A_1A_{10}A_{12})$, $(A_2A_3A_4)$, $(A_2A_{10}A_{11})$, $(A_3A_{11}A_{12})$, $(A_4A_5A_9)$, $(A_4A_6A_8)$, $(A_5A_6A_7)$, $(A_7A_8A_{12})$, $(A_7A_9A_{11})$, $(A_8A_9A_{10})$.

The Desargues finite geometry has only six postulates, it has duality and polarity, it is non-euclidean in that a line may have as many as three lines

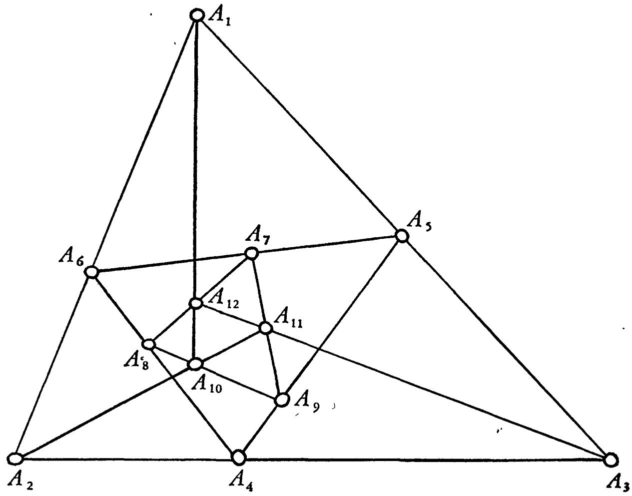


FIG. 5. Four triangles each inscribed in and circumscribed about another triangle of the set.

parallel to it through a given point, and it is associated with a real configuration.

This finite geometry is treated in Part III, Chapter 1 of *Fundamental Mathematics* by Duncan Harkin (Prentice-Hall, 1941).

John E. Darraugh, in his Master's thesis (Brooklyn College, 1940) gives fifty-two theorems for this geometry. Among the most important theorems are the duals of the postulates, and also the following:

THEOREM 1. *If A lies on the polar line of B, then B lies on the polar line of A.*

THEOREM 2. *If b and c are both parallel to a, then b and c intersect in a point.*

THEOREM 3. *There exist precisely ten points.*

THEOREM 4. *There exist precisely ten lines.*

THEOREM 5. *If two triangles are perspective from a point, their corresponding sides intersect in collinear points.* (The Desargues theorem).

ON FREQUENCY DISTRIBUTIONS OF THE QUOTIENT AND OF THE PRODUCT OF TWO STATISTICAL VARIABLES*

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1. Introduction. Given the probability density function (p.f.) for x and y, to find the p.f. for their quotient, and for their product. These problems lie on the surface and in addition to their intrinsic interest they have obvious importance in applications. Surely, even though the mathematical theory of statistics is a relatively new subject, enough study has been given to these two questions to provide the information needed for the more common applications. There is a good deal of literature on the problem for quotients, but nevertheless it is only in some rather special, if important, cases that our knowledge of the distribution of such quantities is satisfactory. But when we are asked, as I have been more than once, by colleagues trying to apply statistical methods to investigations in their own fields, what value can be assigned to the probability that the product of two statistical variables will lie between given values, granting that one knows all about the distributions of the variables separately, even supposing them independent, we simply do not have the information wanted. I do not mean to say that no p.f.'s for products are known; there is at least one important exception; but the state of our ignorance concerning my colleagues's question is embarrassingly complete. My own point of view in considering these two problems is that of a person who would like to see the kind of results which provide satisfactory significance tests for use in applications.

To keep within the space assigned me I shall have to confine my remarks for the most part to three topics chosen from what would amount to a more complete discussion of the subject. First, I shall discuss the situation with regard to the quotient y/x in which x and y obey a normal bivariate p.f. and at the end I shall similarly deal with the product of the same two variables. The second section of the paper will be devoted to quotients of the type of $\alpha_3 = m_3/m_2^{3/2}$, in which m_2 and m_3 are second and third central moments calculated in a sample of N from any infinite universe for which moments of the requisite order exist.

2. Elementary methods for quotients. Two elementary methods for finding the p.f. for a quotient are sufficient for the cases considered here. It is supposed that we are given $f(x, y)$, the joint p.f. for x and y. If x and y are statistically independent, then we are given $f_1(x)$ and $f_2(y)$, and $f(x, y) = f_1(x) \cdot f_2(y)$.

The most obvious thing to do is to set $y = zx$; then the elementary probability for x and y, $f(x, y)\, dx\, dy$, if x and y are continuous variables, is converted to $f(x, zx)\,|x|\, dx\, dz$ (an elementary probability must be positive or zero), the elemen-

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tary probability for x and z. Then the p.f. for z is given by

$$f_3(z) = \int_{\text{range of } x} f(x, zx) \mid x \mid dx,$$

or by

$$f_3(z) = \int_{\text{range of } x} f_1(x) f_2(zx) \mid x \mid dx$$

if x and y are independent.

Professor Huntington has given a derivation of the second form from first principles [1] in case the range for both x and y is the interval $(0, \infty)$. However, the formula is in general valid in the cases in which the sum or integral involved exists.

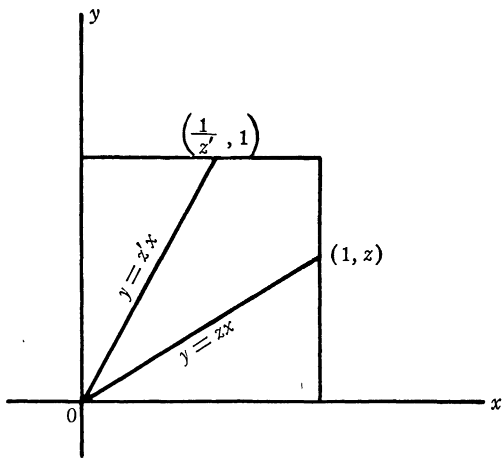

FIG. 1

As a very simple illustration, let x and y be independent and let each obey a rectangular probability law on the interval $(0, 1)$. That is,

$$f_1(x) = f_2(y) = 1, \qquad 0 \leqq x \leqq 1, \qquad 0 \leqq y \leqq 1,$$
$$= 0, \text{ otherwise.}$$

We have

$$f(z) = \int_0^1 x\, dx = \tfrac{1}{2}, \qquad \text{for} \quad 0 \leqq z \leqq 1,$$

$$= \int_0^{1/z} x\, dx = \frac{1}{2z^2}, \qquad \text{for} \quad 1 \leqq z \leqq \infty.$$

(In the case that $z \geqq 1$, $f(zx) \equiv 0$ for $x > 1/z$.)

The second method, only slightly less obvious than the first, is to undertake to find the elementary volume under the frequency surface $p = f(x, y)$ between the two planes $y = zx$ and $y = (z + dz)x$. To do this we try to find the cumulative distribution function (d.f.) for z, $F_3(z)$, which is given by the volume under the frequency surface for which $y \leqq zx$, and then take the derivative (or difference) of $F_3(z)$.

This is also very simply carried out in our hand-picked example. By inspection,

$$\begin{aligned} F_3(z) &= z/2, \qquad 0 \leqq z \leqq 1, \\ &= 1 - \frac{1}{2z}, \qquad 1 \leqq z \leqq \infty. \end{aligned}$$

A more complete discussion of this subject would include an account of the mathematically more sophisticated method of characteristic functions.

3. Quotients of normally distributed variables. Naturally the first important quotient, y/x, studied was that in which x and y are normally distributed. Despite the fact that no fundamental difficulty is encountered, it is nevertheless true that, though there had been earlier papers on the subject, the exact solution in case that x and y obey a normal bivariate p.f. was not published until 1932, by E. C. Fieller [2]. This problem is a bit troublesome and the final solution involves a quadrature, though a tabulated one. Fieller proceeded by finding the d.f. which contains a double quadrature. He remarks, however, that the p.f. could be found more directly by the first method above. If, as we assume,

$$\begin{aligned} f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - \rho^2}} \exp \left\{ -\frac{1}{2(1 - \rho^2)} \left[\left(\frac{x - \mu_x}{\sigma_x} \right)^2 \right. \right. \\ \left. \left. - 2\rho \left(\frac{x - \mu_x}{\sigma_x} \right) \left(\frac{y - \mu_y}{\sigma_y} \right) + \left(\frac{y - \mu_y}{\sigma_y} \right)^2 \right] \right\}, \end{aligned}$$

and if we set

$$\frac{x}{\sigma_x} = u, \qquad \frac{y}{\sigma_y} = v, \qquad \frac{\mu_x}{\sigma_x} = r_1, \qquad \frac{\mu_y}{\sigma_y} = r_2 \qquad \frac{v}{u} = z \qquad \frac{\sigma_x}{\sigma_y} = w,$$

we have, noting that for a fixed x we must take $dy = |x| dz$,

$$f(w) = \frac{\exp\left[-\frac{1}{2(1 - \rho^2)} (r_1^2 - 2\rho r_1 r_2 + r_2^2) \right]}{2\pi\sqrt{1 - \rho^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{au^2}{2} + bu \right) |u| \, du,$$

in which,

$$a = \frac{1 - 2\rho w + w^2}{1 - \rho^2} > 0, \qquad \rho^2 < 1,$$

and

$$b = \frac{r_1 - \rho r_2 + (r_2 - \rho r_1)w}{1 - \rho^2}.$$

With r_1 and r_2 not both zero, the final result is,

$$f(w) = \frac{1}{\pi\sqrt{1-\rho^2}} \left\{ \frac{\exp\left[-\frac{1}{2(1-\rho^2)}(r_1^2 - 2\rho r_1 r_2 + r_2^2)\right]}{a} + \frac{b}{a^{3/2}} \exp\left[-\frac{(r_2 - r_1 w)^2}{2a(1-\rho^2)}\right] \int_0^{b/\sqrt{a}} \exp(-t^2/2)dt \right\}.$$

Of course, this can be calculated from existing tables for particular values of w and of the parameters. Tables for $f(z)$ are almost out of the question since the p.f. for z contains not only the parameters r_1, r_2, and ρ but also the ratio σ_x/σ_y. It may be noted that if $r_1 = r_2 = 0$, *i.e.*, if $E(x) = E(y) = 0$, the above result simplifies considerably so that in this case,

$$f(w) = \frac{1}{\pi\sqrt{1-\rho^2}} \cdot \frac{1}{a},$$

or

$$f(z) = \frac{\sqrt{1-\rho^2}\,\sigma_x\sigma_y}{\pi(\sigma_x^2 - 2\rho\sigma_x\sigma_y z + \sigma_x^2 z^2)}.$$

Now the calculation of values of the d.f. is accomplished merely by use of a set of tables of anti-tangents.

But it is to be noted that if x and y obey a normal bivariate p.f. the mean value of y/x does not exist. If we look back at the $f_3(z)$ obtained in the example with rectangular distributions, we see that in that case, too, $E(z)$ does not exist. One will not jump at the conclusion that in general y/x does not possess a mean value. Rather one will have anticipated that allowing zero to be included in the range for x would have some kind of an uncomfortable effect on the distribution of y/x. Of course the situation is simply that if x is distributed normally, or if it follows a rectangular distribution with zero in its range, $E(1/x)$ does not exist. On the other hand, if x has the p.f.

$$\frac{x^{p-1}e^{-x}}{\Gamma(p)}, \qquad p > 0, \qquad 0 \leqq x \leqq \infty,$$

and if ν is the largest positive integer which is $<p$, the first ν moments of $1/x$ are finite. Thus such ratios as Fisher's t and Snedecor's F do in general possess a finite number of moments.

But if $E(y/x)$ does not exist, not even the Tchebychef inequality which states that if x is any statistical variable, the probability that

$$|x - E(x)| \geqq \lambda\sigma_x$$

is $\leqq 1/\lambda^2$, is available to give us any information about the probability that a value of y/x at random will exceed a given value. The much used phrase, "deviation from expected in standard units," is in this case doubly meaningless.

However, there are cases in which $f_1(0) \neq 0$, in which the range of x can be curtailed so as to exclude $x=0$ without doing much violence to the d.f. for y/x. As an example, in the case in which x and y obey a normal bivariate p.f., Geary [3] concluded that if $r_1 = \mu_x/\sigma_x$ is large, so that $F_1(0)$ is very small, the quantity

$$\frac{\mu_x z - \mu_y}{(\sigma_x^2 - 2\rho\sigma_x\sigma_y z + \sigma_x^2 z^2)^{1/2}}$$

is approximately normally distributed with zero mean and unit variance.

4. Moment characteristics of quotients. Sometimes the p.f. for a y/x of considerable importance cannot be found because the joint p.f. for x and y is quite unknown. It has been a standard procedure in statistics in case we do not know the p.f. for a variable to try to get the information about it afforded by its moment characteristics, in particular, its mean, its variance, its skewness α_3, and its kurtosis $\alpha_4 - 3$ ($\alpha_{k:x}$ is the kth central moment of x divided by σ_x^k). Of course, we ought to have some assurance that the moments sought actually exist, and then we ought to know something about whether or not the actual distribution is such that these first few moment characteristics do enable us to get, at least, first approximations to the probabilities that the variable will fall within a given interval.

A case in point is the ratio $\alpha_3 = m_3/m_2^{3/2}$ of problem (2). Even if the population sampled is normally distributed, we cannot produce the joint p.f. for m_2 and m_3. The most that is possible at present is to calculate values of $E(m_2^k m_3^l)$. The computation grows very tedious and involved as $2k+3l$ increases, but the necessary work has been done for $2k+3l \leqq 12$. Can such values be used to obtain approximations to $E(\alpha_3^j)$, at least for small values of j?

Tschuprow [4] used the following method in such a problem. Write

$$E(m_2) = M_2, \qquad E(m_3) = M_3,$$

and then

$$m_2 = M_2 + \delta_2, \qquad m_3 = M_3 + \delta_3$$

with

$$E(\delta_2) = E(\delta_3) = 0.$$

We have then formally, $(j=1)$,

$$E(m_3/m_2^{3/2}) = E\left[\frac{M_3+\delta_3}{(M_2+\delta_2)^{3/2}}\right]$$

$$= \frac{M_3}{M_2^{3/2}}\left(1+\frac{15}{8}\frac{E(\delta_2^2)}{M_2^2}-\frac{35}{16}\frac{E(\delta_2^3)}{M_2^3}+\cdots\right)$$

$$+\frac{1}{M_2^{3/2}}\left(-\frac{3}{2}\frac{E(\delta_2\delta_3)}{M_2}+\frac{15}{8}\frac{E(\delta_2^2\delta_3)}{M_2^2}-\frac{35}{16}\frac{E(\delta_2^3\delta_3)}{M_2^3}+\cdots\right).$$

The values of $E(\delta_2^k\delta_3^l)$ are the ones actually on record, instead of those of $E(m_2^k m_3^l)$, and they have the advantage of being of $O(N^{1-k-l})$. This circumstance seemed to satisfy Tschuprow without further examination that such a series could be used to give a good approximation to the required value for large N's. There is no difficulty in general with the existence of $E(\alpha_3^j)$, in particular, none if we are sampling from a normal universe. But neither of the two series above is in general convergent; they certainly are not if we are sampling from normal.

However, the following argument shows that, nevertheless, we can get good approximations by means of such series for sufficiently large N's. Consider the first of the two series above. Let us restrict ourselves to samples in which $|m_2-M_2|\leqq kM_2$, with $k<1$ and independent of N. Now it is immediately seen that the series is convergent and, moreover, that the error committed by using a given number of terms can be made arbitrarily small with increasing N. For if $\sigma_{m_2}^2$ exists and is $O(1/N)$, as is ordinarily the case, then the probability that $|m_2-M_2|>kM_2$ becomes small with increasing N, and moreover for a fixed h, the difference between $E(\delta_2^h)$ for the curtate range and for the whole range can be made arbitrarily small. Then a given number of terms of the series with moments from the complete range for m_2 can be made to approximate as closely as we please the same number of terms of the series for the curtate range. Thus we can calculate with accuracy the moments of the ratio $m_3/m_2^{3/2}$ with restricted range, since the argument used is readily adapted to each series that arises. But now if the variance of α_3 is finite and $O(1/N)$, as is also usually the case, the moments of α_3 and of α_3 modified as above can also be made to differ as little as we please with increasing N. Therefore the method, so far as the required computations can be carried out, does give good approximations for sufficiently large N's. J. B. D. Derksen has recently discussed these Tschuprow series [5] using some results due to Slutsky [6] in arriving at substantially the argument given above. The practical difficulty arises in estimating how large an N is finally required in each case.

Coming back once more to the case in which x and y obey a normal bivariate p.f., if $|E(x)|\geqq k\sigma_x$ and if we agree to exclude from our distribution values of y/x for which $|x-E(x)|\geqq k\sigma_x$, as I once did [7], Tschuprow's method leads to some

fairly interesting properties of the moment characteristics of such a distribution since now k may be taken to be as large as one chooses.

5. Products; elementary methods. There is at present less to say about the distribution of $w=xy$. Of course there are a few cases in which the p.f. of xy is simple and is simple to obtain. Such a one is that in which $\log x$ and $\log y$ are both normally distributed. An example of a more important and more complicated problem that leads to a relatively simple result is offered by the case in which once more x and y are normally distributed with $E(x)=E(y)=0$. If x and y are independent, we have [8]

$$f(w) = \frac{\sigma_x\sigma_y}{\pi} K_0\left(\frac{w}{\sigma_x\sigma_y}\right),$$

in which $K_0(x)$ is the Bessel function of the second kind with a purely imaginary argument, of zero order. If x and y are correlated, this result is only slightly modified [9].

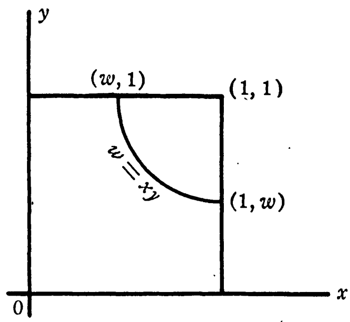

FIG. 2

Of course the same elementary methods which we applied to quotients are also available for use with products. If, again, we suppose that x and y are independent and that each obeys a rectangular probability law on the interval (0, 1), we have

$$f(w) = \int_w^1 \frac{dx}{x} = -\log w, \qquad 0 \leqq w \leqq 1,$$

or

$$F(w) = w + \int_w^1 \frac{w\,dx}{x} = w(1 - \log w).$$

It will be noted that $f(w)$ has an infinite ordinate at the origin, while at that point $F(w)$ is zero and, of course, continuous on the right.

For products there is no question of the existence of moments beyond that of the existence of $E(x^iy^i)$ for the joint p.f. of x and y. That is, the moment char-

acteristics of xy are usually at hand but they do not necessarily afford much information about the probability that $xy \geqq C$.

6. Products of normally distributed variables. I will conclude with sketching briefly the rather incomplete investigation I once made of the case [9], surely important for applications, in which x and y obey a normal bivariate p.f. The simple result given above comes only with the special choice of origin made there. Using the same notation as before, first for x and y independent ($\rho = 0$), it is easy to find the following moment characteristics of xy:

$$\begin{aligned} E(xy) &= \mu_x \mu_y, \\ \sigma_{xy}^2 &= \sigma_x^2 \mu_y^2 + \sigma_y^2 \mu_x^2 + \sigma_x^2 \sigma_y^2, \\ \alpha_{3:xy} &= \frac{6 r_1 r_2}{(r_1^2 + r_2^2 + 1)^{3/2}} \leqq 2\sqrt{3}/3, \\ \alpha_{4:xy} - 3 &= \frac{6[2(r_1^2 + r_2^2) + 1]}{(r_1^2 + r_2^2 + 1)^2} \leqq 6. \end{aligned}$$

Thus in this case the distribution of xy cannot be very asymmetrical, but its kurtosis can be quite excessive.

Here the p.f. can be found as an infinite series. If we set $y = w/x$ and take $dy = dw/|x|$, we obtain, if we put $uv = \zeta$,

$$f(\zeta) = \frac{\exp[-(r_1^2 + r_2^2)/2]}{2\pi} [\psi_1(\zeta) - \psi_2(\zeta)],$$

in which

$$\psi_1(\zeta) = \int_0^\infty \exp[-(x^2/2 - r_1 x - r_2\zeta/x + \zeta^2/x^2)] \frac{dx}{x},$$

and $\psi_2(\zeta)$ is the integral of the same function over the interval $(-\infty, 0)$. Now $\psi_1(\zeta)$ and $\psi_2(\zeta)$ can each be found as a convergent series of a not too simple form. Subtracting them, we find,

$$\begin{aligned} f(\zeta) = \frac{\exp[-(r_1^2 + r_2^2)/2]}{\pi} \Bigg[& \sum_0 (r_1 r_2 \zeta) K_0(\zeta) + (r_1^2 + r_2^2) \frac{|\zeta|}{2!} \sum_2 (r_1 r_2 \zeta) K_1(\zeta) \\ & + (r_1^4 + r_2^4) \frac{\zeta^2}{4!} \sum_4 (r_1 r_2 \zeta) K_2(\zeta) + (r_1^6 + r_2^6) \frac{|\zeta|^3}{6!} \sum_6 (r_1 r_2 \zeta) K_3(\zeta) + \cdots \Bigg], \end{aligned}$$

in which

$$\sum_k (x) = 1 + \frac{x}{k+1} + \frac{x^2}{(k+2)^2 2!} + \frac{x^3}{(k+3)^3 3!} + \cdots,$$

and is simply related to the Bessel function of the first kind with a purely imagi-

nary argument. Here $K_r(x)$ is the Bessel function of the second kind with a purely imaginary argument.

This series converges for any value of ζ except 0. For $\zeta = 0$, the first term becomes logarithmically infinite at the origin. For $r_1 = r_2 = 0$, we have, as above,

$$f(\zeta) = \frac{K_0(\zeta)}{\pi}.$$

For small values of r_1 and r_2 the series converges rapidly and $f(\zeta)$ can be tabulated and plotted. Unfortunately, for larger values of r_1 and r_2 the series converges more slowly and it is laborious to compute many terms of such a series.

The case in which x and y are not independent ($\rho \neq 0$) turns out not to be essentially more complicated. For this fact and for further details for $\rho = 0$, I refer the reader to my paper cited above [9]. It is evident that though we have a mathematical solution to our problem it falls considerably short of what is desirable for applications.

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PROBLEMS ON MAXIMA AND MINIMA*

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In the simpler problems on maxima and minima studied in the calculus we are led to a function $u(x)$ and we seek values of x for which u has a relative maximum or minimum. If $u(x)$ is differentiable in the interval considered and if the maximum or minimum is assumed at an interior point of the interval, then by Fermat's theorem the derivative du/dx must be zero for the values of x sought. In many cases, however, it is awkward to reduce the problem to the above form and we have instead a function $u(x, y)$ whose maxima and minima we seek to determine under an auxiliary condition, $f(x, y)=0$. Differentiating with respect to x, we have again by Fermat's theorem,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}=0, \qquad \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0.$$

The elimination of dy/dx gives $D(u, f)/D(x, y)=0$ and we solve this equation with $f(x, y)=0$ to find the values of x and y necessary for a maximum or minimum of u.

Instead of this customary analytic approach to the problem, we may employ a partially geometric treatment, as follows. Writing m for dy/dx, we have the following equations for the determination of x, y, m.

$$\frac{\partial u}{\partial x}+m\frac{\partial u}{\partial y}=0, \tag{1}$$

$$\frac{\partial f}{\partial x}+m\frac{\partial f}{\partial y}=0, \qquad f(x, y)=0. \tag{2}$$

We may interpret (x, y) as the rectangular coördinates of a point in the plane and m as the slope of a line through this point, and thus (x, y, m) form the coördinates of a *linear element*, *i.e.*, a point and a line through it. Equation (1) places one restriction on the three coördinates and thus determines a two parameter family of linear elements, while equations (2) place two restrictions on the coördinates and determine a one parameter family of linear elements. The problem of determining the extrema of u thus resolves itself into that of finding the linear elements common to the two families.

Equation (1) shows that the linear elements of this family lie at each point (x, y) in the direction perpendicular to the gradient $\nabla u \equiv (\partial u/\partial x, \partial u/\partial y)$ and thus in the direction of the curve $u(x, y)=constant$ which passes through this point. This family will also include as singular elements every linear element at the singular points, if any, of the function $u(x, y)$, since the direction of the element becomes indeterminate there. Equations (2) show that the linear elements

* Presented to the Michigan Section of the Mathematical Association of America at Ann Arbor, Michigan, March 9, 1938.

of this family have their points on the curve $f(x, y)=0$ and their lines tangent to this curve at these points. We shall express this by simply saying that the elements are tangent to the curve. This family will also include as singular elements every linear element at the singular points, if any, of the curve $f(x, y)=0$, since the direction of the element becomes indeterminate there. In particular, family (2) might consist merely of the one-parameter family of linear elements at a point; *e.g.*, $f(x, y)=x^2+y^2$.

The problem of finding values of x and y for which extrema of u may occur is thus reduced to finding those points of the plane at which some curve of the family $u(x, y)=constant$ is tangent to the curve $f(x, y)=0$. Geometry is very rich in theorems concerning the constancy of certain point functions $u(x, y)$ along certain plane curves, and by the above analysis each such theorem may yield interesting problems in maxima and minima. Conversely, many problems of the calculus on maxima and minima are almost instantly solved by this analysis and a great deal of light is thrown on all such problems. We present a few examples, starting with a very simple one.

1. If A and B are any two points on a conic, determine the points P on the conic for which the area of the triangle ABP is an extremum.

In this case the curves traces by P when the area u of the triangle ABP remains constant are evidently the straight lines parallel to AB, while for P anywhere on the line through A and B the direction of increase of u is indeterminate.

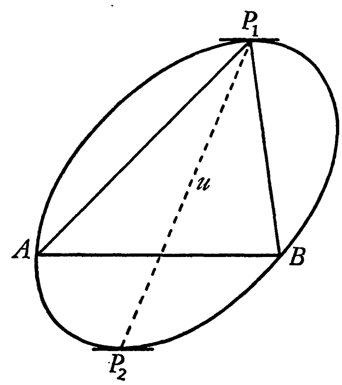

FIG. 1

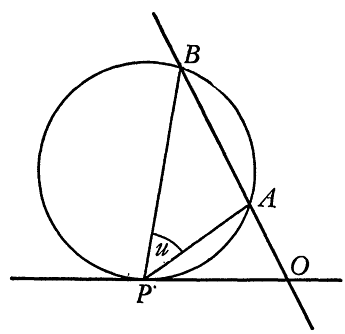

FIG. 2

Family (1) thus consists of all linear elements whose directions are parallel to AB, and includes as singular elements all elements whose points are on AB (extended). Since P must be on the conic, the linear elements of family (2) are those tangent to the conic. The points P_1, P_2, A, B thus constitute the positions of P necessary for an extremum of the area u; P_1 and P_2 being the extremities of the diameter bisecting the chord AB.

The next problem is a generalization of the familiar problem of finding at

what distance one should stand from a tapestry hanging on the wall in order that it subtend the maximum vertical angle at the eye.

2. Two straight roads intersect at any angle at a point O. An enemy column is to traverse one road which is concealed by trees except between two points A and B on the same side of O. At what point P on the other road should a field piece be placed so that the segment AB shall subtend the maximum angle at P?

The curves traced by P when the angle $u=APB$ remains constant are evidently the circular arcs running from A to B, and the regular linear elements of family (1) are those tangent to these arcs. Since P must be a point of the second road, the linear elements of family (2) are tangent to this road. Thus the extrema of u occur for P at the points of tangency of the second road to these circular arcs. But then OP will be the tangent to a circle for which OB and OA are the whole secant and its external segment, and hence $OP^2=OA\cdot OB$. This gives two positions for P, on opposite sides of O and each yielding a maximum of u. The minimum value of u is of course zero, attained when P is at O and yielded by a singular element of family (1), as in Problem 1.

We next consider the familiar "beam around the corner" problem.

3. What is the length of the longest beam that can be moved horizontally from a hall of width a into a hall of width b at right angles to the first?

If we take the outer walls of the halls as coördinate axes, the problem is in effect: What is the length of the shortest line segment with ends on the two axes and passing through the point $A=(a, b)$? Consider any line segment of length u with ends on the two axes. It will, at some point, $P=(x, y)$, be tangent to an

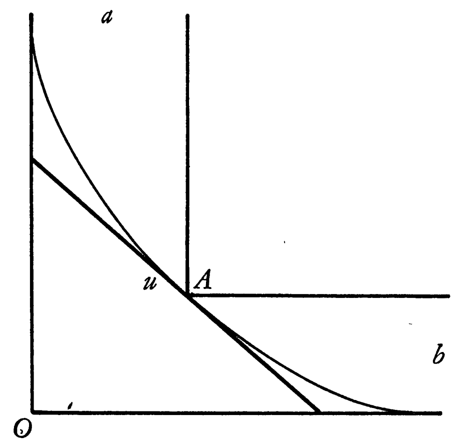

FIG. 3

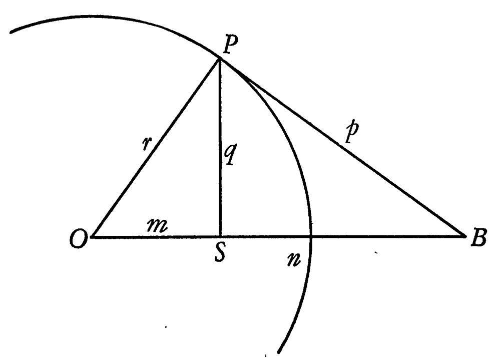

FIG. 4

astroid of the family, $x^{2/3}+y^{2/3}=c^{2/3}$, and it is a familiar property of these astroids that $u^{2/3}=x^{2/3}+y^{2/3}$ and that u consequently remains constant as P traces out the curve. The linear elements of family (1) are thus tangent to the

astroids of this family. The requirement that the line segments pass through A will force P to remain on a certain curve* through A which is the locus of the points P of tangency of lines through A to the astroids of the family, $x^{2/3}+y^{2/3}=c^{2/3}$. The linear elements of family (2) are tangent to this locus. Now let P move along this locus and approach A as a limit point. The limiting position of the line AP is not only the tangent to the locus at A, but also the tangent to the astroid of the family which passes through A. Thus the linear element tangent to this astroid at A belongs to both families and yields the desired minimum value of u. This minimum value is thus given by the equation $u^{2/3}=a^{2/3}+b^{2/3}$.

4. A submarine sights an enemy battleship at a distance and apparently traveling faster than the submarine can go in a direction at an angle to the line of sight. In what direction should the submarine sail to intercept the battleship, if possible?

Let B and S be the initial positions of the battleship and submarine, respectively. Let P be any other point and call $BP=p$, $SP=q$. Evidently the submarine will intercept the battleship, if possible, at the point P of her course at which $u=p/q$ has its maximum value. The curves traced by P when u remains constant are of course the circles having B and S as a pair of inverse points, and the linear elements of family (1) are thus tangent to these circles. Since P must be on the course, supposedly straight, of the battleship, the linear elements of family (2) are tangent to this straight line through B. The position of P for which u is a maximum, being at the common element of these families, must be at the point of tangency of a straight line through B with a circle having B and S as inverse points. Let O be the center of this circle, and call $OS=m$, $OB=n$, $OP=r$. Then since $mn=r^2$, we have $m/r=r/n$ and the triangles OSP and OPB are similar. But then $\angle OSP=\angle OPB=\pi/2$, and the submarine should lay a course at right angles to the bearing of the battleship when sighted.

5. A piece of paper having the form of a right angled triangle of legs a and b has the corner at the right angle folded over to touch the hypotenuse. What is the least possible area of the folded over piece?

Let O and P be the original and final positions of the corner. Then with the notation of the figure we have,

$$OP = \rho = 2m \cos\theta = 2n \sin\theta$$

$$\rho^2 = 4mn \sin\theta \cos\theta = 4u \sin 2\theta,$$

where u is the area of the folded over triangle. The curves traced by P when the area u remains constant are thus the lemniscates $\rho^2=4u\sin 2\theta$ and the linear elements of family (1) are tangent to these lemniscates. Since P must be on the hypotenuse of the triangle, the linear elements of family (2) are tangent to the hypotenuse. For the minimum of u, P must be at the common element of the

* $(x-a)y^{1/3}+(y-b)x^{1/3}=0$.

two families and thus at the point of tangency of the hypotenuse to a member of the family of lemniscates. It is a familiar property of these lemniscates that

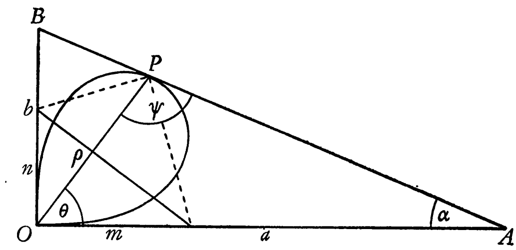

FIG. 5

the angle ψ which the tangent makes with the radius vector ρ is twice the vectorial angle θ. From the triangle OPA we then have,

$$3\theta + \alpha = \pi, \qquad \frac{\rho}{\sin 3\theta} = \frac{a}{\sin 2\theta} \quad \text{where} \quad \tan \alpha = \frac{b}{a} \cdot$$

It follows that $\theta = \frac{1}{3}(\pi - \alpha)$, and the desired minimum value of u is,

$$u = \frac{a^2}{4} \frac{\sin^2 3\theta}{\sin^3 2\theta} \cdot$$

In particular, for $\alpha = \pi/2$, the paper becomes rectangular, $\theta = \pi/6$ and $u = \sqrt{6}a^2/18$. For a simple numerical case, rational throughout, take $a = 36/11$, $b = 16/13$. This yields $m = 1$ and $u = 3/8$.

SUMMABILITY AND THE DEFINITION OF A LIMIT

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A classical and well-known method for summation of a sequence s_n is the Hölder method by which

$$s = \lim_{n \to \infty} H_n^{(r)},$$

where

$$H_n^{(r)} = \frac{1}{n+1} \sum_{n_1=0}^{n} \frac{1}{n_1 + 1} \sum_{n_2=0}^{n_1} \frac{1}{n_2 + 1} \cdots \sum_{n_r=0}^{n_{r-1}} s_{n_r}. \tag{1}$$

The more widely known Cesàro method most naturally presents itself in the following form

$$s = \lim_{n \to \infty} C_n^{(r)},$$

where

$$C_n^{(r)} = \frac{r!}{(n+1) \cdots (n+r)} \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \cdots \sum_{n_r=0}^{n_{r-1}} s_{n_r}. \tag{2}$$

It is to be noticed that

$$\frac{(n+1)\cdots(n+r)}{r!}=\sum_{n_1=0}^{n}\sum_{n_2=0}^{n_1}\cdots\sum_{n_r=0}^{n_{r-1}}1.$$

Formulas (1) and (2) clearly show a kinship between the Hölder and Cesàro sums. From (2) we write

$$\begin{aligned}C_n^{(r)}&=\frac{r!}{(n+1)\cdots(n+r)}\sum_{n_1=0}^{n}\frac{(n_1+1)\cdots(n_1+r-1)}{(r-1)!}C_{n_1}^{(r-1)}\\&=\frac{n!}{(r+1)\cdots(r+n)}\sum_{n_1=0}^{n}\frac{r(r+1)\cdots(r+n_1-1)}{n_1!}C_{n_1}^{(r-1)}.\end{aligned}$$

These are simple expressions for $C_n^{(r)}$ in terms of C_n^{r-1} which together with (1) were used by the author* to prove by mathematical induction that the Hölder and Cesàro methods of summation are equivalent.

Summation by parts according to the formula,

$$\sum_{i=0}^{n}u_i\Delta v_i=u_iv_i\Big]_0^{n+1}-\sum_{i=0}^{n}v_{i+1}\Delta u_i,$$

applied successively to (2) yields

$$\begin{aligned}C_n^{(r)}&=\frac{r!}{(n+1)\cdots(n+r)}\sum_{n_1=0}^{n}\frac{(n-n_1+1)\cdots(n-n_1+r-1)}{(r-1)!}s_{n_1}\\&=\frac{n!}{(r+1)\cdots(r+n)}\sum_{n_1=0}^{n}\frac{(r-1)r\cdots(n-n_1+r-1)}{(n-n_1)!(r-1)}s_{n_1}.\end{aligned}\tag{3}$$

This last is the form in which the Cesàro definition is usually given. There is an advantage over (2) in that non-integral values of r are permitted.

It is to be observed that the Cesàro method is a special case of summation by triangular matrix,

$$s=\lim_{n\to\infty}\sum_{i=0}^{n}K(n,i)s_i,\tag{4}$$

which has been extensively studied.† It is also to be remarked that definitions of summability by triangular matrix are extensions of the ordinary definitions of the limit of a sequence in case the method of summation is regular.

Now if we consider a function $f(x)$ of the continuous real variable x, similar methods of "summation" of $f(x)$ immediately suggest themselves. For the Hölder definition we have;

* See Bull. Amer. Math. Soc., vol. 33, p. 301.

† See, for example, Fort, Infinite Series, p. 241.

The function $f(x)$ is summable to A if $\lim_{x\to\infty} H^{(r)}(x)=A$ where,

$$H^{(r)}(x) = \frac{1}{x}\int_0^x \frac{1}{x_1}\int_0^{x_1} \cdots \frac{1}{x_{r-1}}\int_0^{x_{r-1}} f(x_r)dx_1dx_2 \cdots dx_r. \tag{5}$$

The definition of integration can be, for example, the Riemann definition.

Likewise the Cesàro definition suggests,

The function $f(x)$ is summable to A if $\lim_{x\to\infty} C_1^{(r)}(x)=A$, where

$$C_1^{(r)}(x) = \frac{r!}{x^r}\int_0^x \int_0^{x_1} \cdots \int_0^{x_{r-1}} f(x_r)dx_1dx_2 \cdots dx_r. \tag{6}$$

If integration by parts is applied to formula (6) we have

$$C_1^{(r)}(x) = \frac{r!}{x^r}\int_0^x \frac{(x-\alpha)^{r-1}}{(r-1)!}f(\alpha)d\alpha = \int_0^x \frac{r(x-\alpha)^{r-1}}{x^r}f(\alpha)d\alpha.$$

It is to be observed that this integral is a special case of

$$\int_0^x K(x, \alpha)f(\alpha)d\alpha. \tag{7}$$

Integration by parts will also reduce (5) to the form (7). Form (7) can always be written

$$\psi(x) = \int_0^\infty K(x, \alpha)f(\alpha)d\alpha \tag{8}$$

by simply defining $K(x, \alpha)$ as zero when $x>\alpha$. We shall call this the transformation by integration with kernel K and $\lim_{x\to\infty}\psi(x)$ the limit of $f(x)$ by integration with kernel K. The integral (8) is a form which has been much studied.*

It is proposed to set up an analogous form to (7) from the theory of finite differences. Note that $\int_0^x f(x)dx$ is a continuous function which is a solution of the differential equation $dy/dx=f(x)$. We propose to utilize the solution of the difference equation, $\Delta_\omega y=f(x)$ called by Nörlund† a "sum" and denoted by him by $S_c^x f(t)\Delta_\omega t$. The results obtained are far from complete judged by results long known for summation by triangular matrix. They do not seem to be without interest, however, and may be suggestive for further research. The definition of Nörlund "sum" adopted throughout the present paper is

$$\overset{x}{\underset{c}{S}} f(t)\Delta_\omega t = \lim_{\mu\to 0}\left[\int_c^\infty e^{-\mu t}f(t)dt - \omega\sum_{i=0}^{\infty} e^{-\mu(x+i\omega)}f(x+i\omega)\right],$$

where $\mu\geqq 0$, $\omega>0$. Wherever the symbol or its equivalent is used existence is assumed. The integral of the definition is assumed to be an ordinary improper

* See for example R. P. Agnew, Bull., Amer. Math. Soc., vol. 45, p. 689.

† See N. E. Nörlund, Differenzenrechnung, p. 43.

integral, where the Riemann definition of integration is used for the finite interval. This "sum" is a continuous function of the continuous variable x, having many resemblances to a definite integral.

Consider

$$\phi(x) = \mathop{S}_{0}^{x} K(x, t)f(t)\Delta_\omega t. \tag{9}$$

Throughout this paper we shall assume all variables real. We call (9) the transformation by continuous summation with kernel K and $\lim_{x\to\infty} \phi(x)$ the limit by continuous summation with kernel K. However, whenever the word, limit, is used alone it will refer to the limit in the ordinary (ϵ, δ) sense. For purposes of illustration we define by analogy with (3) the Cesàro limit of order r of $f(x)$ by this definition as $\lim_{x\to\infty} C_2^{(r)}(x)$ where

$$C_2^{(r)}(x) = \frac{r\Gamma\left(\dfrac{x}{\omega}+1\right)}{\Gamma\left(\dfrac{x}{\omega}+r+1\right)} \mathop{S}_{0}^{x+\omega} \frac{\Gamma\left(\dfrac{x-t}{\omega}+r\right)}{\Gamma\left(\dfrac{x-t}{\omega}+1\right)} f(t)\Delta_\omega t.$$

Analogous forms suggest themselves such as $\lim_{x\to\infty} C_3^r(x)$ where

$$C_3^{(r)}(x) = r\frac{Q_r(x)}{B_r^{(r)}(x)}$$

with

$$Q_r(x) = \mathop{S}_{0}^{x+\omega} B_{r-1}^{(r-1)}(t)f(t)\Delta_\omega t$$

where $B_j^{(j)}(x)$ is the Bernoulli polynomial* of order j and degree j with difference intervals, $\omega_1, \omega_2, \cdots, \omega_j$. Forms analogous to the Hölder sum could also be suggested.

At this point we introduce the following definitions.

The transformation (9) *is said to be limit-producing for the function* $f(x)$ *if* $\lim_{x\to\infty} \phi(x)$ *exists. It is said to be regular if, whenever* $\lim_{x\to\infty} f(x) = A$, $\lim_{x\to\infty} \phi(x) = A$ *for those functions for which* $\phi(x)$ *exists*† *for all positive values of* x.

THEOREM I. *Necessary conditions that transformation* (9) *be regular are*

$$\lim_{x\to\infty} \mathop{S}_{0}^{x} K(x, t)\Delta_\omega t = 1, \tag{a}$$

$$\lim_{x\to\infty} \int_{t_0}^{t_0+\delta} K(x, t)dt = 0, \tag{b}$$

* N. E. Nörlund, *loc. cit.*, p. 129.

† See, for example, Nörlund, *loc. cit.*, p. 47.

whenever $t_0 \geqq 0$ *and* $\delta \geqq 0$ *provided that the sum and integral exist.*

Consider the function $f(t) \equiv 1$. Condition (a) results. Consider the function, $f(t)=1$ when $t_0 \leqq t \leqq t_0+\delta$, $f(t)=0$ when $t>t_0+\delta$ or $t<t_0$; condition (b) results.

Next suppose $K(x, t)=0$ if $t>x$, then*

$$\phi(x) = \int_0^x K(x, t)f(t)dt - \omega K(x, x)f(x). \tag{10}$$

Let $f(x) \equiv 1$ and $K(x, x) \equiv 1$, $K(x, t)=1/(t+1)^2$, $t \neq x$; then the limit by integration with kernel K is 1 but the limit by continuous summation is $1-\omega$. In fact, from formula (10) we readily conclude the following theorem.

THEOREM II. *Functions* K *and* f *can be defined such that the limit by integration with kernel* K *is different from the limit by continuous summation with the same kernel* K. *In fact one limit may exist without the existence of the other.*

Likewise in case the expressions written converge†

$$\phi(x) = \int_0^\infty K(x,t)f(t)dt - \omega \sum_{i=0}^{\infty} K(x, x+i\omega)f(x+i\omega).$$

From this we have the following theorem.

THEOREM III. *A sufficient condition that the transformation by summation with kernel* K *be regular is that the transformation by integration with kernel* K *be regular and that*

$$\lim_{x\to\infty} \sum_{i=0}^{\infty} K(x, x+i\omega)f(x+i\omega) = 0.$$

From our definition we write

$$\phi(x) = \lim_{\mu\to 0+} \Bigg[\sum_{i=0}^{\infty} \bigg(\int_{\delta+i\omega}^{\delta+(i+1)\omega} e^{-\mu t}K(x, t)f(t)dt - \omega e^{-\mu(\delta+i\omega)}K(x, \delta+i\omega)f(\delta+i\omega) \bigg)$$
$$+ \omega \sum_{i=0}^{(x-\delta-\omega)/\omega} e^{-\mu(\delta+i\omega)}K(x, \delta+i\omega)f(\delta+i\omega) + \int_0^\delta e^{-\mu t}K(x, t)f(t)dt \Bigg].$$

where $0 \leqq \delta < \omega$ and $(x-\delta) \equiv 0 \pmod{\omega}$. Now in case $tK(x, t)f(t)$ is positive monotonic decreasing in t for each x, each terms of the series

$$\sum_{i=0}^{\infty} \left| \int_{\delta+i\omega}^{\delta+(i+1)\omega} e^{-\mu t}K(x, t)f(t)dt - \omega e^{-\mu(\delta+i\omega)}K(x, \delta+i\omega)f(\delta+i\omega) \right| \tag{11}$$

increases when μ decreases. Moreover, (12) converges when $\mu=0$ as is shown

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* See L. M. Milne-Thomson, Calculus of Finite Differences, p. 201.

† See L. M. Milne-Thomson, *loc. cit.*, p. 201.

by comparison with the telescopic series

$$\sum_{i=0}^{\infty} (K(x, \delta + i\omega)f(\delta + i\omega) - K(x, \delta + (i + 1)\omega)f(\delta + (i + 1)\omega)).$$

Hence (11) converges uniformly in μ by the Weierstrass test* using for comparison series (11) itself when $\mu = 0$. Under these conditions

$$\begin{aligned} \phi(x) = &\sum_{i=0}^{\infty}\left(\int_{\delta+i\omega}^{\delta+(i+1)\omega} K(x, t)f(t)dt - \omega K(x, \delta + i\omega)f(\delta + i\omega)\right) \\ &+ \omega \sum_{i=0}^{(x-\delta-\omega)/\omega} K(x, \delta + i\omega)f(\delta + i\omega) + \int_0^{\delta} K(x, t)f(t)dt, \end{aligned} \tag{12}$$

where $(x-\delta) \equiv 0 \pmod{\omega}$ and $0 \leqq \delta < \omega$.

From formula (12) we draw some conclusions as expressed in the following theorems provided, of course, formula (12) holds.

THEOREM IV. *The following conditions together are sufficient that transformation* (9) *be limit producing*:

(a) *$tK(x, t)f(t)$ be positive monotonic decreasing in t for each x*;

(b) $$\sum_{i=0}^{(x-\delta-\omega)/\omega} K(x, \delta + i\omega)f(\delta + i\omega)$$

approach one and the same limit as $x \to \infty$ through any set of values such that $(x-\delta) \equiv 0 \pmod{\omega}$, *where* $0 \leqq \delta < \omega$;

(c) $$\sum_{i=0}^{\infty}\left(\int_{\delta+i\omega}^{\delta+(i+1)\omega} K(x, t)f(t)dt - \omega K(x, \delta + i\omega)f(\delta + i\omega)\right)$$

approach one and the same limit as $x \to \infty$ through any set of values such that $(x-\delta) \equiv 0 \pmod{\omega}$ *and* $0 \leqq \delta < \omega$;

(d) $$\lim_{x\to\infty} \int_0^{\delta} K(x, t)f(t)dt = 0, \qquad 0 \leqq \delta < \omega.$$

From Theorem IV we conclude the following theorem by showing that the conditions stated in it are sufficient for the conditions of Theorem IV. Details will be omitted.

THEOREM V. *The following taken together are sufficient conditions that transformation* (9) *be limit-producing*:

(a) *$tK(x, t)f(t)$ be positive monotonic decreasing in t for each x*;

(b) $\lim_{x\to\infty} K(x, t) = 0$ *uniformly in each interval*, $0 \leqq t \leqq \delta$, $\delta + i\omega \leqq t \leqq \delta + (i+1)\omega$;

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* See, for example, Fort, Infinite Series, p. 101.

(c) *$d/dtK(x, t)f(t)$ be monotonic increasing in t for each x.*

(d) *The transformation*

$$\sum_{i=0}^{(x-\omega)/\omega} K(x-\omega,\ i\omega)f(i\omega)$$

be limit-producing when $x \equiv 0 \pmod{\omega}$.

Now by the Euler-Maclaurin summation formula in case certain conditions on $K(x, t)f(t)$ are fulfilled* we can write

$$(13)\quad \mathop{S}_{0}^{x+\omega} K(x, t)f(t)\Delta_\omega t = \int_0^x K(x, t)f(t)dt + \sum_{v=1}^{m} \frac{\omega^v}{v!} B_v\left[\frac{\partial^{v-1}}{\partial\alpha^{v-1}} K(x, \alpha)f(\alpha)\right]_{\alpha=x}$$
$$+ \frac{\omega^{m+1}}{m!}\int_0^\infty P_m(1-t)\frac{\partial^m}{\partial t^m}\left[K(x, x+\omega t)f(x+\omega t)\right]dt.$$

From this we conclude immediately the following theorem.

THEOREM VI. *If, in addition to* (13),

$$\int_0^\infty \left|\frac{\partial^m}{\partial t^m} K(x, x+\omega t)f(x+\omega t)\right| dt$$

converges uniformly in x then

$$\lim_{x\to\infty}\phi(x) = \lim_{x\to\infty}\left(\int_0^x K(x, t)f(t)dt + \sum_{v=1}^{m}\frac{\omega^v}{v!}B_v\left[\frac{\partial^{v-1}}{\partial\alpha^{v-1}}K(x, \alpha)f(\alpha)\right]_{\alpha=x}\right).$$

From Theorem VI we conclude the following theorems.

THEOREM VII. *Sufficient conditions, in addition to* (13), *that the transformation* (9) *be limit-producing are the following taken together*;

(a) *The integral transformation* (7) *with kernel K be limit-producing*;

(b)
$$\left.\frac{\partial^n}{\partial\alpha^n}K(x, \alpha)\right]_{\alpha=x}$$

exist and approach a limit when $x\to\infty$, $n=1, 2, \cdots, m$;

(c)
$$\int_0^\infty \left|\frac{\partial^n}{\partial t^n}K(x, x+\omega t)\right| dt,$$

$n=1, 2, \cdots, m$, converge uniformly in x;

(d) *$f(x), f'(x), \cdots, f^{(m)}(x)$ approach limits when $x\to\infty$.*

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* See L. M. Milne-Thomson, *loc. cit.*, p. 212.

THEOREM VIII. *Sufficient conditions, in addition to* (13), *that* $\phi(x)\to A$ *are the following taken together:*

(a) $f(x)\to A$ *when* $x\to\infty$ *and* $f^{(m)}(x)\to 0$ *when* $m\geqq 1$;

(b) $$\frac{\partial^n}{\partial\alpha^n}K(x,\alpha)\Big]_{\alpha=x}$$

approach a limit, when $x\to\infty$, $n=0, 1, \cdots, m$;

(c) $$\int_0^\infty\left|\frac{\partial^n}{\partial t^n}K(x, x+\omega t)\right|dt$$

$n=1,\cdots, m$, *converge uniformly in* x;

(d) *the transformation* (7) *be regular.*

DISCUSSIONS AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A NEW SYSTEM FOR PLAYING THE GAME OF NIM

D. P. MCINTYRE, Victoria, B. C.

The game of Nim is played by two players. Matches or counters are arranged in any number of groups with any number in each. The players alternately draw one or more counters from any one group and the player who draws last wins.

The theory of this game making use of the binary representation of numbers has already been worked out* and shows that the first player, provided he knows the system, can, with few exceptions, win. The rare exceptions occur when the initial arrangement is favorable to his opponent, but in any case a player who knows the system can usually defeat one who does not.

The theory presented here is based on the quaternary representation of numbers, which is somewhat simpler of application than the binary representation since it involves fewer terms.

1. *Definition of the quaternary representation and its coefficients.* Let the number of groups in the game be n, and the number of elements in the ith group be x_i, where $1\leqq i\leqq n$. Then the numbers x_i can be expressed in the form

$$x_i=\sum_{j=0}^{\infty}a_{ij}4^j,\qquad 0\leqq a_{ij}\leqq 3,\qquad 1\leqq i\leqq n.$$

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* See, for example, C. L. Bouton, Annals of Math., ser. II, vol. 3, 1901, p. 35.

2. *Definition of balanced systems of groups.* The set of coefficients a_{ij} for a fixed j is said to be balanced if it can be broken up into one or more sets each of which is either a pair (*e.g.*, 2, 2), the set 1, 2, 3, or the single element 0. For example, the set 1, 3, 0, 1, 1, 2, 0, 3, 3 is balanced since it consists of the pairs 1, 1 and 3, 3 and the set 1, 2, 3 along with two 0's. The system of groups is known as a balanced system if the coefficients a_{ij} are balanced for every fixed value of j. For example, the following system is balanced:

$$6 = 1\cdot 4 + 2,$$
$$5 = 1\cdot 4 + 1,$$
$$3 = 0\cdot 4 + 3.$$

3. *Any unbalanced system can be balanced by the removal of one or more units from one particular group.* We will consider only games of more than two groups since the latter systems can obviously be balanced by evening up the groups. If any of the sets of coefficients a_{ij} for fixed j are unbalanced, the method of cancelling out pairs and sets of 1, 2, 3 will reduce the set to at most two digits. Thus any single set a_{ij} for fixed j can be balanced by subtracting or adding enough from one of the digits to balance the other or by adding enough to a zero digit to make a balanced set or by subtracting the odd digit. The set may also be balanced by altering one and only one previously chosen coefficient. For by keeping this coefficient fixed and cancelling out zero terms, the set can be reduced to at most three terms. If there were more than three terms, there would be pairs or sets of 1, 2, 3 among the unfixed terms. The set can then be balanced by altering the chosen coefficient to complete the other one or two terms if any exist.

Let k be the highest value of j for which the coefficients a_{ij} are unbalanced. Reasoning as before, these can be balanced by the removal of a small multiple of 4^k from one of the groups of x_i units. Denote this group by $i=m$. Thus we have available at least 4^k units to balance the remaining coefficients a_{ij} for each fixed j by altering the coefficients a_{mj}, $(j\leqq k-1)$. The most that we will be required to add to balance any set will be $3\cdot 4^j$ units. Thus the maximum number of units required for the balancing of the remaining k sets is

$$\sum_{j=0}^{k-1} 3\cdot 4^j = 4^k - 1 \text{ units.}$$

But we have available at least 4^k units. Hence it is always possible to balance any given unbalanced set of groups by subtracting one or more units from one of the groups.

4. *A balanced system is a losing game for the first player.* Since all the groups are diminishing in size during the play they must all eventually reduce to digits of less than 4. If a player has been presented with a balanced system at each

turn he will eventually have to play against a system of groups composed of digits none of which is greater than three. Thus the system contains only pairs and sets of 1, 2, 3 which is certainly a loss.

5. *Quick setting up of balanced groups.* In order to be sure of winning it is necessary to present an opponent with a balanced system with every move. The easiest way to do this is to break the numbers down into quaternary form and to decide which one has the largest unbalanced term. This group is the one which must be cut down. Ignore this group and consider only the others of the system, and by cancelling coefficients determine the number which will balance the system. With terms of order higher than 4^0 it is easier to handle the whole term rather than the coefficient; *i.e.*, consider the terms to be 0, 4, 8, 12 or 0, 16, 32, 48 instead of 0, 1, 2, 3.

Example. Balance the system of groups 8, 36, 17, 31, 14.

The group 36 contains the term $2\cdot 4^2$ or 32 which cannot be balanced by any of the other groups; hence this term must be cut down. Considering only the other groups, we have

Order	8	36	17	31	14	Group required to complete system
4^2	0		$\cancel{16}$	$\cancel{16}$	0	0
4^1	8		0	$\cancel{12}$	$\cancel{12}$	8
4^0	0		$\cancel{1}$	$\cancel{3}$	$\cancel{2}$	0
						8

Thus if the group 36 is reduced to 8, the system will be balanced.

If the opponent confronted with a balanced system does not reduce a group below the highest multiple of 4 contained in the group, he can only succeed in deranging the coefficients of 4^0 and the system can again be balanced by considering only these coefficients. For instance, in the example above, if the opponent playing on the system 8, 8, 17, 31, 14 which is balanced decides to reduce the group of 31 by 1, 2, or 3 (*i.e.*, he does not reduce the 31 to a number less than 28), he only succeeds in deranging the coefficients of 4^0 and the first player may again balance the system by removing a sufficient number of units of the same order from the 17 or 14 groups.

A SECOND NOTE ON AUTOPOLAR CURVES

MALCOLM FOSTER, Wesleyan University

1. Introduction. In a recent paper* the author has considered those curves which are autopolar with respect to a parabola. Such curves are here regarded as special solutions of those differential equations which are invariant under the Legendrian dual transformation for which this parabola is the conic of reference. It is the purpose of this brief note to state certain theorems which follow quite readily from my first paper on the subject.

Relative to a given conic of reference, two curves are said to be polar reciprocals if the polars of the points of either curve envelope the other; an autopolar curve is its own polar reciprocal. We shall always use the parabola $2\eta=\xi^2$ as the conic of reference.

2. The conjugate circle. Two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on an autopolar curve C are called a conjugate pair if the tangent to C at either is the polar of the other with respect to the given parabola. Associated with this pair, let us consider the points Q and R in which the pairs of tangents and normals at P_1 and P_2 intersect. The four points P_1, P_2, Q, and R evidently lie on a circle for which QR is a diameter; this circle we shall call a "conjugate circle." The coördinates of its center are†

$$\left(\frac{y_1 - y_2 + y_1^2 - y_2^2 + x_1^2 - x_2^2}{2},\ -\frac{1}{2}\right).$$

Hence we have the following:

THEOREM 1. *For any curve autopolar with respect to the parabola $2\eta=\xi^2$, the locus of the centers of the conjugate circles is the directrix of the conic of reference.*

3. Two metric properties. Let P_1' and P_2' be the points on the parabola of reference whose abscissas are respectively equal to the abscissas of a conjugate pair, P_1 and P_2. The product of the curvatures of the parabola at P_1' and P_2' is readily found to be

$$K_p = \frac{1}{[(1+x_1^2)(1+x_2^2)]^{3/2}};$$

and the product of the curvatures of the autopolar curve at P_1 and P_2 is

$$K_a = \frac{y_1''y_2''}{[(1+x_1^2)(1+x_2^2)]^{3/2}}.$$

But for this dual transformation,‡ $y_1''y_2''=1$. We therefore have the following:

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* Malcolm Foster, Note on autopolar curves, Bulletin of the American Mathematical Society' vol. 47, 1941, pp. 247–253.

† Foster, *loc. cit.* See equations (14) and (15), p. 250.

‡ Foster, *loc. cit.*, equation (2), p. 247.

THEOREM 2. *For any curve autopolar with respect to the parabola $2\eta=\xi^2$, the product of the curvatures at a conjugate pair is equal to the product of the curvatures of the parabola at those points whose abscissas are equal to the abscissas of the conjugate pair.*

Since in the above demonstration we have made no use of the condition of autopolarity,* it follows that Theorem 2 is also valid in the case of corresponding points on pairs of polar reciprocal curves.

We readily find that the equation of the tangent to the parabola of reference at P_1' is $2x_1x-2y-x_1^2=0$, and that the coördinates of the midpoint S of the segment which joins P_1 and P_2 are

$$x = (x_1 + x_2)/2, \qquad y = (y_1 + y_2)/2 = x_1x_2/2.$$

These coördinates, however, satisfy the above equation; hence, since it is evident that the coördinates of S must also satisfy the equation of the tangent to the parabola at P_2', we have the following:

THEOREM 3. *For any curve autopolar with respect to the parabola $2\eta=\xi^2$, the line which joins any conjugate pair P_1 and P_2 is concurrent with the tangents to the parabola of reference at P_1' and P_2' at S, the midpoint of the segment P_1P_2.*

Since Q is the pole of P_1P_2, it follows that the polar of S must pass through Q. Hence Q, P_1', and P_2' are collinear.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

The Development of Rationalism and Empiricism. By G. de Sentillana and E. Zilsel (International Encyclopedia of Unified Science, Volume II, Number 8.) Chicago, University of Chicago Press, 1941. 8+94 pages. $1.00.

The Principles of Financial and Statistical Mathematics. Revised edition. By Maximilian Philip. New York, Prentice-Hall, Inc., 1941. 16+335 pages. $3.50.

A Manual of Problems in Statistics. By S. Dayton. New York, Henry Holt and Company, 1941. 163 pages. $0.90.

Tools. A Mathematical Sketch and Model Book. By R. C. Yates. Baton Rouge, Louisiana State University Press, 1941. 194 pages. $1.60.

Factor Analysis; A Synthesis of Factorial Methods. By K. J. Holzinger and

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* Foster, *loc. cit.*, equation (10), p. 249.

H. H. Harman. Illinois, The University of Chicago Press, 1941. 12+417 pages. $5.00.

What is Mathematics? By R. Courant and H. Robbins. London, New York and Toronto, Oxford University Press, 1941. 19+521 pages.

Fourier Series and Orthogonal Polynomials. By D. Jackson. The Carus Monographs, No. 6. Oberlin, Ohio. The Mathematical Association of America, 1941. 12+234 pages. $2.00; to members, $1.25.

Early Military Books in the University of Michigan Libraries. By T. M. Spaulding and L. C. Karpinski. Ann Arbor, Michigan, The University of Michigan Press, 1941. 16+45 pages; 25 plates.

Lectures in Topology. Edited by R. L. Wilder and W. L. Ayres. The University of Michigan Conference of 1941. Ann Arbor, Michigan, The University of Michigan Press; London, Humphrey Milford and Oxford University Press, 1941. 7+316 pages. $3.00.

REVIEWS

Fundamental Theorems of Orthographic Axonometry and their Value in Picturization. By W. H. Roever. Washington University Studies, New Series, Science and Technology, No. 12. St. Louis, Missouri, Washington University Press, 1941. 47 pages. $1.00.

This little pamphlet gives a vivid picture of the meaning of axonometry by means of a detailed comparison of the representation of a bracket-shaped figure by the Mongean method and by the axonometric method. Then the fundamental theorems of the latter are derived. The foreshortening ratios, axonomatic scales, and the necessary analytic relations are carefully explained and derived, followed by a detailed analysis of the theorems of Schwarz and of Gauss. These are applied to the solutions of two fundamental problems:

I. Given the axonometric projections of the coördinate axes, to find the foreshortening ratios.

II. Given the foreshortening ratios, to find the mutual inclinations of the projections of the coördinate axes.

The style is simple and direct, and the 24 figures are excellent.

VIRGIL SNYDER

Odd Numbers or Arithmetic Revisited. By Herbert McKay. New York, The Macmillan Company, 1940. 1+215 pages. $2.50.

This book is an interesting presentation of arithmetic for the average layman. It is by no means a text-book. Instead, it is written in a so-called popular manner so as to make it especially appealing to those having little knowledge of arithmetic and to arouse in them an appreciation not only of its usefulness and application but also of arithmetic itself. The author points out that "Arithmetic is usually regarded as the Cinderella of Mathematics, the drudge whose duty it is to do everything that is dull." He deplores the opinion that mathematics is

dull, and opines that the understanding and manipulation of numbers is exciting. He endeavors to substantiate this fact and in the reviewer's opinion does it fairly well.

The material covered includes powers, logarithms, proportion, weights and measures, the arithmetic mean, comparisons, approximations, multiplication and division, tables, units, oddities of numbers, and scales of notation. He tries to make large numbers understandable. Numerous practical examples and illustrations assist to clarify the topics discussed and add to the popular interest.

The author mentions the error which caused the Italians to celebrate the bimillenary of Augustus a year too soon, and H. G. Wells's error in allowing his man in the moon to do some fantastic jumping. A simple explanation is given of the basis of logarithms. Arguments are presented both for and against the decimal and duodecimal systems. He explains a number of tricks in reckoning. In connection with weights and measures, the author defines a unit as "a convenient amount of the quantity to be measured." An explanation is given of how the different units came into being.

The book is enjoyable, stimulating, and amusing.

F. M. WEIDA

The Elements of Statistics. By E. B. Mode. New York, Prentice-Hall, Inc., 1941. 16+378 pages. $3.50.

In the announcement of a recently published work on Statistical Bibliography, the author states that the first objective of his book is: "To make students and teachers of statistics more keenly aware of the inadequacy of much of what is now presented in textbooks and classes despite the fact that such statistical techniques are incorrect, inefficient, and obsolete." By stressing these shortcomings, he expresses exactly what so many research workers and teachers in statistics are feeling today.

All the more do we welcome a book like that by E. B. Mode which, elementary as it is, gives a clear and correct presentation of the problems it deals with. These problems are manifold. Besides the basic concepts of descriptive statistics, such as frequency distributions, averages, moments of one-dimensional and bivariate distributions, we find a chapter on index numbers and some remarks on time series. These sections form the main part of the book, whereas the references to theory of probability and to problems of small samples are rather brief. The use of various sorts of graph paper is as welcome as the valuable introduction to approximate computation. The exercises accompanying the different chapters, "none of which have been borrowed from other textbooks," are interesting and stimulating. The proofs of simple mathematical theorems are generally given; in a few other cases, the reader gets a correct statement of the theorem. The book will certainly be very useful for all statisticians not primarily interested in theoretical statistics.

HILDA P. GEIRINGER

Algebra. A Text-book of Determinants, Matrices, and Algebraic Forms. By W. L. Ferrar. Oxford, Clarendon Press, 1941. vii+202 pages. $3.50.

The purpose of this book is to provide material for an undergraduate or a first year graduate course in the elements of the theories of matrices and algebraic forms. The book is in three divisions: I *Determinants*, II *Matrices*, III *Linear and Quadratic Forms*. In a brief treatment, as this volume is, a careful selection of material is necessary. On the whole, the author's choice of subject matter is judicious, particularly in the chapter on invariants in Division III. The omission from Division II of the topics of similarity and matrices with polynomial elements may be questionable.

For the most part, the clarity of the presentation is commendable and the author in preparing the book has kept the student uppermost in his mind. However, the statement of the separate corollaries to Theorem 7, p. 16, would be an insult to the intelligence of even a dull student. Following are listed several features which appealed to the reviewer: a novel definition of a determinant is used in Division I (which later affords a very neat proof of the theorem about the evenness or oddness of a permutation); an elegant proof of the theorem of Frobenius concerning the characteristic roots of a rational function of a matrix; a clear and elementary proof of the theorem of Darboux, that the principal minors of the matrix of a positive definite quadratic form are all positive; the well chosen problems (which do not include many arithmetical examples, however) particularly those in connection with form theory, invariant theory, and manipulation of determinants.

There are a number of features which most mathematicians will find objectionable. Those of a general occurrence are: sporadic use of dummy summation indices; lack of distinction between a square array and a determinant; use of $\times$ for matric multiplication; use of *ordinary* as a synonym for *non-singular*; use of *transformation of* instead of *transform of*; failure to employ matric theory to advantage in the development of form theory. Also notable are the following: the *first step* of p. 9 is superfluous; the laws of exponents for non-singular matrices are awkwardly handled; the lemma on p. 165 could be made much more general without altering the proof; the matrix J_1 on p. 165 is its own inverse, and noting that fact would make the presentation clearer. The language in several places is unjustifiably vague (*cf.* pp. 17, 30, 189, 157 [Ex. 2]).

More serious than the foregoing, however, is the excessive number of errors and slights of rigor which mar the book: *cf.* the definition of field, pp. 2 and 119; p. 35, Ex. 4; p. 107, Ex. 9; p. 60, proof of Theorem 21; p. 115, neglecting to prove that the inverse of an elementary transformation is an elementary transformation; p. 156, vitiation of proof of Theorem 48, due to failure to prove existence of a λ_1; p. 154, equation (4) is not justified by the preceding argument; p. 156, Ex. 1 as stated is false.

The format of the book is pleasing and the typography is excellent (the reviewer noted only three typographical errors). All in all, the book is one which will probably stimulate many students and irritate some teachers.

R. F. Rinehart

Galois Lectures. By Jesse Douglas, Philip Franklin, Cassius Jackson Keyser, and Leopold Infeld. The Scripta Mathematica Library, Number 5. Scripta Mathematica, Yeshiva College, New York, 1941. 124 pages.

The lectures here collected were delivered at the Galois Institute of Mathematics at Long Island University in Brooklyn, New York. The first of them, "Survey of the Theory of Integration" by Jesse Douglas, is a clear, concise presentation of the fundamental definitions and essential properties of Riemann, Stieltjes, and Lebesgue integrals, with some remarks on the Denjoy integral. While avoiding excessive demands on the reader's mathematical background, the author deals clearly with many basic questions in a manner well adapted to the needs of a student of integration. The bibliography and specific references to sources for omitted details make the article a useful guide for a thorough study of the subject.

Philip Franklin's lecture is an exposition of the history and current status of the four-color problem. It contains a simplification of the problem to regular maps, a proof of the five-color theorem, a discussion of reducible configurations, and various specialized theorems on coloration. A few equivalent and inclusive formulations are presented. Franklin treats "perhaps the simplest unsolved problem of mathematics" in an interesting, lucid style, quite comprehensible to a non-specialist. He gives a bibliography which may tempt onward anyone who is lured by the fascinations of this mathematical will-o'-the-wisp.

The third lecture, by Cassius Jackson Keyser, is on "Charles Sanders Peirce as a Pioneer" and contains a very few samples of Peirce's vast and varied contributions to philosophic thought and especially to the development of logic. A brief biographical sketch is followed by sections telling of Peirce's pioneer work in connection with pragmatism, the theory of infinite classes, propositional functions, paradoxes, and the theory of relations. It is to be hoped that readers will be stimulated to consult the *Collected Papers* being published by the Harvard University Press.

The book closes with the text of a radio broadcast given under the auspices of the Galois Institute. It is in the form of a short dialogue between Leopold Infeld and a "clever pupil asking just the right questions" on the subject "The Fourth Dimension and Relativity." Remaining well within the college freshman level of difficulty, Infeld undertakes to cure the customary awe of the layman for those physicists and mathematicians who are supposed to have some intuitive knowledge of a four-dimensional universe.

S. S. CAIRNS

The Second Yearbook of Research and Statistical Methodology Books and Reviews. Edited by Oscar Krisen Buros. The Gryphon Press, Highland Park, New Jersey, 1941. 20+383 pages. $5.00.

In 1938, a set of excerpts from reviews of books on statistics and related fields appeared under the editorship of Professor Buros of Rutgers University

(see this MONTHLY, vol. 46, 1939, pp. 355–356). It consisted of 635 review excerpts from 131 journals. It was, on the whole, warmly received, and the editor has now published a greatly enlarged edition which is here under review. This edition, according to the preface, contains 1652 review excerpts from 283 journals. It is a large, excellently printed volume, very adequately indexed. Indeed, a list of the indices alone gives some idea of the thorough nature of the book; at the beginning, there is a list of the journals used, classified by fields, while at the end, there is a periodical directory and index, a publisher's directory and index, an index of titles, an index of names, and a classified index to books. The author states no less than eleven objectives of the book, or more properly, of the proposed series of similar volumes to appear every two years. These objectives may be summarized by stating that the primary idea is to elevate the standards of statistical literature by giving publicity to criticism written from a broad variety of viewpoints; a secondary aim is to elevate the standards of the criticism itself.

After this catalog of some of the more imposing features of the work under review, it will sound almost frivolous to say that, at least in the opinion of this reviewer, the book is really quite entertaining to read. This is undoubtedly a biased judgment, because several of this reviewer's own reviews are reprinted in juxtaposition to reviews on the same subject by better qualified reviewers, and the resulting comparisons were personally interesting and instructive. But the wide variety of opinions expressed on any one work, and the subtle light which they often throw upon their author's competence, would almost surely provide some entertainment even for a person not directly interested in statistics. Incidentally, the new edition covers an even greater variety of scientific fields than the older one did, which is really saying something. Perhaps this is in line with the definition of statistician which Dr. W. E. Deming likes to give in his lectures, at least by implication: a statistician is absolutely anyone who makes any measurements whatsoever.

The listings still do not include English reviews of foreign language books, an omission which was criticized adversely when it occurred in the earlier edition. In his preface, the editor expresses hope of remedying the situation in later editions, and also suggests two more major improvements toward which he is working. The first is to establish a noncritical abstracting service for the periodical literature on statistics, and the second is to publish original criticisms of articles and papers in the periodical literature. As far as mathematical statistics is concerned, the first of these services appears to be adequately supplied by *Mathematical Reviews*. As to the second part of the program, the plan sounds good at first hearing, but on second thought, the idea of supplementing each of the numerous "important" papers by several critical appraisals and (presumably, to be fair) a spirited rejoinder by the author, all sounds just a little appalling. Considering the amount of work required to establish and maintain *Mathematical Reviews*, one wonders if Professor Buros knows just what he is letting himself in for.

But to return to the present volume, we can say that the work as it stands

is both actually and potentially extremely valuable. Let us urge by all means that the series be continued. Every teacher of statistics courses will find the series indispensable, and it is now rather apparent that scientists engaged in a wide variety of activities will find these volumes useful.

J. H. CURTISS

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT AND J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Brown University, Providence, R. I.

A NIGHT WITH PROBABILITY

E. E. BLANCHE, Michigan State College

If you're looking for a program that will give pep to any club meeting, illustrate a number of mathematical principles, and, incidentally, show why gambling does not pay, then you might let probability do the job for you.

The arrangement of the program is rather simple; the equipment and supplies necessary are inexpensive; and the conduct of the various games at which members try their hand is so easy that even groups which have little mathematical background may understand the laws which "A Night With Probability" illustrates.

During the past year I have had occasion to direct this whirl with Lady Luck at our *Pi Mu Epsilon* chapter at the *University of Illinois*, and later at a number of university and church meetings. The entertainment has been received with such enthusiasm that it seems advisable to describe the entire procedure so that others may enjoy this educational and instructive evening of fun.

The entire program consisted of a number of simple games of chance, similar to those which may be found at county fairs, carnivals, amusement parks, and the "vice dens" of the big city. Before allowing the audience to act as the so-called "suckers" at the various gaming tables, I introduced the program with a twenty- to thirty-minute talk on various forms of amusement and games of chance which I have investigated during the past twelve years—investigation which was first undertaken as a reporter on the Passaic, N. J., *Daily News*, and later as a student and instructor in mathematics.

This sort of introduction acts as a stimulant, and puts the audience in the proper frame of mind. Some of the following games may be described and "debunked," or illustrated rather simply:

1. *The Numbers Racket.* This is very common in large cities and especially in the eastern states, notably New York and New Jersey. The player may select any number from 000 to 999, and place his bet of anything from one cent to one dollar with the local "bookie" (in general, a shady character). This is a daily

game. Based upon the winnings of the horses at some predetermined race track, a certain number is obtained, which is the winner that day. If one is unfortunate enough not to have placed a bet on that number (which is generally the case), he loses. On the other hand, if he has placed a bet on the winning number, then the "ring" conducting the "numbers" game will pay him 500 to 1 or 600 to 1, depending on the locality and the operating "ring." (The New York City area has recently cut its odds to 500 to 1.) If this is the case, then the player with a one-cent winner receives $5.00; but it is an unwritten law that the "bookie" will receive ten per cent of the winnings, so our player actually gets $4.50 for his penny. Actually, however, the odds against our friend winning are 999 to 1. If a wager of one cent were made on each number, totalling ten dollars, then the return would be only $4.50 on the single winning number. The gambling "ring" and the local "bookie" grow fat on the $5.50 which has been left in the till.

2. *Marked Cards.* These may sometimes be secured from local gamblers, local prosecutors' staffs after raids, or may be purchased from well known playing card manufacturers, some of whom do a thriving business in marked cards. (I shall be glad to furnish the names and addresses.) However, marked cards are expensive, the average cost being about five dollars a deck. The marking scheme, once learned, may be illustrated on a blackboard, and the cards may be distributed to the audience for examination.

3. *Carnival Wheel.* The ordinary wheel with twelve to twenty numbers on it sometimes has a brake similar to that on an automobile, and may be slowed down or stopped by the operator when he presses a foot pedal at some point quite distant from the wheel. The wheel may have a movable weight which may be adjusted by a touch of the operator to favor any section.

4. *Pitch-Till-You-Win.* This game allows you to toss hoops at prizes which are set up on cube or parallelopiped bases until a hoop covers the base. The bases of large prizes are such large cubes or parallelopipeds that it is nearly impossible to get a hoop completely over one on a toss from a distance of five to eight feet. Of course, the small prizes (ash trays, whistles, trinkets) have very tiny bases, which may be covered easily.

5. *The Fishing Game.* Here one scoops up a wooden fish from a circular trough of moving water. The moving fish are enticing and deceiving. The trough goes behind the curtain where the operator's confederate either removes or replaces winning fish at certain signals from the operator. Thus, while encouraging a prospect to try his luck, the operator can select a large prize winner; but when the prospect fishes, he always gets a small prize.

6. *Tossing Pennies* for one-inch squares which pay the tosser five or ten or even fifty cents for each penny lodged entirely within the square is a sure way to bankruptcy. Simple experimenting will show that several hundred pennies must be tossed before one is placed exactly inside a square.

7. *The Mouse Game* attracts a goodly crowd. A horizontal wheel is divided into sixty six-degree sections; at the circumference a small hole is drilled in each section; the sections are painted red, blue, green, yellow, orange, gold, and silver.

There are various numbers of sections of each color. The wheel is spun, and a mouse, covered by a small cup, is placed in the center. After a number of whirls, the cup is removed and the dizzy mouse, after regaining his equilibrium, scurries to one of the holes thinking he may find some food. The color of the section into which the mouse disappears determines the winner. Various odds are paid to the backers of the winning colors but these odds are tremendously in favor of the operator. The players are really paying for the privilege of watching the mouse run into a hole.

8. *Covering The Red Spot*, ten inches in diameter, with five small white circles is a task that requires minute exactness and much skill. The average person, without practice, has no idea of the single scheme necessary to blot out all the red color. This game may be mastered, however, by practice and development of skill.

9. The odds against the player in the *Rolling-Ball Game* are unimaginable. The player rolls six balls down an incline toward thirty-six holes, six each numbered one to six inclusive. The total score by the six balls which have fallen into various holes is the criterion by which winners are determined. However, only totals less than 13 or more than 29 will be winners. This presents a nice little problem of determining the probability of winning, namely, $2(924)/1{,}947{,}792 = .0008$ (eight times in ten thousand, approximately).

This type of introduction may be revised, or excluded, or supplemented by other games (I can supply hundreds more), according to the type of audience present or the amount of time allotted.

Following the introduction, a description of the various games to be played on "A Night with Probability" (which I have not yet described) may be given.

The following games have been employed with considerable success in illustrating probabilities and pointing out the truth of the statement "you can't win."

1. *Roulette.* A small or large roulette wheel is necessary for conducting this game, and may be purchased at toy shops, department stores, or borrowed from people who have them. The wheel consists of thirty-eight sections, numbered 0, 00, and 1 to 36. The rules of the game and the playing table may be painted with black ink on white sheets.

Bets of any size may be placed on any number 1 to 36. If the number turns up, the winner receives 35 to 1 or 36 to 1 for his wager. If his number does not turn up, he loses. In addition, the "House" or operator has the numbers 0 and 00, and when either appears, the "House" wins all wagers. Bets may also be placed on red or black, and even money is paid to winners; or sections of sets of numbers may be formed with suitable odds, as illustrated.

2. *Double-Your-Money-Quick.* This is a dice game, found any place that is frequented by dice players and gullible folk. Three dice are thrown by the operator of the game. A cardboard, with numbers 1, 2, 3, 4, 5, 6, is placed on a table. Bets may be placed on these numbers by the players. Now if one bets a unit on

No. 1, and the dice when rolled read 1-4-5, then the operator gives the player on 1 an additional unit. If there are players on 4 and 5, he gives each an additional unit. If there are players on 2, 3, and 6, he scoops up their bets. When all six numbers are covered by a unit each, and the operator tosses three different numbers on the three dice, he simply has to turn over the three units he wins on the losing numbers to the three people who backed the three winning numbers.

Now, when the dice are thrown 1-1-2, the operator gives two units to the player on 1, and one unit to the player on 2. If the board is filled with a single unit on each number, the operator will win a unit each from 3, 4, 5, and 6, and will then pay out two units to 1 and one unit to 2, leaving him a profit of one unit.

When the dice turn up 1-1-1, the operator pays the winner three additional units while he collects all other wagers. Under the same board conditions as above, he wins five units and pays out three.

Hence, whenever doubles or triples turn up on the dice, the operator takes a profit when the game is full. This game always wins for the operator when the players continue for some length of time.

3. *Match a Card.* The betting board has a set of cards exposed. The player simply picks out some card and places his bet upon it. Then, the operator selects a card from two decks shuffled together. If the number of the card is matched but not the suit, the bettor is paid 10 to 1. If both the number and suit are matched, the bettor receives 20 to 1. After the operator of the game has drawn eight or ten cards from the stack, the cards in the stack should be reshuffled so that the players do not have the advantage of having seen too many cards before placing their bets.

4. *Double or Nothing.* Here the persons taking part in the game do all the throwing of the dice while the one conducting the game simply pays the winners and collects from the losers. The idea behind the game is whether the tosser can throw doubles or triples with three dice. If the tosser does not throw doubles (two of a kind) or triples (three of a kind), he loses whatever he has bet. On the other hand, if he tosses doubles or triples, he wins from the operator as much as he has bet, *i.e.*, he doubles what he bet.

5. *Old-Fashioned Horse-Race.* This is a race between six horses, which may be made of wood by some member with a jig-saw. The horses are numbered and painted appropriate colors: (1) Spark Plug, (2) Barney Google, (3) Man O'War, (4) Whirlaway, (5) Flat Feet, (6) Zev. They are set at the starting-point of an oil cloth, ruled with ten spaces of five or six inches in length and two inches in width, in each of six lanes. Only six persons may play. The horses leave the starting-point and move one rectangle each time the number of the horse turns up in a toss of three dice by the operator. At the start of the race each player gives 5 units to the operator. If the first toss happens to be 1-2-2, then horse 1 moves up one rectangle, and horse 2 moves up two rectangles while the other horses remain at the post. Then the dice are tossed again, with various horses moving.

1. Roulette.

Odds 35 to 1								
1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36

1 2 3 4 5	Combination 5 to 1	6 7 8 9 10
11 12 13 14 15	16 17 18 19 20	21 22 23 24 25

Red (Even Money)	26 27 28 29 30	31 32 33 34 35	Black (Even Money)

2. Dice.

1	2	3	4	5	6

Single No. Pays	1–1
Two of a Kind	2–1
Three of a Kind	3–1

3. Match a Card.

Match No. or Card
Clubs A K Q J 10 9 8 7 6 5 4 3 2
Diamonds A K Q J 10 9 8 7 6 5 4 3 2
Hearts A K Q J 10 9 8 7 6 5 4 3 2
Spades A K Q J 10 9 8 7 6 5 4 3 2

Matched No. Pays	10–1
Matched Card Pays	20–1

4. Double or Nothing.

We Double Your Money When You Throw Doubles or Triples

5. Horse Race.

Starting Post

1	2	3	4	5	6
1	2	3	4	5	6

Finish Line

6. Twenty-One.

21 A Total of 19, 20, 21 Doubles Your Money. 2–1 For 21 With Two Cards Picture Cards—10 Points Aces Count 1 or 11 Others Face Value.

The procedure is repeated until one of the horses crosses the finish line. The player backing that horse is paid a win price of 12 units.

The winner is then out of the race, and the other five players continue to move their horses on each toss of the dice until another horse crosses the finish line. Second place wins eight units. The others continue until third place has been decided, paying 5 units. The other three horses are "also-rans," and their backers get nothing.

6. *Twenty-One*, a variation of the well known gambling game, is rather interesting. The conductor of the game is always the dealer of a deck of cards, but as a variation, the dealer does not take a hand. Each person who plays is dealt a single card, face down. Then, having looked at this card, the player may bet up to the limit set by the operator. Picture cards (Jacks, Queens, and Kings) count ten points each, while the Ace may be considered either one or eleven points. All numbered cards are counted their face value.

The player may draw as many cards as he wishes. If his total exceeds 21, he loses. If his total is 19, 20, or 21, he receives from the operator as much as he wagered. Moreover, if he makes 21 with only two cards, *i.e.*, an ace and a picture card or ten, the player is paid two units for every one wagered. Of course, another variation from the regular game is that each player will decide to draw cards until he has won with 19, 20, or 21, or has lost by exceeding 21.

This set of games may be supplemented by games of skill if it seems desirable. Some simple games of skill which are easily set up and which will not prove too expensive to the "House" are these:

1. *Ping Pong Bounce*. Six small glasses (fruit juice size) are placed in a triangle near a wall, with about an inch or two between each glass. A line is drawn about six feet in front of the triangle. Each contestant pays one unit for three ping pong balls, and tries to make them bounce into the glasses. Each time a ball bounces into a glass, the player gets five units. (Winners are rare.)

2. *Darts*. The player gets five darts for one unit of money, and wins a unit each time a dart lodges in the bull's eye.

3. *Milk-Bottle Drop*. The player buys three ping pong balls for a unit, and standing erect over a milk bottle, tries to drop the ping pong balls into the bottle from chin level. He is paid three units for each ball inside the bottle.

4. *Shuffleboard*. Five wooden wafers are given for one unit, and are shuffled down the board by the player. If four wafers lodge in the compartments at the end of the board, the player receives two units; if five wafers go into the compartments, the shuffler receives three units.

There are many other skill games which may be conducted, but the above have been tried with success.

Most of the equipment necessary for conducting the games may be purchased in five-and-dime and department stores. Wager boards similar to those illustrated may be made on large size cardboard with black or red marking crayon or poster inks. The horses may easily be cut, painted and numbered. The

dart game, roulette wheel, ping pong balls, shuffleboard and others may be borrowed from persons who have them, or purchased if there seems to be a likelihood of using them often.

The question of play-money is easily answered. Sheets of thin green and gold paper may be purchased at stationery stores (about ten cents a dozen sheets), and then cut to the appropriate currency size. The green paper may be considered one dollar bills while the gold paper may be considered ten dollar bills. There is no difficulty in recognizing the denomination. Or the numbers 1 and 10 may be written on the respective bills with marking crayon. (My wife and I made $2400 in half an hour from fifty cents worth of paper.)

Each person taking part in the evening's game should be given the equivalent of twenty dollars, one gold-back and ten green-backs. This small amount discourages wild splurging. The "House" should have in the neighborhood of several thousand dollars so that there will be no shortage of cash at any table during the evening.

After an hour or two of playing each person should return his money to the cashier. Prizes may be awarded to the persons returning the highest amount of money, *i.e.*, those who have been most successful at the games. In general, most people will lose their stake. Only a few will show any appreciable gain.

Besides providing plenty of laughs and fun for everyone taking part, "A Night With Probability" will teach those present why gambling does not pay.

Instead of the customary note printed by one of the New York daily newspapers at the bottom of each story on boxing, "Don't Bet on Fights," we should publicize the slogan, "Don't Bet on Anything."

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 501. *Proposed by Daniel Arany, Budapest, Hungary.*

If A, B, C, I, J, X are six points on a conic, while L, M, N are points on the respective sides BC, CA, AB of the triangle ABC, and if further the three pencils $L(BXIJ)$, $M(CXIJ)$, $N(AXIJ)$ are projectively related, prove that the points L, M, N are collinear.

E 502. *Proposed by V. Thébault, San Sebastian, Spain.*

Find a number and its fourth power which together have nine digits, all different.

E 503. *Proposed by N. A. Court, University of Oklahoma.*

Through a point M lines are drawn meeting the pairs of opposite edges of a given tetrahedron in the pairs of points U, X; V, Y; W, Z. Prove that if M bisects each of the three segments UX, VY, WZ, it coincides with the centroid of the tetrahedron.

E 504. *Proposed by J. F. Kenney, University of Wisconsin at Milwaukee.*

If p and q are positive numbers with $p+q=1$, show that

$$\lim_{n\to\infty} \left(pe^{qt/\sqrt{npq}} + qe^{-pt/\sqrt{npq}}\right)^n = e^{t^2/2}.$$

E 505. *Proposed by H. T. R. Aude, Colgate University.*

How many different proper fractions when written in the ternary scale will be repeaters with not more than three digits in their repetends?

SOLUTIONS

E 463 [1941, 210]. *Proposed by N. A. Court, University of Oklahoma.*

Determine the locus of the trilinear pole of a given line with respect to the triangle along which a variable plane through the line cuts a given trihedral angle.

Solution by Howard Eves, Pittsburgh, Pa.

Let V be the vertex of the given trihedral angle, and ABC the section by a

plane passing through the given line p. Let P be the trilinear pole of p with respect to the triangle ABC, and let the plane VBC cut p in A'. Then the planes VAA', VAB, VAP, VAC form a harmonic set. Therefore P lies on a fixed plane through VA. Similarly, P lies on fixed planes through VB and VC. Hence the locus of P is a straight line through the vertex V.

Also solved by the proposer.

E 465 [1941, 210]. *Proposed by L. S. Johnston, University of Detroit.*

Without explicit use of the integral calculus, find the area enclosed by the curve $b^2y^2=(b+x)^2(a^2-x^2)$, where $b \geqq a > 0$.

Solution by Albert Furman, Marmion Military Academy, Aurora, Ill.

The equation written in the explicit form

$$y = \pm (1 + x/b) \sqrt{a^2 - x^2}$$

shows that the curve is a composite of the circle of radius a centered at the origin and the graph of the odd function $(x/b)\sqrt{a^2-x^2}$. On the right of the y-axis this last function adds to the area of the circle an amount equal to that subtracted from the circle on the left of this axis. Hence the required area is πa^2.

Also solved by E. F. Allen, Howard Eves, A. K. Waltz, and the proposer.

E 466 [1941, 266]. *Proposed by W. C. Rufus, Observatory of the University of Michigan.*

A is travelling in a restricted zone at two-thirds the speed limit, and B passes him, going twice as fast. Five minutes later a "speed cop," C, passes A and overtakes B. He spends two minutes giving out a ticket, then starts back at speed limit and meets A one mile back. Find at least one practicable solution. Discuss other possible solutions, limiting B's distance to five miles.

Solution by C. W. Trigg, Los Angeles City College.

If the speed limit is x m.p.h., it takes the "speed cop" t minutes travelling at y m.p.h. after he passes A to overtake B. If the distance from the point where C passes A to where C overtakes B be expressed in terms of B's travel and of C's travel, then

$$\frac{x}{18} + \frac{xt}{45} = \frac{yt}{60} \cdot$$

Equating A's total travel plus one mile to B's total travel, we obtain

$$\frac{x}{18} + \frac{xt}{90} + \frac{x}{45} + \frac{2}{3} + 1 = \frac{x}{9} + \frac{xt}{45} \cdot$$

Thus $t=3(50-x)/x$.

Speed limits are likely to be set only in multiples of 5, so we may tabulate the solutions as follows:

x (m.p.h.)	t (min.)	B's total travel	y (m.p.h.)
20	$4\frac{1}{2}$	$4\frac{2}{9}$	$41\frac{13}{27}$
25	3	$4\frac{4}{9}$	$61\frac{1}{9}$
30	2	$4\frac{2}{3}$	90

The most practicable solution would be the last of these, with A, B, and C travelling at 20, 40, and 90 m.p.h., respectively.

None of these solutions is essentially correct, for no provision has been made for the time and distance for deceleration after C overtakes B, nor for acceleration of C on the return journey.

Also solved by R. K. Allen, D. H. Browne, Howard Eves, Evelyn Hesseltine, E. P. Starke, and the proposer. Browne remarks that 90 m.p.h. is close to the record speed for a motorcycle; therefore he prefers the solution $x=20$, $y=41\frac{13}{27}$.

E 467 [1941, 266]. *Proposed by V. Thébault, San Sebastián, Spain.*

In a given triangle, show that the radical axes of the circumcircle with the respective circles whose diameters are the three medians, meet the corresponding sides in three collinear points.

Solution by Robin Robinson, Dartmouth College.

In the triangle ABC, let L, M, N be the midpoints of the sides, D, E, F the feet of the altitudes, and H the orthocenter. Since ADL is a right angle, the circle on AL as diameter passes through D. This circle, the circumcircle ABC, and the degenerate conic consisting of their radical axis AP and the line at infinity, all belong to the pencil of conics through A, P, and the circular points at infinity. By Desargues's involution theorem, the pairs of points in which the line BC cuts them belong to an involution. If AP meets BC at D', and A_∞ is the point at infinity on BC, these pairs of points are BC, LD, $A_\infty D'$. Thus

$$BCLA_\infty \barwedge CBDD';$$

and since BC, LA_∞ form a harmonic set, so do CB, DD'. Hence D', being the harmonic conjugate of D with respect to B and C, lies on the side EF of the complete quadrangle $AEFH$. Since the triangles ABC and DEF are in perspective from H, we deduce from Desargues's triangle theorem that D' and the analogous points on the other two sides are collinear.

Also solved by W. B. Clarke, H. A. DoBell, Howard Eves, P. D. Thomas, C. W. Trigg, and the proposer.

Editorial Note. The above proof shows that the line containing the three points is the trilinear polar of the orthocenter, namely, the *orthic axis* of the triangle ABC. Clearly, this is also the radical axis of the circumcircle and the nine-point circle. (For, B, C, E, F being concyclic, we have $D'B\cdot D'C=D'E\cdot D'F$.) Hence the orthic axis is perpendicular to the Euler line. In the special case of an equilateral triangle, it is the line at infinity (and the Euler line is indeterminate).

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS AND SOLUTION

4017. *Proposed by Norman Anning, University of Michigan.*

It is easy to show that $D(D^2+16)$ will annihilate $\cos^4 x+\sin^4 x$ and also $\cos^6 x+\sin^6 x$. Show in general that the same differential operator will annihilate similar expressions with exponents $4k$ and $4k+2$, where k is any positive integer.

4018. *Proposed by N. A. Court, University of Oklahoma.*

If four points taken in the four faces of a tetrahedron are collinear, their trilinear polars for the respective faces cannot be (*a*) coplanar, or (*b*) hyperbolic.

4019. *Proposed by Robin Robinson, Dartmouth College.*

Given a triangle ABC. Prove that the bisectors of the interior and exterior angles at C, the side AB and its perpendicular bisector, and the perpendiculars to AC at A and to BC at B, are all tangent to a parabola. Locate its focus.

4020. *Proposed by V. Thébault, Le Mans, France.*

With three consecutive odd digits form a number of five digits whose square has ten distinct digits.

SOLUTIONS

3957 [1940, 245]. *Proposed by Otto Dunkel, Washington University.*

Given a triangle ABC with angles $A<B<C$, show that there are precisely one, two, three straight line segments which bisect both its perimeter and area according as

$$1-\frac{\sin A}{\sin B} \gtreqless 2\tan^2(A/2)\tan^2(B/2),$$

where we may replace B by C. If $B=C$, there are one, two, three such segments, according as $A \lesseqgtr A_0$, where $\sin(A_0/2)=\sqrt{2}-1$.

Solution by E. P. Starke, Rutgers University.

Let P be any one of the three angles, p the opposite side, and q, r the adjacent sides. Let a segment cut off a length θq on side q and $\theta' r$ on side r measured from P, with θ, $\theta'<1$. If the segment bisects the area $\frac{1}{2}qr\sin P$, we must have $\theta\theta'=\frac{1}{2}$.

If the perimeter is also bisected, we have $\theta q+\theta' r=\frac{1}{2}(p+q+r)$ or $2\theta q+r/\theta=p+q+r$. Putting

$$f(\theta)=2\theta^2 q+r-\theta(p+q+r),$$

we require $f(\theta)=0$, $\frac{1}{2}<\theta<1$. Now $f(\frac{1}{2})=\frac{1}{2}(r-p)$ and $f(1)=q-p$. If $f'(\theta_0)=0$, then $\theta_0=(p+q+r)/4q$ and $f(\theta_0)=\{8qr-(p+q+r)^2\}/8q$. Thus if p is taken as b, $f(\frac{1}{2})$ and $f(1)$ are of opposite signs and there is one segment meeting sides a and c. If p is c, then $f(\frac{1}{2})$, $f(1)$, $f(\theta_0)$ are all negative,* and there is no segment meeting a and b. Corresponding to $p=a$, $f(\frac{1}{2})$ and $f(1)$ are both positive and there are no segments, one segment, or two segments meeting b and c, according as $f(\theta_0)\gtreqless 0$, *i.e.*, according as $(a+b+c)^2\lesseqgtr 8bc$†. This can be transformed into the proposed condition by the following steps:

$$(a+b+c)^2(-b-a)\gtreqless 8bc(-b-a)=-8bc(a+b+c)+8bc^2,$$

$$(a+b+c)^2(b-a)\gtreqless 2b(a+b+c)^2-8bc(a+b+c)+8bc^2$$

$$=2b(a+b+c-2c)^2,$$

$$\frac{b-a}{b}\gtreqless\frac{2(a+b-c)^2}{(a+b+c)^2}$$

$$=2\left\{\frac{(a-b+c)(a+b-c)}{(a+b+c)(b+c-a)}\right\}\left\{\frac{(b+c-a)(b-c+a)}{(a+b+c)(a-b+c)}\right\},$$

$$\frac{\sin B-\sin A}{\sin B}\gtreqless 2\tan^2(A/2)\tan^2(B/2).$$

If however, $B=C$, so that $\sin B=\cos(A/2)$, then $(a+b+c)^2\lesseqgtr 8bc$ gives $(a+2b)^2\lesseqgtr 8b^2$ or $(\sin A+2\sin B)^2\lesseqgtr 8\sin^2 B$ or

$$\sin A+2\cos(A/2)\lesseqgtr 2\sqrt{2}\cos(A/2).$$

When $\sin A$ is replaced by $2\cdot\sin(A/2)\cos(A/2)$, this becomes

$$\sin(A/2)\lesseqgtr\sqrt{2}-1=\sin(A_0/2).$$

whence $A\lesseqgtr A_0$.

Editorial Note. The last part, for which $B=C$, requires an examination of the values of θ excluded in the first part. The case $A=B=C$ is easily disposed of, since then each of the three angles is greater than A_0 and there are precisely three segments.

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* $p>q$ implies $(p+q+r)^2>(2q+r)^2=(2q-r)^2+8qr\geqq 8qr$.

† Since this relation involves b and c symmetrically, these letters may be interchanged in what follows.

A theorem useful for the construction of the required segments may be stated as follows. If a transversal of triangle PQR satisfies two of the following requirements, it satisfies the third; namely, it passes through the incenter I, it bisects the perimeter, it bisects the area. If we consider only the requirement that the transversal bisects the perimeter $2s$, then this may be stated in the following manner. On PR lay off the segment PS_q in the direction PR and with the length s; similarly, lay off on PQ the segment PS_r. Then PI passes through N, the midpoint of S_rS_q, and the perpendicular to PN at its midpoint V bisects PS_q, PS_r in V_q, V_r. Now take Q' on the straight line of PR, and R' on PQ, so that $PR'=Q'S_q$ or what is the same, $PR'+PQ'=s$. The two ranges of points R', Q' are projective, and hence $R'Q'$ is tangent to a conic which is tangent to the line at infinity and also to PQ, PR at S_r, S_q. The conic is therefore a parabola tangent to V_rV_q at V the vertex, and with the axis along VN. The perpendicular to PR at V_q cuts VN in F, the focus. There is no tangent through I if I lies outside PV, and just one if it is at V. If I lies within PV, the circle on IF as diameter cuts V_rV_q in two points each of which joined to I by a straight line gives a tangent to the parabola. This construction gives the desired segments $R'Q'$ of the problem, if there are any such for the two considered sides PQ, PR.

We may also consider the single condition that $R'Q'$ bisects the area S of PQR. Then $PQ'\cdot PR'=qr/2=k^2$; the points R', Q' describe projective ranges on PQ, PR, and the line $R'Q'$ is tangent to a conic. Since the conic is tangent to PQ, PR at infinity, it must be a hyperbola with its transverse axis along PI. On PR lay off the segment PV_q' in the direction PR and with the length k, and determine V_r' on PQ similarly. Then the hyperbola is tangent to $V_r'V_q'$ at its midpoint V', and the focus F on the extension of PV' is easily constructed. A construction for the tangents through I for this branch, if there are any such, may be easily obtained. There are other geometric results relating to I and the two conics.

The desired conditions of the problem may be found by considering the rotation of a transversal $R'Q'$ through I with R' on PQ and limited to that segment, while Q' is on and limited to PR. When Q' is at R, let the end R' be at R_i; when R' is at Q, let Q' be at Q_i. Let the transversal through I perpendicular to PI cut PQ, PR in I_r, I_q; then the order of points on PR is P, Q_i, I_q, R, and similarly, for PQ the order is P, R_i, I_r, Q. Let the parallel to PQ through I_q cut R_iR, $R'Q'$, QQ_i in R_i', R'', Q_i'; then R_i' is inside PQR and Q_i' is outside. Denoting the area of a figure by its vertices, we find that $PR'Q'=PR_iR-R''R_i'RQ'$ when Q' is on segment RI_q; and if Q' is on I_qQ_i, then $PR'Q'=PI_rI_q+R''Q'I_q$. Hence, as Q' moves from R to I_q, the area $PR'Q'$ steadily decreases from PR_iR to PI_rI_q, and then steadily increases from PI_rI_q to PQQ_i, as Q' moves from I_q to Q_i. The ratio $PR_iR/S\lesseqqgtr 1/2$, according as $q\lesseqqgtr p$, since RR_i is an internal bisector of angle R. We now consider only the cases where Q' is a desired point within the segment RQ_i; that is, we consider non-vertex positions for the sides of lengths q, r. We have at once three results:

$$(1) \qquad (q, r): \quad \begin{array}{l} \text{case 1, } q \leqq p \text{ and } r \leqq p, \ (0); \\ \text{case 2, } q \leqq p \text{ and } r > p, \ (1); \\ \text{case 3, } q > p \text{ and } r \leqq p, \ (1); \end{array}$$

where the number in the last parentheses denotes the number of non-vertex positions. There remains the case 4, where $q>p$ and $r>p$,and we must now consider the area of PI_rI_q, or of its complement I_qI_rQR, and we select the latter. We have

$$(2) \qquad \begin{aligned} \frac{I_qI_rQR}{S} &= \frac{RR_iQ}{S} + \frac{I_qR_i'R}{S} = \frac{p}{p+q} + \frac{q}{p+q}\left(\frac{I_qR}{q}\right)^2 \\ &= \frac{p}{p+q} + \frac{q}{p+q}\tan^2\frac{Q}{2}\tan^2\frac{P}{2}, \end{aligned}$$

where we have used the relations

$$(3) \qquad q = \rho\frac{\cos\dfrac{Q}{2}}{\sin\dfrac{P}{2}\sin\dfrac{R}{2}}, \qquad I_qR = \frac{\sin\dfrac{Q}{2}}{\sin\dfrac{R}{2}}\left\{\frac{\rho}{\cos\dfrac{P}{2}}\right\}, \qquad \rho = \text{inradius}.$$

Hence for case 4 there are 0, 1, 2 non-vertex positions according as the right side of (2) is less than, equal to, or greater than 1/2, or what is the same

$$(4) \qquad 1 - \frac{\sin P}{\sin Q} \gtreqless 2\tan^2\frac{P}{2}\tan^2\frac{Q}{2}.$$

This leads easily to the results of the problem when no two angles of the triangle are equal. For the case $Q=R$, the consideration of case 4 is much simpler. Let PI cut QR in M; then

$$(5) \qquad \frac{PI_rI_q}{S} = \left(\frac{PI}{PM}\right)^2 = \left\{\frac{1}{1+\dfrac{\rho}{PI}}\right\}^2 = \left\{\frac{1}{1+\sin\dfrac{P}{2}}\right\}^2,$$

and the ratio on the left is greater than, equal to, or less than 1/2 according as $\sin(P/2) \lesseqgtr \sqrt{2}-1 = \sin(A_0/2)$. Considering the three pairs of sides of ABC, we obtain the results given in the problem.

We easily find directly from the figure that $PI_rI_q/S = rq/s^2$, and it will be readily found that this leads in case 4 to the inequalities used by Starke. It will be observed that this ratio may be also written $2(PV_q')^2/(PS_q)^2$, thus connecting it with the two conics in the first part of this note.

A solution by Kwan Chao-chih, Yenching University, was received after the preparation of the above for printing. In this solution the general triangle

$A_1A_2A_3$ was considered first with a required segment cutting the two sides through A_i in the points P_i and Q_i, $A_iP_i=x_i$, $A_iQ_i=y_i$. Then we must have $x_i+y_i=s$ and $2x_iy_i=a_ja_k$. Conditions are then easily set down for two, one, no distinct segments for this pair of sides. Applying this first to the case where no two sides are equal, and then to the case of two equal sides, results were obtained as in the problem. As in Starke's solution, the treatment was mainly an algebraic study of the corresponding quadratic expression.

3958 [1940, 322]. *Proposed by Michael Goldberg, Washington, D. C.*

What is the probability P_n of making a sequence of at least r throws of a number in n throws of a die when the probability of throwing that number in a single trial is p?

Solution by C. Eisenhart, University of Wisconsin.

The probability P_n is given by the coefficient of y^n in the power series expansion of

$$p^r y^r \cdot \frac{(1-py)}{(1-y)^2+(1-p)p^r y^{r+1}(1-y)}. \tag{1}$$

The contrary probability $1-P_n$ is the coefficient of y^n in the expansion of

$$\frac{1-p^r y^r}{1-y+(1-p)p^r y^{r+1}}. \tag{2}$$

This is Proposition XLVI in the 3rd edition (1878) of William Allen Whitworth's *Choice and Chance*—it is Proposition LIII of the 5th edition which is available in reprint form from G. E. Stechert & Co., New York City—expressed in the notation of the present problem. Whitworth's proof is quite direct, and, being readily available, will not be reproduced here.

Since we desire P_n, we must subtract (2) from the function $(1-y)^{-1}$ in the expansion of which the coefficient of y^n is always unity. This gives P_n as the coefficient of y^n in the expansion of

$$\frac{1}{1-y}-\frac{1-p^r y^r}{1-y+(1-p)p^r y^{r+1}}, \tag{3}$$

which, being expressed with a common denominator, gives (1).

That P_n calculated as specified from (1) or (3) satisfies the "boundary conditions" of the problem can be seen as follows:

(*a*) When $r=0$, (3) reduces to $(1-y)^{-1}$, whence $P_n=1$ for all n as it should.

(*b*) When $r=1$, (3) becomes

$$(1-y)^{-1}-(1-[1-p]y)^{-1}=\sum_{n=0}^{\infty}(1-[1-p]^n)y^n,$$

giving $P_n=1-(1-p)^n$ as it should.

(*c*) When $r=n$, (1) may be written

$$p^n y^n\left(1+\sum_{t=1}^{\infty} a_t y^t\right), \tag{4}$$

since the fractional part of (1) has the value 1 when $y=0$. From this it is clear that $P_n=p^n$ in this case.

(*d*) When $r>n$, $P_n=0$ as is clear from (4) with n replaced by $r>n$.

For other values of r it is not so easy to obtain an expression for P_n valid for all n, but P_n can be obtained from (1), or arrangements of it, by various artifices and perseverance with division or differentiation. For example, it is not difficult to see that $P_n=p^{n-1}\ (2-p)$ for $r=n-1$.

Later addition. This problem was originally proposed and solved by De Moivre. A complete solution with discussion of approximations for large n is given on pp. 77–84 of Uspensky's *Introduction to Mathematical Probability.*

Solved also by R. E. Moritz.

Editorial Note. R. E. Moritz gave a development of a generating function for $Q_n=1-P_n$ in the form

$$\frac{p^{n+1}(1-x^r)}{p-x+(1-p)x^{r+1}},$$

and then, by expansion of this function in a power series in x, found Q_n as the coefficient of x^n. This derivation is long and complicated.

It is possible to obtain the desired probability without the use of a generating function, and to simplify the work we define as follows a related probability $p_2(n)$ upon which the desired one depends:

In a sequence of n trials of the same experiment, the success S of each has the same probability μ. The probability of beginning the sequence with at least r successes with no subsequent failure F followed by as many as r successes is denoted by $p_2(n)$.

We derive first the difference equation for $p_2(n)$. If for each favorable case for $p_2(n)$ we follow it by an F, we get all the favorable cases for $p_2(n+1)$ ending in an F. If for each favorable case in $p_2(n-j)$, $1\leqq j\leqq r-1$, we follow it by an F and j of the S's, we get all the favorable cases in $p_2(n+1)$ ending in j of the S's. The only favorable case in $p_2(n+1)$ ending in r of the S's is the one for which each of the $n+1$ results is an S. Hence we have

$$\begin{aligned} p_2(n+1) = (1-\mu)\left[p_2(n)+\mu p_2(n-1)+\mu^2 p_2(n-2)+\cdots\right. \\ \left.+\mu^{r-1}p_2(n-r+1)\right]+\mu^{n+1}. \end{aligned} \tag{1}$$

We have obviously $p_2(n)=0$, $1\leqq n\leqq r-1$; $p_2(r)=\mu^r$; and we then find easily from (1) that $p_2(0)=0$. Hence, we may take as initial conditions for (1) that $p_2(n)=0$, $0\leqq n\leqq r-1$. If we write the equation (1) replacing n by $n-1$, multiply each term of it by μ, and subtract from (1), we get a simpler equation of order one unit

higher. We write this equation in the form

$$(2)\qquad p_2(n+1) - p_2(n) = \Delta p_2(n) = -\lambda p_2(n-r), \qquad \lambda = (1-\mu)\mu^r,$$

with the initial conditions $p_2(n)=0$, $0\leqq n\leqq r-1$; $p_2(r)=\mu^r$.

Equation (2) may be derived directly. If we follow each favorable case in $p_2(n)$ by either an F or an S, we obtain all favorable cases in $p_2(n+1)$ and some unfavorable cases. The latter are those of $p_2(n)$ which have at the end an F followed by precisely $r-1$ of the S's, so that the additional S makes it unfavorable. The probability of this latter is $(1-\mu)\mu^r\, p_2(n-r)$, and then (2) follows immediately.

The formula for $p_2(n)$ will be found by successive steps and in each step we use the known formula $\Delta x^{(k)}=kx^{(k-1)}$, where x is any number and $x^{(k)}=x(x-1)\cdots(x-k+1)$, $x^{(0)}=1$. First we have $\Delta p_2(n)=0$, $r\leqq n\leqq 2r-1$; hence $p_2(n)=\mu^r$ for $r\leqq n\leqq 2r$. Next we have $\Delta p_2(n)=-\lambda\mu^r$, $2r\leqq n\leqq 3r$; and hence $p_2(n)=\mu^r[1-\lambda(n-2r)]$ for $2r\leqq n\leqq 3r+1$, where the constant of summation has been determined. For the third step we have $\Delta p_2(n)=\mu^r[-\lambda+\lambda^2(n-3r)]$, $3r\leqq n\leqq 4r+1$. Then

$$p_2(n) = \mu^r[1-\lambda(n-2r)+\lambda^2(n-3r)^{(2)}/2], \qquad 3r\leqq n\leqq 4r+2,$$

where we have naturally preserved the terms already found, and the constant of summation is properly determined as we now show. If we set in this last expression $n=3r$, $3r+1$, we get results already determined in the second step, and thus no other constant is to be added. This suffices to indicate that

$$(3)\qquad p_2(n) = \mu^r\sum_{j=1}^{i}(-1)^{j-1}\frac{(n-jr)^{(j-1)}}{(j-1)!}\lambda^{j-1}, \qquad ir\leqq n\leqq (i+1)r+i-1, \quad (i=1,2,\cdots).$$

We now prove that this is correct by assuming that (3) has been proved for a given $i\geqq 1$ and then showing that it is also true for $i+1$. By hypothesis,

$$\Delta p_2(n) = \mu^r\sum_{j=1}^{i}(-1)^j\frac{[n-(j+1)r]^{(j-1)}}{(j-1)!}\lambda^j, \quad (i+1)r\leqq n\leqq (i+2)r+i-1.$$

Then by the method of the above steps we have, after slight changes in the limits of the summation,

$$p_2(n) = \mu^r\sum_{j=1}^{i+1}(-1)^{j-1}\frac{(n-jr)^{(j-1)}}{(j-1)!}\lambda^{j-1}, \qquad (i+1)r\leqq n\leqq (i+2)r+i.$$

No further constant is to be added, since for $(i+1)r\leqq n\leqq(i+1)r+i-1$, the term for $j=i+1$ in this last expression vanishes and the expression is then the one of the hypothesis. Also, this last interval for n is common to the hypothesis interval and the derived interval. The formula (3) is now completely proved. We may avoid the overlapping of consecutive intervals by writing (3) in a form more convenient for computation, namely,

$$(4) \qquad p_2(n) = \mu^r \sum_{j=1}^{i} (-1)^{j-1}{}_{n-jr}C_{j-1}\lambda^{j-1}, \qquad i = \left[\frac{n+1}{r+1}\right],$$

where ${}_mC_t$ denotes a binomial coefficient.

We now define $p_1(n)$ as the probability that in n trials there is at least one F followed by at least r of the S's. Then $p_3(n) = p_1(n) + p_2(n)$ is the probability desired in the present problem. Also, $q_1(n) = 1 - p_1(n)$ is the probability that in n trials no F is followed by as many as r of the S's. If we consider each favorable case for $q_1(n)$ following r of the S's, we get the favorable cases for $p_2(n+r)$. Hence $p_2(n+r) = \mu^r q_1(n)$, and then we have

$$(5) \qquad p_3(n) = 1 + p_2(n) - \mu^{-r}p_2(n+r);$$

this completes the solution of the present problem, where $\mu = p$ and $p_3(n) = P_n$.

We now give an evaluation of $p_2(n)$ which requires neither the generating function nor the difference equation. Consider a straight line divided into n spaces for receiving a letter in each space. In each of the first r spaces we place an S, and then j blocks each consisting of an F followed by r of the letters S, the blocks occupying $j(r+1)$ spaces. The number of ways the j blocks may be placed on the $n-r$ spaces is ${}_{n-(j+1)r}C_j$, where $j \leqq i-1$, $i = [(n+1)/(r+1)]$. The remaining $n-r-j(r+1)$ free spaces are then filled in any manner by the letters S, F. This gives a representation of n experiments with at least r of the S's at the start and with at least j of the F's each followed by at least r of the S's. The probability of n experiments resulting this way is

$$(6) \qquad \nu_j = \mu^r \, {}_{n-(j+1)r}C_j\lambda^j.$$

We shall prove that

$$(7) \qquad \sum_{j=0}^{i-1} (-1)^j\nu_j = p_2(n) = \mu^r \sum_{j=1}^{i} (-1)^{j-1} \, {}_{n-jr}C_{j-1}\lambda^{j-1}.$$

Consider the probability of a definite arrangement of this kind, where there are precisely u blocks in given positions. Then this probability is counted ${}_uC_j$ times in ν_j, $j \leqq u$; and the number of times it is counted in the sum (7) is

$$\sum_{j=0}^{u} (-1)^j \, {}_uC_j = 0.$$

Hence in (7) the probability of an F followed by at least r of the S's is eliminated, and the proof is complete.

The generating function is desirable in the computation of an approximating expression for $p_3(n)$ when n is large. In this case it is more convenient to obtain such a function for $p_2(n)$. The auxiliary algebraic equation for (2) is $x^{r+1} - x^r + \lambda = 0$, and we set

$$\frac{\phi(y)}{1 - y + \lambda y^{r+1}} = \sum_{n=0}^{\infty} p_2(n) y^n,$$

and then find easily that $\phi(y) = \mu^r y^r$.

The special case of $\mu = 1/2$ is considered in the solution of 3046 [1924, 403], where the difference equation for $p_3(n)$ is solved in a manner similar to the above. For $n=50$, $r=5$, it is there stated that $p_3(50) = .55188$. In an article entitled *Two types of probabilities and their difference equations*, by Otto Dunkel (Washington University Studies, Scientific Series, No. 2, 1925, pp. 119–136), there is a small table $\mu = 1/2$, $r=5$, for $p_1(n)$, $p_2(n)$, $p_3(n)$ with $n = 5, 10, 15, \cdots, 50$. In this MONTHLY in Dunkel's article on *Solutions of a probability difference equation* (vol. 32, 1925, pp. 354–370), there is given the development of the generating function for $P_2(n) = 2^n p_2(n)$, the computation of approximate formulas for large values of n, and a method for computing the root of the associated algebraic equation which is used in the approximate formula. In the solution of 3363 [1930, 319] the case of μ equal to the reciprocal of an integer greater than unity is considered. A discussion of the absolute values of the roots of the auxiliary algebraic equation is given in the solution of 3162 [1929, 105].

3965 [1940, 491]. *Proposed by H. S. Wall, Northwestern University.*

Show that for a real or complex x, $|x| \leqq 1$,

$$\frac{|x|}{1+|x|} \leqq |\log (1+x)| \leqq \frac{|x|(1+|x|)}{|1+x|}.$$

I. *Solution by S. E. Warschawski, Washington University.*

To prove the right-hand inequality we make use of the Taylor series for $\log(1+z)$ which represents the principal value of the logarithm for $|z| < 1$. We write

$$\begin{aligned}(1+z)\log(1+z) &= (1+z)\left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots\right)\\ &= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots\\ &\quad + z^2 - \frac{z^3}{2} + \frac{z^4}{3} - \cdots\\ &= z + \frac{z^2}{2} - \frac{z^3}{2\cdot 3} + \frac{z^4}{3\cdot 4} - \cdots\\ &= z\left\{1 + z\left[\frac{1}{2} - \frac{z}{2\cdot 3} + \frac{z^2}{3\cdot 4} - \cdots\right]\right\}.\end{aligned}$$

Hence, since $|z|<1$, we have

$$\begin{aligned}|(1+z)\log(1+z)| &\leqq |z|\left\{1+|z|\left[\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\right]\right\}\\ &= |z|\left\{1+|z|\left[\tfrac{1}{2}+(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{4})+\cdots\right]\right\}\\ &= |z|\left\{1+|z|\right\},\end{aligned}$$

or

$$|\log(1+z)| \leqq \frac{|z|(1+|z|)}{|1+z|}.$$

We now consider the left-hand inequality. The function

$$w=\log(1+z)=\int_0^z \frac{dt}{1+t}$$

(which is the principal value of the logarithm) is regular in the region D obtained from the z-plane by cutting it along the negative real axis from -1 to $-\infty$. The map of D in the w-plane is the strip $-\pi<I(w)<\pi$. Take a fixed point z_0 in D; let its map in the w-plane be $w_0=\log(1+z_0)$. Connect w_0 with the origin in the w-plane by the straight line segment $\overline{Ow_0}$. The map of this line in the z-plane by means of the inverse function $z=e^w-1$ is a certain curve C which connects $z=0$ with $z=z_0$. The length of the line $\overline{Ow_0}$ may now be expressed as

$$|w_0|=\int_C \frac{|dt|}{|1+t|}.$$

Let C' be the arc obtained when C is traversed from $z=0$ to its first point of intersection with the circle $|z|=|z_0|$. Then certainly

$$|w_0| \geqq \int_{C'} \frac{|dt|}{|1+t|}.$$

Since $|1+t| \leqq 1+|t| \leqq 1+|z_0|$ on C', we have

$$|w_0| \geqq \int_{C'} \frac{|dt|}{1+|z_0|} = \frac{1}{1+|z_0|}\int_{C'} |dt| \geqq \frac{|z_0|}{1+|z_0|}.$$

The left-hand inequality has now been proved for the whole z-plane.

II. *Solution by A. M. Gleason, New Haven, Conn.*

(1) The upper inequality. Since

$$\frac{\log(1+x)}{x}=\int_0^1 \frac{dt}{1+tx},$$

$$\left|\frac{\log(1+x)}{x}\right| \leqq \int_0^1 \frac{dt}{|1+tx|} \leqq \max_{0\leqq t\leqq 1} \frac{1}{|1+tx|} \equiv M.$$

(*i*) If x is in the right half plane, then $M=1\leqq(1+|x|)/|1+x|$.

(*ii*) If $|x+\frac{1}{2}|\leqq\frac{1}{2}$, then $M=1/|1+x|\leqq(1+|x|)/|1+x|$.

(*iii*) If x falls within neither of these regions, then if $R(x)\geqq -1$, the triangle OAP, $\{O\ (0,0), A\ (-1,0), P(x)\}$ has all acute angles, and hence the sum of the sines of the angles is greater than 2. Using the law of sines, we have

$$M = \frac{1}{\sin O} = \frac{OA+OP}{AP}\,\frac{1}{\sin P + \sin A} \leqq \frac{OA+OP}{AP} = \frac{1+|x|}{|1+x|}.$$

This establishes the upper inequality throughout the half-plane $R(x)\geqq -1$.

(2) The lower inequality. This is apparent graphically along the real axis from -1 to $+\infty$. Along the unit circle, we have

$$|\log(1+e^{i\theta})| = |\log 2\cos\tfrac{1}{2}\theta + \tfrac{1}{2}\theta i| \geqq \tfrac{1}{2},$$

since if $\theta \geqq 1$ and if $\theta < 1 < 60°$, then $\log 2\cos\frac{1}{2}\theta > \log\sqrt{3} > \frac{1}{2}$.

Consider a path along which $|1+x|$ is constant, extending from the real axis to the unit circle. The functions $|\log(1+x)|$ and $|x|/(1+|x|)$ are both continuous along this path, and differentiable within the interior. Using $\phi=\arg(1+x)$ as independent variable, we have

$$|x|^2 = |1+x|^2 + 1 - 2|1+x|\cos\phi, \qquad 2|x|\frac{d|x|}{d\phi} = 2|1+x|\sin\phi.$$

Then

$$\begin{aligned}\frac{d}{d\phi}\left\{|\log(1+x)|^2 - \left(\frac{|x|}{1+|x|}\right)^2\right\} &= \frac{d}{d\phi}\left\{\log^2|1+x| + \phi^2 - \frac{|x|^2}{(1+|x|)^2}\right\}\\ &= 2\phi - 2\frac{|x|}{1+|x|}\,\frac{1}{1+|x|)^2}\,\frac{d|x|}{d\phi}\\ &= 2\phi - 2\frac{|1+x|}{1+|x|}\,\frac{1}{(1+|x|)^2}\sin\phi,\end{aligned}$$

which cannot vanish for $\phi\neq 0$. Applying Rolle's theorem, and remembering that the inequality has been shown at either end-point, we know that if it failed anywhere in the middle the derivative must vanish somewhere in the interior. This establishes the lower inequality.

Solved also by E. S. Pondiczery.

Editorial Note. Pondiczery's solution considered $|x|<1$, and then extended the results to $|x|=1$, excluding $x=-1$. His proof of the right-hand side of the

inequality is similar to the one in solution I. For the left-hand side, after setting $x=|x|e^{i\theta}$, we have

$$\log(1+x)=\int_0^{|x|}\frac{e^{i\theta}dr}{1+re^{i\theta}},$$

$$|\log(1+x)|=\left|\int_0^{|x|}\frac{dr}{1+re^{i\theta}}\right|\geqq\left|\Re\left\{\int_0^{|x|}\frac{dr}{1+re^{i\theta}}\right\}\right|$$

$$\geqq\int_0^{|x|}\frac{dr}{1+|x|}=\frac{|x|}{1+|x|}.$$

The last parts of the reduction follow from the inequalities

$$\Re\left\{\frac{1}{1+re^{i\theta}}\right\}=\frac{1+r\cos\theta}{1+r^2+2r\cos\theta}\geqq\frac{1}{1+r}\geqq\frac{1}{1+|x|}.$$

The trigonometric theorem used in solution II may be obtained from the identity $\sin^2A+\sin^2B+\sin^2C=2(1+\cos A\cos B\cos C)$, where $A+B+C=\pi$. For an actual acute angled triangle the sum of squares of the sines is greater than 2, and this must also be true for the sum of the sines.

In the last part of this solution it follows from the final steps that $dy/d\phi$, where y is the expression in brackets, is zero if $\phi=0$ and positive if $\phi>0$ when $|1+x|$ is constant. When $\phi=0$, $y\geqq 0$, and hence for $\phi>0$ we must have $y>0$. This proves the left-hand side for the whole upper half-plane without use of the unit circle. Then for the lower half-plane it must also be true by considerations of symmetry with respect to the axis of reals.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Decatur, May 8–9, 1942
INDIANA, Crawfordsville, May 1–2, 1942
IOWA, Mt. Pleasant, April 17–18, 1942
KANSAS, Hays, March 27–28, 1942
KENTUCKY, Lexington, April 11, 1942
LOUISIANA-MISSISSIPPI, Jackson, Miss., March 6–7, 1942
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Ashland, Va., May 1942
METROPOLITAN NEW YORK, New York, April 18, 1942
MICHIGAN
MINNESOTA
MISSOURI, Kansas City, April 17, 1942
NEBRASKA, Omaha, May 9, 1942
NORTHERN CALIFORNIA, Berkeley, Jan. 31, 1942
OHIO, Columbus, April 2, 1942
OKLAHOMA, Oklahoma City, Feb. 13, 1942
PHILADELPHIA, Philadelphia, Nov., 28, 1942
ROCKY MOUNTAIN, Golden, Colo., April 17–18, 1942
SOUTHEASTERN, Emory, University, Ga., March 26–27, 1942
SOUTHERN CALIFORNIA, Los Angeles, March 14, 1942
SOUTHWESTERN, State College, N. M., April 27–28, 1942
TEXAS, Lubbock, April 3–4, 1942
UPPER NEW YORK STATE, Rochester, May 2, 1942
WISCONSIN, Oshkosh, May 2, 1942

Twenty-fifth Summer Meeting, Ithaca, New York, September 7–9, 1942.

SELECTIVE SERVICE FOR MATHEMATICIANS

The object of this open letter is to give to interested mathematicians a report concerning developments in the problem of the use of mathematicians of the draft age.

At a July meeting of the Roster of Scientific and Specialized Personnel in Washington these matters were discussed at length and a number of those present, including the writer, urged that the Roster is the body best equipped to take up the problem of the proper use of scientists. It should be understood at the beginning of this letter that the Roster has no power to defer scientists, but is concerned only with making recommendations as to whether or not a man be regarded as "necessary" in the sense of the law.

In order to make proper recommendations it was necessary that the Roster obtain considerable additional information concerning men of draft age on its lists. The matter of procedure has now been systematized. Any man who is on the Roster and who finds himself likely to be called may write the Roster for a standard questionnaire. Among other things this questionnaire seeks to find out the man's present occupation and scientific status and the names of references. These references and the man's employer are then sent questionnaires calculated to assist in determining whether the man may be regarded as "necessary."

The Roster then calls in men in the various scientific fields to review the assembled data and advise the Roster on each individual case. The mathematicians were the first so treated, and Murnaghan, Hotelling, and Morse recently served as consultants for the Roster. The Roster had assembled excellent data bearing on 300 cases, all of which was carefully reviewed by each of the three mathematicians. In general we tried to follow the resolutions which were voted at the recent Chicago meeting of the mathematicians and are to be published in the *Bulletin of the American Mathematical Society*.

The Roster makes its recommendations to the Selective Service Headquarters, which in turn *may* pass on the recommendations to the local boards. The local boards are free to follow the recommendations or not.

In case a man is inducted, the Roster acts in an advisory capacity to the personnel department of the Adjutant General's office, offering information which will aid the personnel officers in properly using the scientists on the Roster.

In addition to making recommendations as to individual men, the Roster also seeks information concerning shortages in various scientific fields and passes on this information to the Selective Service Headquarters and the Labor Office. Our Committee on the Supply and Demand for Mathematicians will materially aid the Roster in this respect, and will help to clarify the situation.

In conclusion I wish to recommend that problems of the use of scientists on the Roster be taken up with the Roster whenever they arise. With the advice of consultants in the various fields, the Roster is the group best organized and equipped to handle these questions.

December 26, 1941

MARSTON MORSE, *Chairman*
War Preparedness Committee

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THE CALCULUS OF VARIATIONS FOR MULTIPLE INTEGRALS*

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Introduction. On page 158 of his article on the calculus of variations in the Encyclopédie des Sciences Mathématiques Lecat [1916]† makes a statement concerning the theory for multiple integrals which may be translated as follows:

"The theory of the calculus of variations for multiple integrals is much less advanced than for simple integrals. Some of the theory for simple integrals is easily carried over, it is true, but for the most part difficulties increase, largely as a result of the fact that ordinary differential equations must be replaced by partial differential equations. One cannot develop a satisfactory theory of the problems of Lagrange or Mayer for multiple integrals until one has established numerous properties of partial differential equations which are now lacking. Multiple integrals have been made the objects of very few rigorous researches."

These statements are still to a large extent applicable in 1941. Though progress has been made in the twenty-five years which have elapsed since 1916, the theory of the calculus of variations for multiple integrals is still far from complete relative to that for simple integrals. The purpose of this paper is to describe some of the progress which has been made, and to call attention to some of the hiati in the theory which still remain. The field is a large one and the author, in confining himself to those portions of the theory which have interested him especially, must omit much of importance which has been of great interest to others.

1. **Formulation of the problem.** The integral to be studied is

$$I = \int_{\mathfrak{X}} f(x, y, y')d\mathfrak{X} \tag{1}$$

with the understanding that

$$x = (x_1, \cdots, x_m),\ y = (y_1, \cdots, y_n),\ y' = \|\partial y_i/\partial x_\alpha\| = \|y_{i\alpha}\|,$$
$$(\alpha = 1, \cdots, m;\ i = 1, \cdots, n),$$
$$d\mathfrak{X} = dx_1 \cdots dx_m.$$

The manifolds on which the values of the integral are to be taken are defined

* Presented to a joint session of the Mathematical Association of America and the American Mathematical Society at Chicago, Illinois, September 3, 1941.

† The references in the text are to the chronological bibliography at the end of this paper. The references there are taken from a "Bibliography for the theory of multiple integrals in the calculus of variations" which is scheduled to appear about the time this paper will be printed. This Bibliography is the last section of a book entitled "Contributions to the calculus of variations 1938–1941" edited by members of the Department of Mathematics of the University of Chicago. It contains about 350 titles and is divided into three parts in which the papers are listed chronologically, alphabetically, and by subject matter, respectively.

by functions of the form

$$y(x), \qquad (x \text{ on } \mathfrak{X}), \tag{2}$$

where $\mathfrak{X}$ is the range in x-space on which the set of functions $y(x)$ is defined. The boundary of $\mathfrak{X}$ will be denoted by $\mathfrak{B}$. The problem is then to find in a class of m-dimensional spaces (2) defined over the same range $\mathfrak{X}$ and having identical values on the boundary $\mathfrak{B}$ of $\mathfrak{X}$, one which minimizes the integral I.

The first variation of the integral I on a space (2) is

$$I_1 = \int_{\mathfrak{X}} (f_i \eta_i + f_{i\alpha}\eta_{i\alpha})\, d\mathfrak{X}, \tag{3}$$

with the notations

$$f_i = \partial f/\partial y_i, \quad f_{i\alpha} = \partial f/\partial y_{i\alpha}, \quad \eta_{i\alpha} = \partial \eta_i/\partial x_\alpha.$$

It is found as usual by replacing $y(x)$ by $y(x)+a\eta(x)$ in I, differentiating with respect to a and setting $a=0$. In formula (3) the repeated indices i and α indicate, as customary in tensor analysis, that sums for $i=1, \cdots, n$ and $\alpha=1, \cdots, m$ are to be taken. The symbol $\eta(x)$ stands for a set of "admissible" functions $\eta_i(x)$ defined on the range $\mathfrak{X}$ and vanishing on the boundary $\mathfrak{B}$ of $\mathfrak{X}$. Similarly the second variation, found by differentiating twice with respect to a and setting $a=0$, has the form

$$I_2 = \int_{\mathfrak{X}} 2\omega(x, \eta, \eta')d\mathfrak{X} \tag{4}$$

in which $\eta' = \|\eta_{i\alpha}\|$ and 2ω is the homogeneous quadratic form in the elements of η, η' whose coefficients are the usual second derivatives of the function f with respect to the elements of y and y'. It is evident that on a minimizing space (2) the conditions $I_1=0$, $I_2 \geqq 0$ must be satisfied for all choices of admissible sets η which vanish on the boundary $\mathfrak{B}$ of $\mathfrak{X}$.

It should be noticed that the very general problem formulated in the preceding paragraphs includes, when $(m, n)=(1, n)$, the classical problem of the calculus of variations with fixed end points for simple integrals, and when $(m, n)=(m, 1)$ the simplest problem with fixed boundaries for multiple integrals.

It is understood in the following pages that without further specification the integrand function f is supposed to have continuous partial derivatives in a region $\mathfrak{R}$ of values (x, y, y') of as many orders as may be needed at any time to carry through the part of the theory which is being studied. A set (x, y, y') interior to $\mathfrak{R}$ is called "admissible." All of the sets (x, y, y') beonging to the spaces (2) under consideration are supposed to be admissible.

2. Conditions for a minimum associated with the first variation. The Lagrange equations for the integral I are the equations

$$df_{i\alpha}/dx_\alpha = f_i, \qquad (i = 1, \cdots, n). \tag{5}$$

A space (2) having continuous second derivatives and satisfying these equations is called an extremal. It has long been known that each piece of a minimizing space (2) on which the functions $y(x)$ have continuous second derivatives must be an extremal. The proof for the problem of Plateau, for which $(m, n)=(2, 1)$ and $f=[1+y_{11}^2+y_{12}^2]^{1/2}$, was made by Lagrange [1762], and for the general case $(m, n)=(2, 1)$ also by Lagrange [1806]. The proofs for more general values m, n are quite similar and have been made by many writers. The second derivatives of the space (2) appear both in the transformations of the first variation used in the proof, and in the resulting equations (5) themselves.

For a piece of a minimizing space only assumed to have continuous first derivatives Haar [1919, 1930] has deduced independently some interesting equations. His results are also a consequence of a paper by Mason [1905]. The simplest form of these results for application to the general integral I was suggested by Coral [1937]. Let $\mathfrak{X}_0$ be a sub-region of the region $\mathfrak{X}$ with boundary $\mathfrak{B}_0$. Let the direction cosines of the outer normal to $\mathfrak{B}_0$ be designated by l_α, and let $d\mathfrak{B}$ be the generalized element of area on $\mathfrak{B}_0$. Then on every region $\mathfrak{X}_0$ on which a minimizing space (2) has continuous first derivatives the equations

$$\int_{\mathfrak{X}_0} f_i d\mathfrak{X} = \int_{\mathfrak{B}_0} f_{i\alpha} l_\alpha d\mathfrak{B}, \qquad (i = 1, \cdots, n), \tag{6}$$

must hold. The equations of Lagrange are consequences of these when the minimizing space has continuous second derivatives.

Haar deduced his results especially for the cases when $m=2$ and $m=3$. For the general integral I the Haar-Coral equations (6), and a number of other interesting results, are consequences of methods introduced by Hestenes [1941] and pursued by Carson [1941]. The latter considers the equation

$$\int_{\mathfrak{X}} (w\eta + v_\alpha \partial\eta/\partial x_\alpha) d\mathfrak{X} = \int_{\mathfrak{B}} v_\alpha l_\alpha \eta d\mathfrak{B} \tag{7}$$

in which w, v_α are given functions of the variables x, and the other symbols have the same significance as before. Following the procedure of Hestenes for the two-dimensional case Carson finds a variety of necessary and sufficient conditions that the equation (7) shall hold for all admissible sets of functions η. From these he shows that the Haar-Coral equations are necessary and sufficient in order that the first variation (3) shall have the value zero for all admissible sets η vanishing on the boundary $\mathfrak{B}$, and he finds also an independent proof of the analogue of Green's theorem for higher spaces. His results are applicable further to the deduction of the ridge condition mentioned in the following paragraph.

A ridge on a continuous space (2) is an $(m-1)$ dimensional boundary between two pieces of the space on each of which the space has continuous first derivatives. On the ridge itself these derivatives are not continuous. The so-called ridge condition says that across a ridge on a minimizing space (2) the sums $f_{i\alpha} l_\alpha$ must be continuous, where the arguments of the derivatives $f_{i\alpha}$ are

those belonging to the minimizing space and the l_α are the direction cosines of the normal to the ridge. The ridge conditions seem to have been deduced first for multiple integrals by Kobb [1892, 1893] for the parametric case in 3-space. A more recent proof for non-parametric problems when $(m, n)=(2, 1)$, with integrands of the form $f(x, y, z, p, q)$, has been given by Bliss [1939]. By transforming this latter problem into parametric form and applying the conditions of Kobb it is found that across a ridge on a minimizing surface $z(x, y)$ the three determinants of the matrix.

$$\left\| \begin{matrix} f_p & f_q & pf_p + qf_q - f \\ dx & dy & dz \end{matrix} \right\|$$

must be continuous when dx, dy, dz define the tangent to the ridge curve and the arguments of f are $x, y, z, p=z_x, q=z_y$. The general case for arbitrary values (m, n) has been treated by Powell [1931]. For the problems of Dirichlet and Plateau the ridge conditions imply that a minimizing surface can have no ridges. The conditions have an important application in the proof of the analogue of the condition of Jacobi by the method suggested by Bliss (1939).

3. Conditions analogous to those of Weierstrass and Legendre. The function $E(x, y, y', Y')$ of Weierstrass is by definition

$$E = f(x, y, Y') - f(x, y, y') - (Y_{i\alpha} - y_{i\alpha})f_{i\alpha}(x, y, y').$$

It is provable that at every element (x, y, y') of a minimizing space (2) the condition $E \geqq 0$ must be satisfied for all sets Y' such that (x, y, Y') is admissible and the matrix $\| Y_{i\alpha} - y_{i\alpha} \|$ has rank 1. The proof for multiple integrals of this so-called condition of Weierstrass was first made for the case $(m, n)=(2, 1)$ by Levi [1915], and for the general (m, n) case by McShane [1931]. Simpler proofs have been given for $(m, n)=(2, n)$ by Coral [1937] and for arbitrary (m, n) by Graves [1939].

The requirement that the matrix $\| Y_{i\alpha} - y_{i\alpha} \|$ shall be of rank 1 is no restriction when either of the integers m or n is unity, since the matrix has then only one row or one column. The requirement is a restriction, however, when $m \geqq 2$ and $n \geqq 2$, and at first it may seem an unnatural one. But the following simple example for the case $(m, n)=(2, 2)$, proposed by Reid, shows that from the nature of the problem the restriction is unavoidable. The integral I whose integrand is $f=y_{11}y_{22}-y_{12}y_{21}$ is independent of the path, since its Lagrange equations are identically satisfied. It is therefore minimized by every space (2) with suitable continuity properties. Its E-function, which has the form

$$E = (Y_{11} - y_{11})(Y_{22} - y_{22}) - (Y_{12} - y_{12})(Y_{21} - y_{21}),$$

vanishes when the matrix $\| Y_{i\alpha} - y_{i\alpha} \|$ is of rank 1, and when y' is given E may be made either positive or negative by properly choosing Y'. Thus it is hopeless to expect to prove the necessity of the condition that on a minimizing space (2) we must have $E \geqq 0$ for unrestricted values Y'.

The necessary condition of Legendre states that at every set (x, y, y') belonging to a minimizing space (2) the quadratic form $f_{i\alpha,k\beta}\pi_{i\alpha}\pi_{k\beta}$ must be non-negative for all sets of values $\pi_{i\alpha}$ such that the matrix $\|\pi_{i\alpha}\|$ has rank 1. This condition is a consequence of that of Weierstrass by a well-known method which uses Taylor's formula (see Bliss [1939]). It was proved for the case $(m, n) = (2, 1)$ by Mason [1907]. If the quadratic form is everywhere positive for all of the sets $\pi \neq 0$ specified, the space (2) is said to satisfy the "strengthened" condition of Legendre. From a rather complicated transformation of the second variation by Clebsch [1859] Hadamard [1902, 1905] inferred that the necessity of the non-negativeness of the quadratic form of Legendre could be proved only for matrices $\|\pi_{i\alpha}\|$ of rank 1. It was this remark which called attention to the similar restriction in the case of the condition of Weierstrass.

4. Conditions analogous to that of Jacobi. For the problem formulated in section 1 above the so-called accessory equations belonging to a space (2) are the Lagrange equations formed from the integrand 2ω of the second variation,

$$J_i(\eta) = d\omega_{i\alpha}/dx_\alpha - \omega_i = 0 \tag{8}$$

with the notations

$$\omega_i = \partial\omega/\partial\eta_i, \quad \omega_{i\alpha} = \partial\omega/\partial\eta_{i\alpha}.$$

They are linear and homogeneous in the functions η and the first and second derivatives of these functions with respect to the variables x. The coefficients are expressible in terms of derivatives of the integrand f which are well-defined functions of the variables x when the functions $y(x)$ occurring in them and belonging to the space (2) have continuous second derivatives. The necessary condition of Jacobi states that on a minimizing extremal space (2) there can be no solution $\eta(x)$ of the accessory equations (8) defined over a sub-region $\mathfrak{X}_0$ of $\mathfrak{X}$, vanishing on the boundary $\mathfrak{B}_0$ of $\mathfrak{X}_0$, and having derivatives $\eta_{i\alpha} = \partial\eta_i/\partial x_\alpha$ not all zero at a point of $\mathfrak{B}_0$ interior to $\mathfrak{X}$.

A relatively simple proof of this theorem for the case $(m, n) = (2, 1)$ was given by Sommerfeld [1899] following a method suggested by Schwarz [1885]. Perhaps the simplest modern proof is analogous to one suggested for simpler cases by Bliss. Assume that there exists a solution of the accessory equations with the properties described in the condition. Extend the definition of $\eta(x)$ to be $\eta \equiv 0$ over the part of $\mathfrak{X}$ outside of $\mathfrak{X}_0$. For this extended $\eta(x)$ it can be proved that the second variation has the value zero, and that the set $\eta(x)$ does not satisfy along $\mathfrak{B}_0$ the ridge condition for a minimum of the second variation. Thus zero is not the smallest value possible for the second variation on the space (2), and this space cannot minimize I since on a minimizing space the second variation is necessarily positive. Proofs have been made by this method for the case $(m, n) = (2, 1)$ by Kubota [1916], and for arbitrary (m, n) by Raab [1932]. Reid [1939] proves the condition with more general regions $\mathfrak{X}_0$ and boundaries $\mathfrak{B}_0$. Haar [1930] replaces the accessory equations (8) by equations analogous to those deduced by him from the first variation.

The early literature of the calculus of variations concerning the conditions of Legendre and Jacobi, in forms not always clearly stated, is voluminous and complicated. Its purpose is for the most part to attain necessary conditions for a minimum by transformations of the second variation or other methods similar to those used by Legendre and Jacobi themselves in simpler cases. For this literature the reader is referred to the sections of the Bibliography mentioned above on the Jacobi condition and the second variation.

Boundary value problems associated with the second variation have been formulated in a number of related forms. One of them is the problem of finding a solution η, λ of the equations and boundary conditions

$$J_i(\eta) + \lambda\eta_i = 0, \quad \eta \equiv 0 \text{ on } \mathfrak{B}, \tag{9}$$

where λ is a parameter. A second necessary condition for a minimum, related to that of Jacobi, states that on a minimizing space (2) satisfying the Lagrange equations and the strengthened condition of Legendre there can be no solution of the boundary value problem (9) with a negative value of the parameter λ. The first proof of a theorem like this seems to have been given by Schwarz (1885) for the Plateau problem. The case $(m, n) = (2, 1)$, with a different boundary value problem, has been discussed by Lichtenstein [1915, 1917, 1920] and Picone [1922]. Reid [1932] relates the different points of view hitherto adopted, but further possibilities might also well be studied. When the minimum problem of section 1 is replaced by one with variable boundaries a necessary condition similar to that of the first paragraph of this section is in general lacking. Conditions arising from boundary value problems of the second variation have, however, been considered for the case $(m, n) = (2, 1)$ by Lichtenstein [1919] and Simmons [1926], and for the case $(m, n) = (m, 1)$ by Simmons [1934].

5. Fields and sufficiency proofs. A field for the integral I of section 1 is defined to be a region F of xy-space with a set of slope functions $p_{i\alpha}(x, y)$ and an invariant integral

$$I^* = \int D(x, y, y')d\mathfrak{X}$$

such that

$$D(x, y, p) = f(x, y, p), \quad D_{i\alpha}(x, y, p) = f_{i\alpha}(x, y, p). \tag{10}$$

An integral is said to be invariant if it has the same value on all spaces (2) having a common boundary. Since every space (2) is a minimizing space for an invariant integral, and therefore satisfies the Lagrange equations of the integral, it is provable with the help of equations (10) and a very simple argument that every solution (2) of the equations

$$\partial y_i/\partial x_\alpha = p_{i\alpha}(x, y) \tag{11}$$

is a solution also of the equations of Lagrange for the integral I. A space (2)

which satisfies equations (11) is called an *extremal of the field*, and the equations (11) are called the differential equations of the field. It follows readily from the first equation (10) that on an extremal space $\mathfrak{E}$ of the field $I(\mathfrak{E}) = I^*(\mathfrak{E})$.

The importance and use of fields in making sufficiency proofs was first emphasized by Weierstrass for simpler cases. In general let $\mathfrak{E}$ be an extremal space imbedded in a field $\mathfrak{F}$, and let $\mathfrak{S}$ be an arbitrary space of the form (2) in the field having the same boundary as $\mathfrak{E}$. Then from the properties of the integral I^* it follows readily that

$$(12)\qquad I(\mathfrak{S}) - I(\mathfrak{E}) = I(\mathfrak{S}) - I^*(\mathfrak{E}) = I(\mathfrak{S}) - I^*(\mathfrak{S}) = \int_{\mathfrak{S}} (f - D) d\mathfrak{X}.$$

It is evident from the last equation that if the function $f - D$ is everywhere non-negative then $I(\mathfrak{E})$ is a minimum. If a space $\mathfrak{E}$ whose minimizing properties are to be established is given in advance one may seek to imbed $\mathfrak{E}$ in a field $\mathfrak{F}$ as an extremal of the field, and then to show that the integrand $f - D$ is non-negative. For the case of simple integrals, when $(m, n) = (1, n)$, it is well known that suitably strengthened conditions of Lagrange, Weierstrass, Legendre, and Jacobi insure the possibility of carrying through this process. For the case $(m, n) = (2, 1)$ some of the corresponding theorems have been established by Lichtenstein [1917] and other writers, but in general the theory for multiple integrals is still only fragmentary.

The differential equations (11) of a field $\mathfrak{F}$ may be incompatible and have no solution, in which case the field seems to have little interest, or they may have a particular space $\mathfrak{E}$ as a solution but not be compatible everywhere in $\mathfrak{F}$, or they may be "completely integrable" in $\mathfrak{F}$ and define an n-parameter family of extremals

$$(13)\qquad y_i = y_i(x_1, \cdots, x_m, c_1, \cdots, c_n) = y_i(x, c)$$

simply covering the region $\mathfrak{F}$. A well-known necessary and sufficient condition for this last to happen is that the integrability conditions

$$\partial p_{i\alpha}/\partial x_\beta + p_{k\beta}\partial p_{i\alpha}/\partial y_k = \partial p_{i\beta}/\partial x_\alpha + p_{k\alpha}\partial p_{i\beta}/\partial y_k$$

are satisfied identically in $\mathfrak{F}$. The field is then called a *field of extremals*. In that case the equations (13) have solutions $c_i(x, y)$ and the slope functions of the field are the slope functions

$$p_{i\alpha}(x, y) = y_{i\alpha}\left[x, c(x, y)\right]$$

of the family.

6. The construction of fields. One of the first problems which arises in the construction of fields is the determination of the possible forms for the invariant integral I^*. Two ingenious special types of such integrals and the fields associated with them have been studied by Carathéodory [1922, 1929] and his associates, and by Weyl [1935]. The integrand D for the invariant integral of

Carathéodory is a functional determinant, and for that of Weyl is a divergence. The most general integrand D of an invariant integral was determined by Smiley in 1938 as a sum of determinants of all orders of the matrix $\|y_{i\alpha}\|$ with coefficients functions of the variables x, y satisfying a certain system of partial differential equations. The integrand D is evidently a polynomial in the variables $y'=\|y_{i\alpha}\|$ of degree equal to the smaller ν of the integers m and n. The conditions of Smiley are consequences of the fact that a necessary and sufficient condition for an integral I^* to be invariant is that the Lagrange equations formed with the integrand D of I^* shall be satisfied identically in the variables x, y, y', y''.

Every integrand function D belonging to a field must be related to the slope functions $p(x, y)$ of the field by the equations (10). If D is expanded in powers of the variables $y'-p$ by Taylor's formula the terms of degree less than 2 are completely determined by the conditions (10), and D has the form

$$D = f(x, y, p) + (y_{i\alpha} - p_{i\alpha})f_{i\alpha}(x, y, p) + D_2 + \cdots, \tag{14}$$

where D_k $(k \geqq 2)$ is a symbol for the terms of order k in the expansion of D and is a sum of determinants of order k of the matrix $y'-p$ with coefficients which are functions of the variables x, y. The functions $p(x, y)$ and these coefficients must satisfy the equations of Smiley deduced for this form of D. The difference $f-D$ has the form

$$f - D = E(x, y, p, y') - D_2 - \cdots. \tag{15}$$

If the E-function is also expanded in powers of the variables $y'-p$ it is found that

$$f - D = (y_{i\alpha} - p_{i\alpha})(y_{k\beta} - p_{k\beta})(f_{i\alpha,k\beta} - D_{i\alpha,k\beta}) + (f_3 - D_3) + \cdots \tag{16}$$

where the arguments of the second derivatives of f and D are x, y, p, and $f_k (k \geqq 2)$ is a symbol for the terms of order k in the expansion of f.

For simple integrals when $(m, n) = (1, n)$, and for multiple integrals with a single dependent variable when $(m, n) = (m, 1)$, the value of ν is unity, D is uniquely determined as the sum of the terms of orders <2 in (14), and the sign of $f-D=E$ in (15) is that of E. For the case of simple integrals it can be proved from Smiley's equations that every field is a field of extremals. Except when $n=1$ not every n-parameter family of extremals simply covering a region $\mathfrak{F}$ of xy-space has slope functions belonging to a field. The families belonging to fields are called Mayer families and are special in character, as is well known.

When $(m, n) = (m, 1)$ the integrals involved are multiple integrals whose integrands depend upon only one function y. In this case there may be fields which are not fields of extremals, but every one-parameter family of extremals simply covering a region $\mathfrak{F}$ of xy-space has slope functions forming a field over $\mathfrak{F}$.

The case $(m, n) = (2, 2)$ is the one which inspired the remarks of Hadamard mentioned above concerning the form of the conditions of Legendre. It was

studied in some detail by Alaoglu in 1938. The expansion (14) for D has in this case no terms of order greater than 2, and D_2 has the simple form

$$D_2 = B(x, y)\left[(y_{11} - p_{11})(y_{22} - p_{22}) - (y_{12} - p_{12})(y_{21} - p_{21})\right].$$

The slope functions p and the coefficient B must satisfy four partial differential equations of the first order. These equations doubtless have solutions not belonging to any field of extremals. But by a pleasantly skillful manipulation Alaoglu shows that when the functions $p(x, y)$ are given as the slope functions of a 2-parameter family of extremals two of the equations are identically satisfied and the remaining two are linear in B and its first derivatives and compatible. Thus the slope functions $p(x, y)$ of every two-parameter family of extremals simply covering a region $\mathfrak{F}$ in xy-space are the slope functions of an infinity of fields associated with the family.

The analysis of the invariant integrals associated with fields of extremals becomes much simpler if the coordinates x, y are replaced by x, c, where the c's are the parameters of the extremal family, and if the integrand f is replaced by $f-D$ where D is the integrand of some particular invariant integral. This last modification does not affect the solutions of the problem since the integral I^* is invariant. When these changes have been made and the new coordinates have been designated again by x, y the family of extremals of the field is the family $y_i = c_i$, the slope functions $p(x, y)$ of the family are all identically zero, and it turns out that

$$f(x, y, 0) \equiv 0, \quad f_{i\alpha}(x, y, 0) \equiv 0.$$

The equations (15) and (16) for the case $(m, n) = (2, 2)$ take the forms

$$(17) \quad \begin{aligned} f - D &= f(x, y, y') + B(y_{11}y_{22} - y_{12}y_{21}) \\ &= y_{i\alpha}y_{k\beta}f_{i\alpha,k\beta}(x, y, 0) + B(y_{11}y_{22} - y_{12}y_{21}) + f_3 + \cdots. \end{aligned}$$

The equations of Smiley are satisfied if and only if B is a function of the variables y alone. They will certainly be satisfied if B is a constant.

In order to find a field in which $f-D$ is non-negative we seek to determine the coefficient $B(y)$ so that the terms of the second order in the second expression (17) for $f-D$ form a positive definite quadratic form. So far this has not been done for the whole of the region $\mathfrak{F}$. But it can be shown that when the strengthened condition of Legendre is satisfied by an extremal space $\mathfrak{E}$ of a field then there exists for each particular point $(x, y)_0$ of $\mathfrak{E}$ a constant B such that at $(x, y)_0$ the quadratic terms in (17) are positive definite. The difference $f-D$ will then be non-negative at all points (x, y) of the field $\mathfrak{F}$ sufficiently near to $(x, y)_0$ and for all sufficiently small values of the variables y'. From the formula (12) it follows that each piece $\mathfrak{E}'$ of $\mathfrak{E}$ sufficiently near to $(x, y)_0$ will minimize the integral I relative to other spaces having sufficiently small values of the variables y' and passing through the boundary of $\mathfrak{E}'$. This means, in other words, that each sufficiently small piece $\mathfrak{E}'$ of an extremal $\mathfrak{E}$ of a field gives to the integral I a weak relative minimum.

The existence of a constant B which will made the second order terms in the expression (17) positive definite at a particular point $(x, y)_0$ is a consequence of a theorem on quadratic forms which has interested a number of writers.* This theorem says that if a quadratic form $P(z)$ in variables $z_1, \cdots, z_n$ with real coefficients is positive at every set of real values z not all zero making a second quadratic form $Q(z)$ vanish, then there exists a constant B such that the quadratic form $P(z)+BQ(z)$ is positive definite. The quadratic form in the second expression (17) with coefficients derivatives of f is positive for all sets $y_{i\alpha}$ not all zero making the determinant $y_{11}y_{22}-y_{12}y_{21}$ vanish, on account of the strengthened condition of Legendre which is supposed to hold on the extremal space $\mathfrak{E}$ under consideration. Thus by a suitable selection of a constant value for B the quadratic terms in the second expression (17) can be made positive definite at a particular point $(x, y)_0$, and $f-D$ will be positive for all sets (x, y, y') with (x, y) sufficiently near to $(x, y)_0$ and elements y' sufficiently small.

The theorem on quadratic forms mentioned in the last paragraph has been generalized by Hestenes and McShane [1940] so that similar arguments are effective for the case $(m, n)=(2, n)$, and one can perhaps treat the case $(m, n) =(m, 2)$ in a similar manner. An example has been given by Terpstra [1938] to show that when $m \geqq 3$, $n \geqq 3$ the generalization of the quadratic form theorem which might be expected to hold in general is no longer true. For these cases therefore the arguments made above would need to be modified in some way not yet discovered.

7. Conclusion. From what has been said above it is clear that the theory of the calculus of variations for multiple integrals is still very imperfect. For the cases $(m, n)=(1, n)$ involving simple integrals the equations of the extremals are ordinary differential equations for which existence and imbedding theorems are well known. If an extremal arc $\mathfrak{E}$ in the $(n+1)$ dimensional xy-space satisfies suitably strengthened conditions of Weierstrass, Legendre, and Jacobi, then $\mathfrak{E}$ belongs to an n-parameter family of extremal arcs whose slope functions form a field surrounding $\mathfrak{E}$ in which the difference $f-D$ is non-negative, and $I(\mathfrak{E})$ is therefore a minimum.

For the case $(m, n)=(2, 1)$ Lichtenstein [1917] proved a similar succession of theorems, using his boundary value problem form of Jacobi's condition. In this case therefore, and for this form of the condition of Jacobi, the theory is comparable to that for simple integrals.

Let us consider for the moment, in the general case (m, n), an extremal space $\mathfrak{E}$ satisfying the strengthened condition of Legendre and imbedded in an n-parameter family of extremals simply covering a region $\mathfrak{F}$ of xy-space. From the results of Smiley and Landers [1939] it follows that when $m \geqq 2$, $n \geqq 2$ there is an infinity of invariant integrals I^* which form fields in $\mathfrak{F}$ with the slope functions $p(x, y)$ of the extremal family. But only for special values (m, n) and

* Finsler [1937], Albert [1938], Reid [1938], Terpstra [1938], Hestenes and McShane [1940], Dines [1941].

for limited portions $\mathfrak{E}'$ of $\mathfrak{E}$ is it known that among the possible integrals I^* there are some such that the condition $f-D\geqq 0$ is satisfied in the field near $\mathfrak{E}'$, with the consequence that we can be sure that $I(\mathfrak{E}')$ is a minimum. This seems, however, a much too restricted result in any case. One would hope to be able to prove that the whole space $\mathfrak{E}$ has the minimizing property.

Weyl [1935] has given a proof that an extremal space $\mathfrak{E}$ with suitable properties is imbedded in a field of the type which he devised, and Boerner (1936) has a similar theorem for fields of the Carathéodory type. If the condition $f-D\geqq 0$ holds near $\mathfrak{E}$ in one of these fields then $I(\mathfrak{E})$ will be a minimum. But so far as I know this condition has not been proved to be necessary for a minimum in either case. The question as to whether $\mathfrak{E}$ can be imbedded in a field in which $f-D\geqq 0$ near $\mathfrak{E}$ is still unanswered.

There are many topics of interest and importance in connection with the theory of the calculus of variations for multiple integrals which are not discussed in the preceding pages. The reader is referred to the fourth section of the bibliography mentioned in the introduction above, in which the classification of papers is by subject matter. Among the topics there mentioned and recently studied, some of them in a large number of papers, are problems with variable edges, the inverse problem, the character of the solutions of the equations of Lagrange and Haar, existence theorems, the problem of Dirichlet, and especially the problem of Plateau.

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RATIONAL CURVES DEFINED BY AN ALGEBRAIC CORRESPONDENCE

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1. Introduction. Let $F(\lambda_1, \lambda_2, \lambda_3)$ be an irreducible homogeneous polynomial of degree μ in $\lambda_1, \lambda_2, \lambda_3$, of degree m in λ_1, and of degree n in λ_2. Then the equation

$$F(\lambda_1, \lambda_2, \lambda_3) = 0 \tag{1}$$

establishes an (m, n) algebraic correspondence* between the ratios $(\lambda_1:\lambda_3)$ and $(\lambda_2:\lambda_3)$; that is, any value assigned to the ratio $(\lambda_1:\lambda_3)$ determines n values of the ratio $(\lambda_2:\lambda_3)$, and any value assigned to $(\lambda_2:\lambda_3)$ determines m values of $(\lambda_1:\lambda_3)$. When $(\lambda_1:\lambda_3)$ and $(\lambda_2:\lambda_3)$ are interpreted in homogeneous rectangular coordinates as the two pencils of parallel lines $x_1/x_3 = \lambda_1/\lambda_3$ and $x_2/x_3 = \lambda_2/\lambda_3$, the locus of the intersection of corresponding lines as established by (1) is the most general algebraic plane curve; the equation of the locus is $F(x_1, x_2, x_3) = 0$. Thus, every algebraic plane curve is defined, as is well-known, as the intersection of corresponding lines of two mutually perpendicular pencils of lines, where the correspondence is established by (1).

Let $P_1(\lambda_1:\lambda_3)$ give a point range on the line $A_1(1, 0, 0)$, $A_3(0, 0, 1)$, and let $P_2(\lambda_2:\lambda_3)$ give a point range on the line A_3, $A_2(0, 1, 0)$, where

$$\begin{aligned} P_1 &\equiv \lambda_1 A_3 - \lambda_3 A_1 = (-\lambda_3, 0, \lambda_1) \text{ on } A_1A_3, \quad \text{and} \\ P_2 &\equiv \lambda_2 A_3 - \lambda_3 A_2 = (0, -\lambda_3, \lambda_2) \text{ on } A_2A_3. \end{aligned} \tag{2}$$

Then the equation of the line joining P_1 to a corresponding point P_2 will be

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0;$$

that is, the line coordinates of P_1P_2 (*i.e.*, the parametric line equations of the envelope) are

$$\sigma u_i = \lambda_i, \qquad (i = 1, 2, 3). \tag{3}$$

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* M. Chasles, Comptes Rendus, vol. 58, 1864, p. 1175; W. L. Edge, The Theory of Ruled Surfaces, Cambridge, 1931, p. 9.

Thus it follows that an (m, n) correspondence established by (1) defines the most general algebraic plane curve* whose line equation is $F(u_1, u_2, u_3)=0$. Since F was assumed to be irreducible, then $m+n\geqq\mu$; let $m+n-\mu=\gamma$. Then since F is of degree μ, the envelope established by this correspondence is of class μ; since F is of degree m in u_1 and of degree n in u_2, the lines $[1, 0, 0]$ and $[0, 1, 0]$ are tangents of the envelope and of multiplicities $(n-\gamma)$ and $(m-\gamma)$, respectively.

A non-degenerate line conic is a simple example of a curve defined by an (m, n) correspondence, with $m=n=1$; it is the totality of lines joining corresponding points of a $(1, 1)$ correspondence between two projective, non-perspective ranges of points. The constructions of other particular types of curves by means of (m, n) correspondences have appeared frequently in the journals.

In §2, we obtain parametric line equations of the envelope R of a line joining corresponding points of an (m, n) correspondence established by two involution-ranges of degrees n and m on two fixed lines; every rational plane curve can be so defined. The existence of line singularities is considered in §3, and the Plücker characteristics of R are obtained. It is shown in §4 that every rational curve is the envelope of the Pascal line of a hexagon inscribed in a conic when two of the vertices of the hexagon are allowed to vary by being rational functions of a parameter, and that every algebraic plane curve is the envelope of the Pascal line when the two vertices are related through a general algebraic correspondence.

2. A rational envelope and its parametric equations. If $(t_1:t_2)$ is the parameter pair of a point on a line and $f_1(t_1, t_2)$ and $f_2(t_1, t_2)$ are relatively prime homogeneous polynomials each of degree n in t_1, t_2, then the set of n points determined by $\lambda_2 f_1(t_1, t_2)=\lambda_1 f_2(t_1, t_2)$ is said to trace out an involution-range† of degree n as λ_1/λ_2 varies. We shall now consider the envelope R of the line P_1P_2 when the (m, n) correspondence between $P_1(\lambda_1:\lambda_3)$ and $P_2(\lambda_2:\lambda_3)$ is established by two involution-ranges in the following way.‡ Let

$$\rho\lambda_i = \phi_i(t_1, t_2), \qquad (i = 1, 2, 3),$$

where each ϕ is a homogeneous polynomial of degree μ. Let d_1 be the greatest common divisor of ϕ_2 and ϕ_3, let d_2 be the G. C. D. of ϕ_3 and ϕ_1, and let d_3 be the G. C. D. of ϕ_1 and ϕ_2, so that

$$\begin{aligned} \rho\lambda_1 &= \phi_1 = d_2d_3K_1,\\ \rho\lambda_2 &= \phi_2 = d_3d_1K_2, \qquad (4)\\ \rho\lambda_3 &= \phi_3 = d_1d_2K_3,\end{aligned}$$

with each d prime to each other d and each K prime to each other K. Let d_1 be of degree $\mu-m$, and d_2 of degree $\mu-n$, and K_3 of degree γ, so that $m+n-\gamma=\mu$.

* Virgil Snyder, Bulletin of the National Research Council, no. 63, p. 166.

† H. Hilton, Plane Algebraic Curves, Oxford, 1920, p. 375.

‡ The following representation was suggested by Professor W. B. Carver.

Equations (4) represent a rational curve of class μ since its line coordinates are rational functions of degree μ. The correspondence between P_1, P_2 of (2) is established by the ratios,

$$\frac{\lambda_1}{\lambda_3} = \frac{d_3K_1}{d_1K_3} \equiv \frac{f_1(t_1, t_2)}{f_2(t_1, t_2)},$$
$$\frac{\lambda_2}{\lambda_3} = \frac{d_3K_2}{d_2K_3} \equiv \frac{g_1(t_1, t_2)}{g_2(t_1, t_2)}, \tag{5}$$

where the f's are relatively prime homogeneous polynomials each of degree n, and the g's are relatively prime homogeneous polynomials each of degree m. These functions establish involution-ranges of degrees n and m on the lines A_1A_3 and A_2A_3, respectively, and an (m, n) correspondence between the points P_1, P_2, on these two lines. For example, a fixed value of λ_1/λ_3 determines a polynomial equation $\lambda_1 f_2 = \lambda_3 f_1$ of degree n whose n roots $(t_1:t_2)$ determine n values of λ_2/λ_3. Since the parametric line equations of any rational plane curve are of the form (4), we have the following theorem.

THEOREM 1. *Every rational plane curve is defined by an (m, n) correspondence between the points of two lines in a plane where the correspondence is established by the two involution-ranges* (5).

It is apparent, from (2) and (5), that the point A_3 corresponds to the parameter pair $(\lambda_1:\lambda_3) = (1:0)$ on A_1A_3 and $(\lambda_2:\lambda_3) = (1:0)$ on A_2A_3. Consequently, the point A_3 is a self-corresponding point of multiplicity γ in the correspondence and its parameter pairs $(t_1:t_2)$ on each of the two lines are the γ roots of $K_3(t_1, t_2) = 0$. It is apparent from (4) that the lines $A_1A_3 = [0, 1, 0]$ and $A_2A_3 = [1, 0, 0]$ are tangents of multiplicities $(m-\gamma)$ and $(n-\gamma)$, respectively, with parameter pairs $(t_1:t_2)$ which are the roots of $d_2 = 0$ and $d_1 = 0$.

3. Other multiple tangents of the rational curve R and its Plücker characteristics. It is possible to obtain* immediately the Plücker characteristics of the envelope when the correspondence is established by (1) when it is assumed that the only line singularities are A_1A_3 and A_2A_3. A rational curve of class $\mu = 3$, for example, can have at most one bitangent; if A_1A_3 is a bitangent, the line A_2A_3 can be at most an ordinary tangent. When the value of μ is larger, however, it will be seen that R can have other multiple tangents.

The deficiency of a curve of order ν, class μ, with δ nodes, κ cusps, i stationary tangents, and τ bitangents is

$$p = \tfrac{1}{2}(\nu - 1)(\nu - 2) - \delta - \kappa = \tfrac{1}{2}(\mu - 1)(\mu - 2) - \tau - i. \tag{6}$$

For the curve R we have $\mu = m + n - \gamma$ and $p = 0$; consequently,

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* A. Emch, The symmetric (n, n)-correspondence and some geometric applications, American Jour. of Math., vol. 54, no. 2, 1932, pp. 285–292; Hilton, *loc. cit.*, pp. 372–374.

$$\tau + i = \tfrac{1}{2}(m + n - \gamma - 1)(m + n - \gamma - 2). \tag{7}$$

The two known multiple tangents of R account for*

$$\tau_1 = \tfrac{1}{2}[(m - \gamma)(m - \gamma - 1) + (n - \gamma)(n - \gamma - 1)] \tag{8}$$

bitangents and stationary tangents. When the roots of $d_2=0$ and $d_1=0$ are distinct the corresponding contact points of R on A_iA_3 will be distinct ordinarily, and these two tangents will account for τ_1 bitangents and no stationary tangents. Then it can be seen from (7), (8) that the remaining† number to account for is

$$(m - 1)(n - 1) - \tfrac{1}{2}\gamma(\gamma - 1). \tag{9}$$

Ordinarily, the variable point P_1 will occupy any fixed position A on A_1A_3 for n different parameter pairs $(t_1:t_2)$ and, consequently, the n corresponding positions of P_2 on A_2A_3 are usually distinct. If two of the parameter pairs for which P_1 is at A $(A \neq A_3)$ are $t_1:t_2=x:z$ and $t_1:t_2=y:z$, with $x \neq y$, and also if P_2 is at B on A_2A_3 for these two ratios, then the line AB is a bitangent‡ of R. From (5), this relationship will exist for certain pairs of solutions, $(x:z)$ and $(y:z)$ of the equations,

$$\begin{cases} f_1(x, z)f_2(y, z) - f_1(y, z)f_2(x, z) = 0, \\ g_1(x, z)g_2(y, z) - g_1(y, z)g_2(x, z) = 0. \end{cases} \tag{10}$$

In these equations, since the terms of highest degree in x, y cancel, the factor z can be removed from each, and since any pair $x=y$ is a formal solution the factor $(x-y)$ can be removed from each. Then the resulting pair of equations,

$$\theta_1(x, y, z) = 0 \quad \text{and} \quad \theta_2(x, y, z) = 0$$

will be of degrees $(2n-2)$ and $(2m-2)$ in x, y, z. But θ_1 is of degree $(n-1)$ in x and of degree $(n-1)$ in y so that $\theta_1=0$, as a curve, has the points (1, 0, 0) and (0, 1, 0) as points of multiplicity $(n-1)$. Similarly the curve $\theta_2=0$ has these same two points as points of multiplicity $(m-1)$. In general these two multiple points account for $2(m-1)(n-1)$ solutions of (10) which do not determine bitangents. Furthermore, the γ roots of $K_3=0$ are the parameters of A_3 on each line and they account for $\gamma(\gamma-1)$ solutions of (10). Since each of the equations $\theta_i=0$ is symmmetrical in x and y and the solutions (x, y, z) and (y, x, z) give parameters of the same bitangent, the remaining $2(m-1)(n-1)-\gamma(\gamma-1)$ solutions of (10) determine

$$(m - 1)(n - 1) - \tfrac{1}{2}\gamma(\gamma - 1) \tag{11}$$

bitangents of R. By comparing (9) and (11), it is seen that the solutions of (10)

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* Any k-tuple tangent with α distinct points of contact is equivalent to $[\frac{1}{2}k(k-3)+\alpha]$ bitangents and $(k-\alpha)$ stationary tangents, a total of $\frac{1}{2}k(k-1)$. See Hilton, *loc. cit.*, p. 116.

† In an $(m, 1)$ or a $(1, n)$ correspondence, only one of the two lines is a multiple tangent, and it accounts for all of the bitangents and stationary tangents of the curve.

‡ The line AB would be a stationary tangent only if the coefficients of f_i and g_i were such that the two pairs $(x:z)$ and $(y:z)$ give the same contact point of R on AB.

determine the remaining bitangents and stationary tangents.

If of the solutions there is a pair such that $x=y$, the two points of tangency of R on AB will be coincident, and the line AB will be a stationary tangent of R. If the coefficients of f_i and g_i satisfy the conditions necessary to produce β pairs of solutions $x=y$, the solutions of (10) will determine

$$\tau_2 = (m-1)(n-1) - \beta - \tfrac{1}{2}\gamma(\gamma-1)$$

bitangents and β stationary tangents. Consequently the number τ of bitangents and the number i of stationary tangents of R has been determined to be

$$\tau = \tau_1 + \tau_2 = \tfrac{1}{2}(m+n-\gamma-1)(m+n-\gamma-2) - \beta$$

and

$$i = \beta.$$

The remaining Plücker characteristics of the curve can now be determined and are the following:

$$\begin{aligned}
\nu &= 2(m+n-\gamma-1) - \beta,\\
\kappa &= 3(m+n-\gamma-2) - 2\beta,\\
\delta &= 2(m+n-\gamma-2)(m+n-\gamma-\beta-3) + \tfrac{1}{2}\beta(\beta+3).
\end{aligned}$$

4. The curve R as the envelope of a Pascal line. If six points are selected on a conic and these points are taken as the vertices of an ordered hexagon, then the three pairs of opposite sides meet in three collinear points, P_1, P_2, and P_3. This is Pascal's theorem; the points P_i $(i=1, 2, 3)$ are called the Pascal points and the line through them is called the Pascal line.

If five of the vertices are fixed and the sixth is then allowed to move around the conic, the Pascal line will revolve about a fixed Pascal point. Certain other interesting loci will be generated, however, and they have been investigated by Bunch.* When two of the vertices, say Q_1 and Q_2 are related by having parameters on the conic which are functions of a common parameter pair $(t_1:t_2)$, the Pascal line will envelop a curve as $(t_1:t_2)$ varies. This curve will be merely the fixed Pascal point when Q_1 and Q_2 are opposite vertices of the ordered hexagon. We shall consider in this section the envelope which is generated when Q_1 and Q_2 are not opposite vertices of the hexagon.

Any non-degenerate conic through any four fixed points B_1, B_2, B_3, and B_4, no three collinear, is the locus of a point P which moves so that the cross ratio of the four lines PB_i is a constant k $(k\neq 0, 1, \infty)$. The four points B_i may be projected into the four fixed points $A_1(1, 0, 0)$, $A_2(0, 1, 0)$, $A_3(0, 0, 1)$ and $A_4(1, 1, 1)$, and if the cross ratio of the lines PA_i joining any point P on the conic to the A_i is equal to a constant k, the equation of the conic is

$$x_2x_3 - kx_1x_3 + (k-1)x_1x_2 = 0. \tag{12}$$

* W. H. Bunch, this MONTHLY, vol. 40, 1933, pp. 251–260.

Then it follows that the parametric point equations of any conic may be written as follows:

$$
\begin{aligned}
\rho x_1 &= s_2(s_1 + s_2),\\
\rho x_2 &= k s_1 s_2, \qquad\qquad (13)\\
\rho x_3 &= (k - 1)s_1(s_1 + s_2).
\end{aligned}
$$

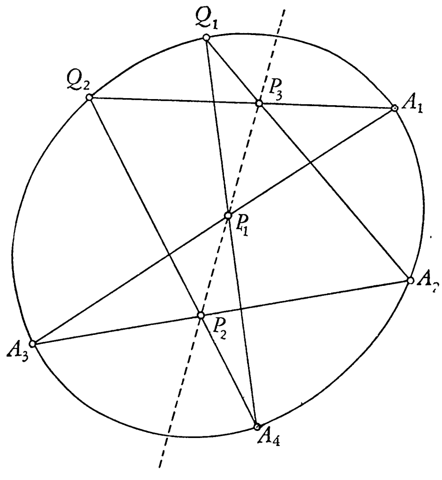
The three conics of the pencil which correspond to $k=0,\ 1,\ \infty$ are degenerate. The parameter pairs $(s_1:s_2)$ corresponding to A_1, A_2, A_3, and A_4 on the conic are $(0:1)$, $(1:-1)$, $(1:0)$, and $(1:k-1)$, respectively. Now let the variable points Q_1 and Q_2 on the conic correspond to the parameter pairs $(s_1^{(1)}:s_2^{(1)})$ and $(s_1^{(2)}:s_2^{(2)})$, respectively. We establish an (m, n) correspondence between Q_1 and Q_2 by letting

$$
\frac{s_1^{(1)}}{s_2^{(1)}} = \frac{f_1(t_1, t_2)}{f_2(t_1, t_2)} \quad\text{and}\quad \frac{s_1^{(2)}}{s_2^{(2)}} = \frac{g_1(t_1, t_2) + g_2(t_1, t_2)}{-\ g_2(t_1, t_2)} \qquad (14)
$$

where the f's are relatively prime homogeneous polynomials each of degree n, and the g's are relatively prime homogeneous polynomials each of degree m, as in §2. We shall now prove the following theorem.

THEOREM 2. *Every rational plane curve is defined as the envelope of the Pascal line of a hexagon inscribed in a conic by an* (m, n) *corresponding between two variable vertices, where the correspondence is established as in* (14).

In order to show the relationship between the envelope of the Pascal line and the locus R of §2, it is convenient to choose the vertices of the ordered hexagon inscribed in the conic to be A_1, A_3, A_2, Q_1, A_4, Q_2, respectively. Corresponding



to each point Q there is *first* (see the figure) a Pascal point P_1 on A_1A_3 and, *second*, there are n values of $(t_1:t_2)$ and therefore n positions of Q_2; these n positions

of Q_2 determine n Pascal points P_2 on A_2A_3. Consequently, the (m, n) correspondence between Q_1 and Q_2 on the conic established by (14) also establishes an (m, n) correspondence between the Pascal points P_1 and P_2 of A_1A_3 and A_2A_3. With the equations (13) of the conic, the parameters (14) of the Q's, and the parameters (2) of the P's, the coordinates of the points P can be determined to be $P_1(-\lambda_3, 0, \lambda_1)$ and $P_2(0, -\lambda_3, \lambda_2)$, where as in (5)

$$\lambda_1/\lambda_3 = f_1/f_2 \quad \text{and} \quad \lambda_2/\lambda_3 = g_1/g_2.$$

Then as the Q's move along the conic, the Pascal points P_1, P_2 move along the two fixed lines exactly as in §2, and the Pascal line will envelope any rational plane curve.

Of the 60 ordered hexagons whose vertices are the preceding six points in some order, there are 12 in which the Q's are opposite vertices. The locus which corresponds to each of the remaining 48 orders is essentially the same as the one for the ordered hexagon described above. In each of these 48 cases there are at least three fixed consecutive vertices, say A, B, C. The (m, n) correspondence between the Q's which is determined by (14) establishes an (m, n) correspondence between the Pascal points on AB and BC provided there are exactly three fixed consecutive vertices. When there are four fixed consecutive vertices, A, B, C, D, the (m, n) correspondence will be established between the Pascal points of AB and CD. The (m, n) correspondence will be established by parameter pairs which are rational homogeneous functions of degrees n and m, respectively.

It can be noted that *every* algebraic curve, whether rational or not, can be defined as the envelope of the Pascal line of the hexagon A_1, A_3, A_2, Q_1, A_4, Q_2, by means of an (m, n) correspondence between Q_1 and Q_2 on the conic when the other four vertices are fixed. Let $F(\lambda_1, \lambda_2, \lambda_3)=0$ be the line equation of the curve where F is of degree m in λ_1, of degree n in λ_2, and μ in $\lambda_1, \lambda_2, \lambda_3$. Then as in §2 this equation establishes an (m, n) correspondence between the point ranges $P_1=\lambda_1A_3-\lambda_3A_1$ and the point range $P_2=\lambda_2A_3-\lambda_3A_2$. When the parameters of Q_1 and Q_2 on the conic (13) are chosen to be $(\lambda_1:\lambda_3)$ and $(\lambda_2+\lambda_3: -\lambda_3)$, the Pascal line will envelop any algebraic plane curve $F(\lambda_1, \lambda_2, \lambda_3)=0$.

THE NUMERICAL INTEGRATION OF $y''+g(x)y=f(x)$

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This note presents a simple procedure for the numerical integration of the linear differential equation

$$y'' + g(x)y = f(x), \tag{1}$$

in which primes denote differentiation with respect to x, and g and f are known functions.

1. Formal derivation. To derive a suitable formula for our purpose we differentiate Stirling's central difference interpolation formula twice, set the argument equal to the central value, and obtain (using Sheppard's central difference operator δ)

$$\begin{aligned} y'' &= \frac{1}{h^2}\left[2\sinh^{-1}\frac{\delta}{2}\right]^2 y \\ &= \frac{1}{h^2}\left[\delta^2 y - \frac{\delta^4 y}{12} + \frac{\delta^6 y}{90} - \cdots\right], \end{aligned} \tag{2}$$

in which h denotes the length of the interval between equally spaced values of x. Next we operate on (1) and (2) with the operator δ^2, which gives

$$\delta^2 y'' + \delta^2[g(x)y] = \delta^2 f(x) \tag{3}$$

and

$$\delta^2 y'' = \frac{1}{h^2}\left[\delta^4 y - \frac{\delta^6 y}{12} + \cdots\right]. \tag{4}$$

From equations (1), (2), (3), and (4) we eliminate the three quantities y'', $\delta^2 y''$, and $\delta^4 y$.

The resulting equation is

$$\delta^2\left[y + \frac{h^2 g(x)}{12}y\right] + h^2 g(x)y = h^2 f(x) + \frac{h^2\delta^2 f(x)}{12} - \frac{\delta^6 y}{240} + \cdots. \tag{5}$$

The substitutions

$$\left[1 + \frac{h^2 g(x)}{12}\right]y = z, \qquad \frac{h^2 g(x)}{1 + \dfrac{h^2 g(x)}{12}} = G(x),$$

and

$$h^2 f(x) + \frac{h^2\delta^2 f(x)}{12} = F(x),$$

carry this equation into

$$\delta^2 z + G(x)z = F(x) - \frac{\delta^6 y}{240} + \cdots . \tag{6}$$

Now if the interval is small enough so that the term $\delta^6 y/240$ is negligible, equation (6) reduces to a simple linear difference equation of the second order.

2. Tabular arrangement. The procedure in solving (6) numerically is obvious. First we select a suitable value for h, then tabulate the values of $G(x)$ and $F(x)$ according to the scheme shown below:

x	$F(x)$	$G(x)$	z	Δz	$\Delta^2 z$
x_0	F_0	G_0	z_0		
x_1	F_1	G_1	z_1	Δz_0	
x_2	F_2	G_2	z_2	Δz_1	$\Delta^2 z_0 = \delta^2 z_1$
x_3	F_3	G_3			
x_4	F_4	G_4			

The first two values z_0 and z_1 are assumed to be known, and from them we get $\Delta z_0 = z_1 - z_0$. Then from (6) we have

$$\Delta^2 z_0 = \delta^2 z_1 = - G_1 z_1 + F_1,$$

and we complete the line for x_2 with the formulas,

$$\Delta z_1 = \Delta z_0 + \Delta^2 z_0,$$
$$z_2 = z_1 + \Delta z_1.$$

The integration proceeds by repetitions of this process. When it is completed, the values of y, or at least as many of them as are desired, are obtained from

$$y = \frac{z}{1 + \dfrac{h^2 g(x)}{12}} .$$

If the general solution of (1) is required it may be obtained by solving

$$\delta^2 u + G(x)u = 0, \qquad u_0 = 0, \qquad u_1 = h,$$
$$\delta^2 v + G(x)v = 0, \qquad v_0 = v_1 = 1,$$

and

$$\delta^2 w + G(x)w = F(x), \qquad w_0 = w_1 = 0.$$

The general solution of (6) is then

$$z = C_1 u + C_2 v = w,$$

and from z we readily obtain y.

3. Example. Consider the differential equation

$$y'' + \frac{3+4x}{16x^2}\, y = 0, \quad y = 1, \quad y' = 0, \quad \text{at} \quad x = 1.$$

Since $f(x)=0$ we need calculate only $G(x)$, which for $h=0.1$ proves to be

$$G(x) = \frac{12(3+4x)}{19200x^2 + (3+4x)}\cdot$$

The values of $G(x)$ are calculated from this expression and tabulated as shown below. We find by a few terms of Taylor's series that $y_1=0.997911$, so that $z_0=1.000364$, $z_1=0.998228$. Everything needful is now known and the calculation is completed to $x=2.0$ as follows:

x	$G(x)$	z	Δz	$\Delta^2 z$	y
1.0	.0043734	1.000364			1.000000
1.1	38211	.998228	−2136		
1.2	33845	.992278	−5950	−3814	
1.3	30318	.982970	−9308	−3358	
1.4	27417	.970682	−12288	−2980	
1.5	24995	.955733	−14949	−2661	
1.6	22945	.938395	−17338	−2389	
1.7	21190	.918904	−19491	−2153	
1.8	19673	.897466	−21438	−1947	
1.9	18349	.874262	−23204	−1766	
2.0	17185	.849454	−24808	−1604	.849333

The equation chosen above can be solved explicitly, and the particular solution desired proves to be

$$y = x^{1/4}[\cos(x^{1/2}-1) - 0.5\sin(x^{1/2}-1)].$$

The substitution of $x=2$ in this expression gives $y=0.8493293$ which differs only by about four units in the sixth decimal place from the value found by numerical ntegration.

AN INVERSE PROBLEM IN CORRELATION THEORY

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1. Introduction. The present problem, more of mathematical interest than practical, grew out of a question which may confront the instructor in statistics. He might, in concocting exercises in the computation of multiple and partial correlation coefficients, give the students a square array of numbers r_{ij} such that $r_{ij}=r_{ji}$, $r_{ij}^2 \leqq 1$, $r_{ii}=1$. If these have not actually been computed from data (observed or manufactured), *how can he be sure of the existence of a set of data which would give rise to such hypothetical correlation coefficients* r_{ij}? Under the restriction, usually desired in such exercises, that the determinant of the array be not zero (non-singular case), a simple criterion is found in Corollary 4 below. Otherwise, the general case (possibly singular) is considered. Sections 2 and 3 may be of some general interest inasmuch as the language of matrices is used to exhibit the dependence of the coefficients r_{ij} on the data and to state necessary conditions on these coefficients. In the course of proving the existence theorem in section 4, particular solutions are constructed and the most general solution is characterized. Some corollaries are drawn in section 5.

2. Matrix formulation. m sets of measurements on each of n variables may be written in the form of a matrix,*

$$Y^{m\times n} = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{pmatrix}, \tag{1}$$

where y_{ij} denotes the ith observation on the jth variable. A matrix (1) will be referred to as *a statistics problem*.

The present approach employs the concepts of unitary vectors and orthogonal vectors. A vector, ξ, that is, a column matrix, is unitary if $\xi'\xi=1$. Two vectors, ξ, η, are orthogonal if $\xi'\eta=0$. At this point it is convenient to define the unitary vector $\psi^{m\times 1}$ all of whose elements are $m^{-1/2}$. Then ψ' applied to the left of a matrix gives a row matrix equal to $m^{-1/2}$ times the sum of the rows of the original. $M=\psi\psi'$ is an $m\times m$ matrix all of whose elements are m^{-1}; applied to the left of a matrix it yields a new one all of whose rows are the same, and the mean of the rows of the old. *The matrix of means* is defined as

$$\bar{Y}^{m\times n} = MY. \tag{2}$$

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* Some of the notation differs from standard practice in order that the following scheme may be rigidly followed: Upper case Latin letters denote matrices; lower case Latin, scalars; lower case Greek (except the Kronecker delta), vectors. A superscript $s\times t$ attached to a matrix signifies that it has s rows and t columns. *All matrices and scalars are real.* A' means the transpose of A.

The matrix of deviations is defined to be

$$X^{m\times n} = Y - \overline{Y}, \tag{3}$$

or

$$X = DY, \quad \text{where} \quad D^{m\times m} = I^{m\times m} - \psi\psi',$$

and I is the identity matrix. We note

$$\psi' X = 0, \quad \text{since} \quad \psi' D = \psi' - \psi' = 0.$$

In order to obtain unitary vectors from the columns of X, we define $S^{n\times n}$ as that diagonal matrix whose jth diagonal element is the non-negative square root of the jth diagonal element of $X'X$, that is, $s_{jj} = (\xi_j' \xi_j)^{1/2}$, where ξ_j is the jth column of X. The diagonal matrix S^{-1} exists unless one of the columns of Y has all identical elements, and we now exclude this trivial case. *The matrix of normalized deviations** is now defined as

$$Z^{m\times n} = XS^{-1}. \tag{4}$$

We note again,

$$\psi' Z = \psi' XS^{-1} = 0. \tag{5}$$

Finally, *the correlation matrix* is defined to be

$$R^{n\times n} = Z'Z, \tag{6}$$

in agreement with the usual definition of r_{ij}, the element in the ith row and the jth column. The matrix R is thus a function of the statistics problem (1),

$$R = F(Y) = S^{-1}Y'D'DYS^{-1}. \tag{7}$$

3. Necessary conditions. It will be convenient to refer to the following as

Conditions $(\mathcal{C})$: $(\mathcal{C}-1)$: *R is symmetric.*
$(\mathcal{C}-2)$: *The diagonal elements of R are unity.*
$(\mathcal{C}-3)$: *R is positive, i.e., rank R = index R.*

A test for $(\mathcal{C}-3)$ if $(\mathcal{C}-1)$, $(\mathcal{C}-2)$ hold is found in Corollary 3 below.

LEMMA 1. *The conditions $(\mathcal{C})$ are necessary if $R = F(Y)$.*

The necessity of $(\mathcal{C}-1)$ follows from the rule for the transpose of a product, $(AB)' = B'A'$, applied to (6), that of $(\mathcal{C}-2)$ from the unitary property of the columns of Z. To prove $(\mathcal{C}-3)$, we note first that the determinants of the principal minors of R are non-negative, since† they are Gramian determinants formed

---

* The matrix of deviations in standard units is simply $T = m^{1/2}Z$, but we shall not need this in the present discussion. Also, the diagonal elements of S are $s_{jj} = m^{1/2}s_j$, where s_j is the standard deviation of the jth variable.

† Courant-Hilbert, Methoden der mathematischen Physik, Berlin, 1931, vol. 1, ch. 1.

from certain columns of Z. This in turn is sufficient* to insure that R is positive.

The lemma suggests the question, are the conditions $(\mathcal{C})$ sufficient that $R=F(Y)$? The theorem below contains the answer. To prove it we need another

LEMMA 2. *Rank* $R=$ *rank* $Z<m$.

Since the rank of a product cannot exceed the rank of a factor, $p(R)\leqq p(Z)$, where $p(A)$ denotes rank A. But we can pick $p(Z)$ independent columns from Z, and their Gramian matrix appears as a non-singular† minor in R. Hence $p(R)\geqq p(Z)$. This proves the equation in the lemma; the inequality follows from (5) which implies that the m rows of Z are dependent.

4. The existence theorem: *Given any matrix* $R^{n\times n}$ *of rank* p *and satisfying the conditions* $(\mathcal{C})$, *consider the possibility of a solution* $Y^{m\times n}$ *of the equation* $R=F(Y)$. *There is no solution with* $m\leqq p$, *but there are* ∞^q *solutions for each* $m>p$, *with* $q=2n+p(2m-p-3)/2$.

The negative aspect of the theorem follows immediately from our Lemma 2. To prove the positive statement we transform the problem. By a finite number of rational operations on the elements of R, and n square root extractions, a non-singular matrix $P^{n\times n}$ can‡ be constructed such that

$$P'RP = E,$$

a matrix whose first p diagonal elements are unity, and all other elements zero. A 1:1 correspondence is established between matrices $Z^{m\times n}$ and $V^{m\times n}$ by the equations

$$V = ZP, \qquad Z = VP^{-1}. \tag{8}$$

If

$$Z'Z = R \quad \text{and} \quad \psi'Z = 0, \tag{9}$$

then

$$V'V = E \quad \text{and} \quad \psi'V = 0, \tag{10}$$

and conversely.

Contemplating the transformed problem (10), we see that the first p columns of V, say $\kappa_1, \kappa_2, \cdots, \kappa_p$, must be unitary and mutually orthogonal, $\kappa_i'\kappa_j=\delta_{ij}$, and the remaining columns all zero. Hence if we write V as a composite matrix

$$V = (K^{m\times p}0^{m\times(n-p)}), \tag{11}$$

where K is the matrix whose columns are $\kappa_1, \kappa_2, \cdots, \kappa_p$, then

$$K'K = I^{p\times p}, \qquad \psi'K = 0. \tag{12}$$

* L. E. Dickson, Modern Algebraic Theories, N. Y., 1930, ch. 4.

† Courant-Hilbert, *loc. cit.*

‡ Dickson, *loc. cit.*

Consider the $(m-1)$-dimensional subspace $\mathcal{L}$ of m-space composed of vectors κ orthogonal to ψ, $\psi'\kappa=0$. A unitary orthogonal basis for $\mathcal{L}$ may be constructed as follows: Define the matrix $L^{m\times(m-1)}$ as that obtained by bordering the identity matrix of order $m-1$ with a row of $m-1$ elements each of which is -1. Obviously $\psi'L=0$. The $m-1$ independent columns of L form a basis for $\mathcal{L}$. To achieve a unitary orthogonal basis we can select* suitable linear combinations of these columns by a finite number of non-tentative steps (Schmidt procedure). If the coefficients in each linear combination are written as a column in a square matrix C of order $m-1$, then the columns of

$$U^{m\times(m-1)} = LC$$

form the desired basis for $\mathcal{L}$,

$$U'U = I \quad \text{and} \quad \psi'U = \psi'LC = 0.$$

Particular solutions of the problem (12) can now be obtained by choosing any p columns of U for the columns of K. The most general solution is of the type

$$K = UW^{(m-1)\times p},$$

where

$$W'W = I.$$

First, note that any such K is a solution,

$$K'K = W'U'UW = W'W = I, \qquad \psi'K = \psi'UW = 0;$$

secondly, that any solution is of this form: Suppose K satisfies the equations (12). Then its columns are in the space $\mathcal{L}$ and hence are linear combinations of the columns of U, $K=UW$. Furthermore, $I=K'K=W'U'UW=W'W$.

The number of distinct solutions for K is the same as the number of distinct matrices $W^{(m-1)\times p}$ for which $W'W=I$, since there is a 1:1 correspondence between possible K and W, given by

$$K = UW, \qquad W = U'K.$$

The restriction $W'W=I^{p\times p}$ puts p^2 conditions on the $p(m-1)$ elements of W. But since $W'W$ is automatically symmetric, only $p(p+1)/2$ of these conditions are independent. Hence the number of solutions for K is ∞^{q_1} where $q_1=p(m-1)-p(p+1)/2$. V is then uniquely determined by (11) and Z by (8). The n diagonal elements of S can now be chosen as arbitrary positive numbers, after which X is determined by

$$X = ZS.$$

Finally, the n elements of the equal rows of $\overline{Y}$ may be chosen arbitrarily, determining

* Courant-Hilbert, *loc. cit.*

$$Y = X + \overline{Y}.$$

The number of solutions for Y is thus ∞^q with $q = q_1 + 2n$.

5. Some corollaries. Corollaries of some inherent algebraic interest may now be deduced from our theorem by noting necessary conditions other than $(\mathcal{C})$ for $R = F(Y)$ and then asserting these as properties of any matrix satisfying $(\mathcal{C})$. We illustrate by using familiar properties of the multiple and partial correlation coefficients for the problem (1).

If η_1, η_2 are any vectors of m elements, then $\xi_1 = D\eta_1$, $\xi_2 = D\eta_2$ are the respective deviations from the means, and the correlation coefficient between the vectors is $r(\eta_1, \eta_2) = \xi_1' \xi_2 / (\xi_1' \xi_1 \cdot \xi_2' \xi_2)^{1/2}$, providing the denominator does not vanish. The Schwartzian inequality yields immediately $r^2 \leqq 1$. Since the coefficients of multiple and partial correlation may† be defined as $r(\eta_1, \eta_2)$, where η_1 and η_2 are certain linear combinations of the columns of Y, they satisfy the relation $0 \leqq r^2 \leqq 1$ whenever they are defined. Using well-known formulas for these coefficients, we get inequalities on determinants of minors of R of order $n-1$ and n. Further inequalities may be obtained by the device of considering a "sub-problem Y^*" of (1), where the columns of Y^* are a subset of the columns of Y. Then $R^* = F(Y^*)$ is a principal minor of R, say of order n_1, and the corresponding inequalities involve determinants of minors of R of order $n_1 - 1$ and n_1.

We now define *bordering sequences*

$$d_1, d_2, \cdots, d_n \tag{13}$$

as follows: Pick any principal diagonal element R_1^* of R, then any 2×2 principal minor R_2^* containing R_1^*, then any 3×3 principal minor R_3^* containing R_2^*, etc. A sequence (13) consists of the determinants of a sequence $R_1^*, R_2^*, \cdots, R$. In terms of

Conditions $(\mathcal{D})$: *Every bordering sequence* (13) *formed from R has the property*

$$1 = d_1 \geqq d_2 \geqq \cdots \geqq d_n \geqq 0,$$

we may state

LEMMA 3. *The conditions* $(\mathcal{D})$ *are necessary if* $R = F(Y)$.

That all determinants of principal minors are non-negative was noted in connection with Lemma 1. If $d_{j-1} > 0$, then $1 - d_j/d_{j-1}$ is the square of a multiple correlation coefficient r for a sub-problem Y^*, and since $r^2 \geqq 0$, $d_{j-1} \geqq d_j$. If $d_{j-1} = 0$, then $d_j = 0$, since d_{j-1} is the Gramian determinant of a subset of those columns of Z which lead to the Gramian determinant d_j. Hence in every case $d_{j-1} \geqq d_j$.

COROLLARY 1. *Any matrix R satisfying the conditions* $(\mathcal{C})$ *satisfies the conditions* $(\mathcal{D})$.

† Yule and Kendall, An Introduction to the Theory of Statistics, London, 1937, pp. 277 and 266.

By similar† reasoning applied to the partial correlation coefficients of sub-problems Y^*, we are led to the following: Let R^* be any principal minor of R (including R as a special case). Let d_{ij}^* be the determinant of the minor of R obtained by striking out the ith row and jth column of R.*

Conditions ($\mathcal{E}$): *For every* R^* *and every* $i, j \leqq$ *order* R,*

$$d_{ij}^{*2} \leqq d_{ii}^* \, d_{jj}^*.$$

LEMMA 4. *The conditions* ($\mathcal{E}$) *are necessary if* $R = F(Y)$.

COROLLARY 2. *Any matrix* R *satisfying the conditions* ($\mathcal{C}$) *satisfies the conditions* ($\mathcal{E}$).

We conclude with a procedure for answering the question raised in section 1:

COROLLARY 3. *A method for testing whether or not a given symmetric matrix* R *with main diagonal elements unity is a possible correlation matrix,* $R = F(Y)$, *is the following: Start constructing a bordering sequence* (13). *Suppose a stage is reached where* $d_1, d_2, \cdots, d_p$ *are all positive. If a* d_{p+1} *is picked such that* $d_{p+1} < 0$, R *is ruled out. If the chosen* $d_{p+1} = 0$, *others have to be evaluated.*‡ *If all* $d_{p+1} = 0$ *the test is always concluded at the next step: if some* $d_{p+2} \neq 0$, R *is ruled out; if all* $d_{p+2} = 0$, R *is a possible correlation matrix of rank* p.

An important special case of this is

COROLLARY 4. *Necessary and sufficient conditions that a symmetric matrix* R *with main diagonal elements unity be a non-singular correlation matrix are the* $n-1$ *conditions that the determinants*

$$\begin{vmatrix} 1 & r_{12} \\ r_{21} & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{vmatrix}, \cdots, |R|$$

be all positive.

It is clear that the determinants in Corollary 4 may be replaced by those from any other bordering sequence (13). This again implies that if R is to be a non-singular correlation matrix all off-diagonal elements must be numerically less than unity.

† We now use the second of the inequalities $0 \leqq r^2 \leqq 1$. A separate argument for Lemma 4 must again be made in the case where d_{ii}^* or $d_{jj}^* = 0$.

‡ If $p+1 =$ order R, there is of course a unique $d_{p+1} = |R|$. Then if $|R| = 0$, $R = F(Y)$ and rank $R = p$.

MEANS AND ENDS IN MATHEMATICS

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It seems to the writer that the outstanding weakness of mathematical instruction is its failure to make clear to the students the purpose, the aims, of various parts of the curricula, and its failure to show how the *means* used in mathematics follow naturally from those *ends*. In other words, while our courses are organized more or less logically from the postulational point of view, they are not so organized from the point of view more important to the student, that of meaning and purpose.

Although trigonometry or analytic geometry would exemplify these statements, let us use instead the pervasive subject of algebra. Much of the prevalent fear and dislike of algebra arises from our failure to make clear to students and to prospective students the *aims* of algebra, and from the inevitable resultant feeling that algebra is artificial, unnatural. A brief discussion of the purposes of algebra may help to substantiate this idea.

First, let us make a distinction between "basic arithmetic" and "applied arithmetic" and between "basic algebra" and "applied algebra." The purpose of basic arithmetic can be simply stated: it is to develop ability in performing the four fundamental operations, on whole numbers and on fractions. (This omits reference to taking roots, and to irrational numbers in general.) It is important for our discussion to note that, as far as the better students are concerned, basic arithmetic completely achieves its purpose. What, then, is the purpose of basic algebra? It must be different from that of basic arithmetic. Our students often think that the aim of algebra is to deal with a *new* kind of number, the literal number. But we know that letters are not a new kind of number, in fact are not numbers at all, but new names for numbers. What new purpose creates the need for these new names?

Sometimes our students think that the aim of algebra is to solve problems and find unknown numbers. (I have often been told by students with two or more years of college mathematics that algebra deals with unknown numbers, arithmetic with known numbers, as if the answer to an arithmetic problem were more "known" than that of an algebra problem!) They, of course, are thinking of applied algebra. This is a very important branch, but before one can have applied algebra, he must have some basic algebra to apply.

In any text we find a clue to the answer to our query. We see the equation $x+x=2x$, which is not a statement about unknown numbers, but about *all* numbers, just as "Man is born to die," though it uses the singular form of the noun, is a statement about all men. Most of algebra is, in a sense, an elaboration of the distributive law, $x(y+z)=xy+xz$, which is a truth about all triples of numbers (real or complex, at least). The purpose of basic algebra is to discover such truths, and to state them in a convenient language. Stated postulationally, one's aim is to discover certain statements, and deduce others from these, such that the totality forms a system suitable for "ordinary" applications.

Not even in college courses do students ordinarily learn to look upon "identities" as actual truths about numbers. This can be shown by a simple test. A group which apparently has mastered $4x+8x=12x$, i.e., which "knows" that it is an identity, is asked "If, for some reason, you wished to add 5 times 17 to 3 times 17 without first multiplying, what would you get?" A large majority of the answers are "8 times 34." Other examples could be given, but the fact is too well known to require further illustration.

A student who understands that basic arithmetic has achieved its aim, and that algebra has this new aim, hardly touched before, of discovering the general truths or "laws" of numbers sees readily the *need* for new words "x," "y," etc., in algebra. He sees how clumsy it is to write or state in English that "If I take any two numbers and add them together, and multiply the result by a third, $\cdots$," i.e., the distributive law. Understanding the need for new words eliminates the apparent artificiality of algebra.

When we remember that it took thousands of years for mankind to discover such algebraic truths, and to formulate them in a convenient language, we become skeptical of the speed with which we expect our students to grasp these truths, and their formulation. What reason is there to expect such a grasp at all, if we do not state over and over the purpose of algebra, and of its identities, if we do not train our students to read the abstract language of algebra, with time and patience comparable to that which we use in their English training? If we do not help them, over a period of months or years, to assimilate the *truth* of equations such as $x(y+z)=xy+xz$, rather than to look upon it as a *problem* in multiplication, when read from left to right, or in factoring, when read from right to left?

A final conclusion from our analysis of the purpose of algebra is: algebra and arithmetic should be integrated. In no other subject do we spend eight years teaching specific facts, without ever hinting at the general truths underlying them. In no other field do we so artificially divorce theory and practice, when the goal is to integrate them. The language of algebra, when introduced in a week, with no comprehension of its purpose, is not understood by our students, and they fall back on memorizing. If introduced gradually, starting in the early grades, with emphasis always on the *meaning* of statements, it is not difficult, as the writer has verified. A ten-year old boy accepts the statements that "α is a new word, which means 'every number'," and "α plus α is always the same as 2 times α" as interesting and meaningful, and he can verify the latter for the numbers in his range of experience. Algebra, to him, has a purpose.

This note has used algebra as an example of its thesis. In other fields of mathematics the emphasis on meaning and purpose does not, in the writer's opinion, imply as drastic a reorganization of methodology but, in every case, such emphasis seems to be required by the very goals of teaching.

UNIVERSAL FUNCTIONS OF POLYGONAL NUMBERS*

LOIS W. GRIFFITHS, Northwestern University

1. The four-square theorem of Lagrange. Among all positive integers an interesting subset, from the point of view of additive number theory, is the set of squares, that is, the integers 1, 4, 9, 16, 25, $\cdots$. Thus, for example, the integer 7, which is not in the subset of squares, is a sum of four integers in the subset, since $7=4+1+1+1$. No fewer than four summands would yield 7, and no other summands, apart from the order in which they are written in the equation, would yield 7. The integer 14, however, requires only three summands, and can not be obtained with exactly four summands. The integer 19 can be obtained with three summands or with four summands. Other positive integers require only two summands. The integer 7, for which three summands do not suffice, is not an isolated case. It was first proved by Gauss that three squares do not suffice for any of the positive integers in the infinite arithmetic progression $8n+7$. The general fact that four squares suffice for all positive integers was proved by Lagrange in 1770. This theorem, and many similar theorems, can be stated more simply if the integer 0 is permitted as a summand. Then the fundamental Lagrange theorem is that every positive integer is a sum of four squares.

This Lagrange theorem will be restated in terms that are more useful for purposes of generalization. The fact that a positive integer A is a sum of four squares is equivalent to the fact that the equation $A=x^2+y^2+z^2+w^2$ has a solution in integers which are greater than or equal to zero. It is then said that A is represented by the function $x^2+y^2+z^2+w^2$. A function which represents every positive integer is said to be *universal*. Hence the Lagrange theorem states that $x^2+y^2+z^2+w^2$ is *universal*.

2. Fermat-Cauchy theorem on polygonal numbers. The polygonal numbers of order 4 are precisely the squares, 0, 1, 4, 9, 16, $\cdots$. In general, if m is a positive integer, fixed but arbitrary, then the polygonal numbers of order $m+2$ are the integers which are the values of the expression $x+m(x^2-x)/2$ when x takes the values 0, 1, 2, 3, $\cdots$. Thus the polygonal numbers of order $m+2$ are the integers 0, 1, $m+2$, $3m+3$, $6m+4$, $10m+5$, $\cdots$.

The Lagrange theorem on representation of integers by sums of squares is a particular case of a general theorem on representation of integers by sums of polygonal numbers of order $m+2$. The first published proof of this general theorem was by Cauchy, but the theorem was first stated by Fermat. The Fermat-Cauchy theorem will be stated after it is illustrated further.

If $m=1$, the polygonal numbers of order 3 are the integers 0, 1, 3, 6, 10, 15, $\cdots$. They are the triangular numbers. The equations $14=10+3+1$, $7=6+1+0$, $19=15+3+1$ illustrate the general fact that every positive integer

* Delivered at a meeting of the Mathematical Association of America, Chicago, Illinois, September 1, 1941.

is a sum of three triangular numbers. It is not possible to use fewer than three triangular numbers to obtain 14.

If $m=6$ the polygonal numbers of order 8 are the integers 0, 1, 8, 21, 40, 65, $\cdots$. The equations $15=1\cdot 8+7\cdot 1$ and $19=2\cdot 8+3\cdot 1$ illustrate the general fact that every positive integer is a sum of 8 polygonal numbers of order 8, and that there are integers for which fewer polygonal numbers of order 8 would not suffice.

The expression $x+m(x^2-x)/2$ which defines the polygonal numbers of order $m+2$ is denoted by $p(x)$. Also p_1 denotes $p(x_1)$, p_2 denotes $p(x_2)$, and so on, but p_2 is not necessarily different from p_1. In these $p_1, p_2, \cdots$ the same m is used, although it is not explicitly indicated in the notation. The general Fermat-Cauchy theorem states that *every positive integer is a sum of $m+2$ polygonal numbers of order $m+2$, and that no fewer summands suffice for all positive integers.* These facts may be stated by saying that the function $p_1+p_2+\cdots+p_{m+2}$ is universal, and that no such function in fewer that $m+2$ variables is universal.

3. Dickson's generalization of the theorem of Lagrange. One generalization of the Lagrange theorem is illustrated by the function $x^2+y^2+z^2+2w^2$ and by the function $x^2+y^2+2z^2+3w^2$. Each of these functions is universal. In 1927 Dickson proved that there are exactly fifty-four such functions in four variables which are universal. If a function is obtained from one of these fifty-four functions by adjoining a term ct^2, in which c is a positive integer and t is a fifth variable, then the new function is trivially universal. Dickson proved that there are exactly six such functions in five variables, and no such functions in six or more variables, which are universal and which are such that no function obtained by deleting one or more terms is universal. Since no such function in three or fewer variables is universal, the problem of representation of integers by such functions $ax^2+by^2+cz^2+\cdots$ was completely solved.

4. Generalization of the Fermat-Cauchy theorem. In 1930 the author of this paper published the complete solution of the analogous generalization of the general Fermat-Cauchy theorem on polygonal numbers if $m\geqq 3$ and the sum of the coefficients is not greater than $m+2$. The universal functions $ap_1+bp_2+ep_3$ of triangular numbers were determined by Liouville in 1862. The general results for $m\geqq 3$ will be illustrated for $m=5$ and for $m=6$ before they are stated.

If $m=5$ the polygonal numbers of order 7 are the integers 0, 1, 7, 18, 34, 55, $\cdots$. The function $p_1+p_2+p_3+2p_4+2p_5$ is universal. This function will be designated by (1, 1, 1, 2, 2), since the coefficients are under investigation. That this function represents 13, for example, is indicated by the equation $13=1\cdot 7+1\cdot 1+1\cdot 1+2\cdot 1+2\cdot 1$. The other universal functions for $m=5$ are the Fermat function (1, 1, 1, 1, 1, 1, 1) and the functions (1, 1, 1, 1, 1, 2) and (1, 1, 1, 1, 3). That the function (1, 1, 1, 4) is not universal is illustrated by the fact that $17=2\cdot 7+3\cdot 1$ is the only representation of 17 as a sum of seven or fewer non-zero polygonal numbers of order 7.

If $m=6$ the polygonal numbers of order 8 are, as stated previously, 0, 1, 8, 21, 40, 65, $\cdots$. The universal functions are (1, 1, 1, 2, 3), (1, 1, 1, 1, 1, 3), (1, 1, 1, 1, 2, 2), (1, 1, 1, 1, 1, 1, 2), and the Fermat function (1, 1, 1, 1, 1, 1, 1, 1). It is clear that the representation of a positive integer A by the function (1, 1, 1, 2, 3) implies its representation by the function (1, 1, 1, 1, 1, 3), because the solution of the equation $A=p_1+p_2+p_3+2p_4+3p_5$ in integers $x_1, \cdots, x_5$ implies the solution of the equation $A=p_1+p_2+p_3+p_4+p_5+3p_6$ in integers $x_1, \cdots, x_6$. Hence the universality of the function (1, 1, 1, 2, 3) implies that of each of the other universal functions. It was not possible to use this fact in the proof, because the details of the proof were much more complicated for functions not having the first four coefficients each unity. The fact is mentioned, however, to emphasize the fact that the universality of each of the functions different from the Fermat function implies that of the Fermat function.

These results for $m=6$ constitute five theorems, including the Fermat theorem. Each of the other four theorems implies the Fermat theorem. The results for $m=5$ constitute four theorems, including the Fermat theorem. Each of the other three theorems implies the Fermat theorem. A contrast in the cases $m=5$ and $m=6$ is to be noted. The universal functions for $m=6$ can all be obtained from the one universal function having the fewest number of variables, that is from (1, 1, 1, 2, 3) by the method of partition illustrated above. However, for $m=5$ there are two universal functions, namely (1, 1, 1, 2, 2) and (1, 1, 1, 1, 3), each having the fewest number of variables, neither of which can be derived from the other by partition.

If m is a fixed but arbitrary integer which is greater than or equal to three, and n is the number of variables in a function, then the function is written $(a_1, a_2, \cdots, a_n)$. The coefficients $a_1, \cdots, a_n$ are integers each greater than or equal to unity, and have been arranged for convenience so that $a_1 \leqq \cdots \leqq a_n$. The sum $a_1+\cdots+a_k$ of the first k coefficients is denoted by w_k. The positive integer A is represented by the function $(a_1, \cdots, a_n)$ if and only if there exist integers $x_1, \cdots, x_n$, each of which is greater than or equal to zero, such that $A=a_1p_1+\cdots+a_np_n$. The function is universal if and only if it represents every positive integer.

The general results on universal functions of polygonal numbers, if $m \geqq 3$ and the sum of the coefficients is not greater than $m+2$, are easily stated, using these notations. If $w_n<m+2$ then there are no universal functions. If $w_n=m+2$ then the universal functions are the following functions: (1, 1, 1, 2); (1, 1, 1, 1, 3); $(1, 1, 1, 1, a_5, \cdots, a_n)$, with $a_5=1$ or 2, and with $a_k \leqq w_{k-1}-1$ $(5 \leqq k \leqq n)$; and $(1, 1, 1, 2, a_5, \cdots, a_n)$, with $a_k \leqq w_{k-1}-1$ $(5 \leqq k \leqq n)$.

The author of this paper has just published a determination, for $m \geqq 3$, of the minimum value N of n and a characterization of m such that there is a unique universal function having $n=N$. The results are that the minimum n is the integer N uniquely defined by the inequalities $2^{N-3}-1<m \leqq 2^{N-2}-1$, and that there is a unique universal function having $n=N$ if and only if $m=3, 4, 2^{N-2}-2, 2^{N-2}-1$. If $m \geqq 3$ then the integers $1, 1, 1, 2, 2^2, \cdots, 2^{N-4}, m+1-2^{N-3}$, after

they have been arranged in order of increasing magnitude, are the coefficients of a universal function having $n=N$. If $m=5$ then $N=5$ and the two universal functions having $n=N$ were noted to be (1, 1, 1, 2, 2) and (1, 1, 1, 1, 3). If m is odd and greater than 5 and not equal to $2^{N-2}-1$, then a set of integers which yield, after they have been arranged in order of increasing magnitude, a universal function having $n=N$ which is different from the function mentioned above, is the set 1, 1, 1, 2, $2^2, \cdots, 2^{N-5}, b, c$ with $b=(m+1-2^{N-3})/2$ and $c=m+1-2^{N-4}-b$. If m is even and greater than 4 and not equal to $2^{N-2}-2$ or 2^{N-3}, then this last set of integers with $b=(m+2-2^{N-3})/2$ and $c=m+1-2^{N-4}-b$ yields such a function. If $m=2^{N-3}$ then this last set of integers with $b=2$ and $c=m+1-2^{N-4}-b$ yields such a function.

The author has investigated universal functions of polygonal numbers of order $m+2$ in which $w_n \geqq m+3$. The methods of proof for the case $w_n \leqq m+2$ in general do not apply to the case $w_n \geqq m+3$. The proofs of the preceding facts for the case $w_n \leqq m+2$, and other references, will be found in the author's papers *A generalization of the Fermat theorem on polygonal numbers* and *A note on representation by polygonal numbers*.*

DISCUSSIONS AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

DIFFERENTIALS

MARK KAC AND J. F. RANDOLPH, Cornell University

As easy and desirable as it is to get along without differentials, they seem to be with us to stay. The fact that differentials enable one to arrive at correct results when he thinks of dx and dy as "little pieces of x and y" makes him believe he is thinking correctly and he therefore will not give up his concept of differentials because some pedantic mathematician tells him he is just lucky.

Granted, then, that we are not going to dispense with differentials, can we not do something to help students understand what they are? The unsophisticated undergraduate who tries to believe everything he reads and his instructor tells him is hopelessly confused. One day he tries to believe (but does not succeed) that dy/dx is *not* dy divided by dx. The next day he may have momentary comfort when he learns that it is true after all that dy/dx is dy divided by dx and that $\lim_{\Delta x \to 0} \Delta y/\Delta x = dy/dx$. However, he is then told or, heaven forbid, sees "proved" that $dx = \Delta x$. He may admire the cleverness of dx for being able to

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* Annals of Mathematics, (2), vol. 31, 1930, pp. 1–12, and a current issue of the Bulletin of the American Mathematical Society.

"remain constant on one side of an equation and (since $dx = \Delta x$) approach zero or the other," but it bothers him just the same.

In the usual way of representing a function (please do not fight with us about this functional notation) let x be the independent variable and $y = f(x)$ the dependent variable. Then let Δx be a number $\neq 0$ and such that $f(x + \Delta x)$ is defined but otherwise let Δx be arbitrary. Consider the number $f(x + \Delta x) - f(x)$ (if x insists on living up to its reputation as a variable, replace x by x_0), let

$$\Delta y,\ \Delta f,\ \Delta f(x)$$

be alternative notations for this number, and define each of the symbols

$$D_x y,\ D_x f,\ D_x f(x),\ y',\ f',\ f'(x)$$

to be the following limit, assumed to exist,

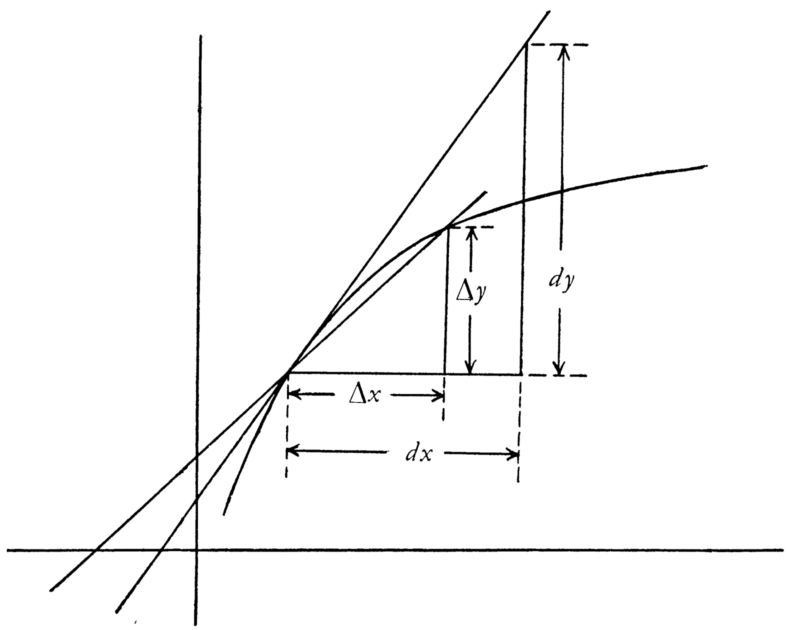
$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\,.$$

Above all things do not define dy/dx to be this limit.

Now let dx be an absolutely arbitrary number and define

$$dy = f'(x)dx.$$

Clearly if $dx = 0$ then $dy = 0$, and if $dx \neq 0$ then $dy/dx = f'(x)$. Then from the very



beginning dy/dx, $dx \neq 0$, is dy divided by dx.

Since Δx and dx are quite arbitrary, there is no inherent reason why they

should be the same, and we therefore first take them different. Consequently, instead of the usual picture illustrating the relation of dy and Δy we have the geometric interpretation given in the figure. Since the slope of the chord approaches the slope of the tangent as $\Delta x \to 0$ we clearly have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx},$$

and there is no worry about a strain on dx as $\Delta x \to 0$.

If now we wish to use differentials as approximations to increments, it should be clear to the student that dy is an approximation to Δy if we choose, as we may, $dx = \Delta x$ and "small." Also if it is thought that infinitesimals are desirable in this connection, then write

$$\lim_{dx=\Delta x \to 0} \frac{dy - \Delta y}{\Delta x} = \lim_{dx=\Delta x \to 0} \frac{dy}{dx} - \lim_{dx=\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
$$= \lim_{dx=\Delta x \to 0} f'(x) - f'(x) = f'(x) - f'(x) = 0,$$

and thus have that $dy - \Delta y$ is an infinitesimal of higher order than $\Delta x (= dx)$.

This definition of dx and dy amounts to what is essentially the Leibnitz definition of differentials; namely, dx is arbitrary and

$$dy = \lim_{t \to 0} \frac{f(x + tdx) - f(x)}{t}.$$

Differentials are, to be sure, not a great help when dealing with a function of one variable, but they become more important in the theory of functions of two or more variables. However, if differentials are introduced early in the course, their use will seem more natural to the student in later more complicated situations. It should, from the very beginning, be emphasized that a differential of an independent variable is itself independent, but a differential of the dependent variable is not independent but is determined.

In a course in mechanics when "virtual displacement", i.e., displacement along the tangent to a path, is introduced, the symbols δx and δy are generally used for the x- and y-components of this displacement. The reason why two symbols, δ's and d's, should be introduced when the algebraic manipulations are the same is never clear to the student. The student is right in this case; the δ's are only differentials and this symbol is therefore superfluous.

AN INDUCTIVE PROOF OF BUDAN'S THEOREM

M. F. Smiley, Lehigh University

We shall be concerned with Budan's theorem in the following form:

If $f(x)$ is a polynomial of degree n with real coefficients, and $V(x)$ is the number of variations in sign of the sequence

$$f(x), f'(x), f''(x), \ldots, f^{(n)}(x),$$

then the number of real roots of $f(x)=0$ *on the interval* $a<x\leqq b$ *(a root of multiplicity* μ *being counted as* μ *roots) is* $V(a)-V(b)-2k$, *where* k *is a positive integer or zero.*

The proofs of this theorem which are given in the elementary textbooks usually involve a rather intricate examination of the behavior of the derivatives of $f(x)$. We present here an inductive proof involving a minimum of such detail.

Budan's theorem, as stated, is an immediate consequence of the following lemma.

LEMMA. *If* α *is a root of* $f(x)=0$ *of multiplicity** μ $(\mu\geqq 0)$, *then* $V(\alpha+0)=V(\alpha)$ *and* $V(\alpha-0)=V(\alpha)+\mu+2k(\alpha)$, *where* $k(\alpha)$ *is a positive integer or zero.*

Proof. We employ induction on the degree, n, of $f(x)$. If $n=1$, the lemma is obviously true. Suppose that it holds for all polynomials of degree at most $k-1$. Consider a polynomial $g(x)$ of degree k. Let $V_1(x)$ be the number of variations in sign of the sequence (1) with the first term suppressed and the f's replaced by g's. We divide the proof that $g(x)$ has the property of our lemma into two cases.

Case 1. $\mu>0$. The hypothesis of our induction applies to $g'(x)$, $V_1(x)$, and the root $x=\alpha$ of $g'(x)=0$ of multiplicity $\mu-1\geqq 0$. We obtain $V_1(\alpha+0)=V_1(\alpha)$ and $V_1(\alpha-0)=V_1(\alpha)+\mu-1+2k_1(\alpha)$, with $k_1(\alpha)$ a non-negative integer. But it is clear that $g(x)$ and $g'(x)$ have opposite signs just before and equal signs just after $x=\alpha$. Hence $V_1(\alpha-0)+1=V(\alpha-0)$ and $V_1(\alpha+0)=V(\alpha+0)$. But, since $g(\alpha)=0$, we have $V(\alpha)=V_1(\alpha)$. The desired result for $g(x)$ follows easily.

Case 2. $\mu=0$. Here $g(\alpha)\neq 0$, and the application of the hypothesis of our induction to $g'(x)$ and $V_1(x)$ yields $V_1(\alpha+0)=V_1(\alpha)$ and $V_1(\alpha-0)=V_1(\alpha)+\nu+2k_1(\alpha)$, with $k_1(\alpha)$ a non-negative integer and $\nu\geqq 0$ the multiplicity of the root $x=\alpha$ of $g'(x)=0$. If $\nu=0$, then $g'(\alpha)\neq 0$, and hence $V(\alpha-0)-V_1(\alpha-0)=V(\alpha)-V_1(\alpha)=V(\alpha+0)-V_1(\alpha+0)$; from which the conclusion of our lemma for $g(x)$ follows easily. On the other hand, if $\nu>0$ we note that $g^{(\nu+1)}(\alpha)\neq 0$ and that $g'(\alpha)=g''(\alpha)=\cdots=g^{(\nu+1)}(\alpha)=0$, while $g'(x), g''(x), \cdots, g^{(\nu+1)}(x)$ alternate in sign just before $x=\alpha$ and all have the same sign just after $x=\alpha$. Hence $V(\alpha+0)-V_1(\alpha+0)=V(\alpha)-V_1(\alpha)$, which proves that $V(\alpha+0)=V(\alpha)$. If ν is even we see that $g'(x)$ and $g^{(\nu)}(x)$ have the same sign just before $x=\alpha$. Whether or not this is the sign of $g(\alpha)$, we obtain the conclusion of our lemma with $2k(\alpha)=\nu+2k_1(\alpha)$. If ν is odd and $g(x)$ and $g'(x)$ have the same sign just before $x=\alpha$, then $V(\alpha-0)=V_1(\alpha-0)$ and $V(\alpha)=1+V_1(\alpha)$, and it follows that $2k(\alpha)=\nu-1+2k_1(\alpha)$ is effective. Finally, if ν is odd and $g(x)$ and $g'(x)$ have opposite signs just before $x=\alpha$, then $V(\alpha-0)=V_1(\alpha-0)+1$ and $V(\alpha)=V_1(\alpha)$; and consequently $2k(\alpha)=\nu+1+2k_1(\alpha)$ is effective. This completes the induction and the proof of our lemma.

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* For economy of statement we have designated a value $x=\alpha$ at which $f(x)\neq 0$ as "a root of $f(x)=0$ of multiplicity zero." The notation $V(\alpha+0)$ $[V(\alpha-0)]$ means the right-hand [left-hand] limit of $V(x)$ at $x=\alpha$.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

A New System of Reckoning which Turns at 8. By Emanuel Swedberg. Translated from a photostat copy of the original Swedish MS. by Alfred Acton. Philadelphia, Pa., Swedenborg Scientific Association, 1941. 34 pages. 60 cents.

This translation from an original manuscript written in 1718 will be of interest to two groups. Collectors of Swedenborgia will welcome this addition for its own sake while mathematicians will be interested from a theoretical and practical point of view. Students of Swedenborg will read the translator's preface with greater attention than the actual translation because of the generous quotations from two other papers of Swedenborg, setting the historical scene for this particular work. They will notice that the earlier spelling Swedberg is used since the change in spelling did not occur until after this paper was written.

Mathematicians will concentrate on the paper itself which contains a short preface by the author followed by a detailed discussion of the new system consisting of eight characters made up of letters. Directions for reading and writing the corresponding numbers are carefully given. Addition and multiplication tables are compiled with diagrams for Napier's sticks using the octonary system. There follow sections on applications to monetary problems and problems on weights and measures. Although the arguments are quite convincing, they would be more so if the translator had not confessed in his preface that Swedberg himself advocated in 1719 the use of the decimal system. It would have been apropos here to state that the decimal system for all weights and measures was introduced into Sweden in 1879 and became obligatory in 1889.

The translator has been very diligent in discovering errors in the manuscript. Unfortunately, there are several mistakes in the translation, some of them typographical. For the sake of those students of Swedenborg who may not be trained mathematicians, we list the mistakes noted in reviewing the paper:

p. 18: in the *Addition Table*, the end of the second line of the second column should read *ll* [*lyl*] instead of [*ll*] *lyl*.

p. 19: the last two problems in multiplication should be arranged as follows:

$$\begin{array}{r} n\,m \\ s\,n \\ \hline l\,s\,m \\ v\,o \\ \hline l\,o\,s\,m \end{array} \qquad\qquad \begin{array}{r} m\,v \\ l\,s \\ \hline l\,l\,f \\ m\,v \\ \hline f\,o\,f \end{array}$$

p. 22: in the example at bottom of page. The last two lines should have l and 1 under *Riksd.* instead of under *Mark.*

p. 28: line 2 from top. The expressions $l;m,nn;nm$; should be $(l);(m,nn);(nm)$. Also in line 6 from top the expressions $l;nn;mn$; should be $(l);(nn);(mn)$.

p. 28: footnote marked †. The computation should be arranged as follows:

$$\begin{array}{r}
\textit{Riksd.} \\
n\ n \\
m\ n \\
\hline
l\ s\ l \\
t\ n \\
\hline
f\ o\ f
\end{array}$$

p. 29: fourth line above footnotes: $1\frac{25}{64}$ should be $1\frac{29}{64}$.

p. 30: fourth line above footnotes: 3065 should be 3675.

p. 30: footnote marked ‡, next to last line should have the number 1 under the column headed 1/2 *quart.*

p. 33: line 10 should read: the cube root of l_0 is s.

HARRIET F. MONTAGUE

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT AND J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest, to J. S. Frame, Brown University, Providence, R. I.

BIBLIOGRAPHY ON METHODS OF APPORTIONMENT IN CONGRESS*

On November 15, 1941, President Roosevelt approved the automatic apportionment bill (H.R.2665, now Public Law 291) which replaces the method of major fractions (devised by Professor W. F. Willcox of Cornell in 1910) by the method of equal proportions (devised by Professor E. V. Huntington of Harvard in 1921). The twenty-year controversy thus brought to an end affords an instructive example of the application of the scientific method to a political problem. The following bibliography may therefore be a timely convenience for students of this problem.

For the most comprehensive list of numerical examples illustrating all known methods, see Huntington's *A Survey of Methods of Apportionment in Congress*, Senate Document No. 304, 76th Congress, 3rd Session, 1940 (Government Printing Office, 41 pages, $0.10). On legal aspects, see Z. Chafee, Jr.'s article on *Congressional reapportionment* in the Harvard Law Review, vol. 12, pp. 1015–1047,

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* Compiled from data supplied at our request by E. V. Huntington.

1929, and his forthcoming paper on *Reapportioning the house of representatives under the 1940 census* in the Proceedings of the Massachusetts Historical Society, vol. 66, 1942. See also L. F. Schmeckebier's book on *Congressional Apportionment*, Brookings Institution, Washington, 1941. On the theoretical side, the fundamental paper is Huntington's *The apportionment of representatives in congress*, Transactions of the American Mathematical Society, vol. 30, pp. 85–110, 1928. The text of the main part of the bill is given in the Congressional Record for October 20, 1941, p. 8261.

Hearings before the House Committee on the Census have been held as follows, but must be read with caution: December 1920 to January 1921 (H.R. 14498, etc.); January to February, 1927 (H.R. 13471); February 1928 (H.R. 130); February 1931 (H.R. 15983, etc.); February to March, 1940. Also, before the Senate Committee on Commerce, February to March, 1941 (H.R. 2665).

For the report of the census advisory committee, see the *Quarterly Publication of the American Statistical Association*, December, 1921, pp. 1004–1013; or the *Congressional Record* for April 7, 1926, pp. 6840–6842; or the 1927 *Hearings*, pp. 53–58; or the 1928 *Hearings*, pp. 80–85. For the report of the National Academy of Sciences, see the *Annual Report of the Academy* for 1928–1929, pp. 20–23; or the *Congressional Record* for March 2, 1929, p. 5059; or the 1940 *Hearings*, pp. 70–71.

For the origin of the method of equal proportions, see Huntington, *A New Method of Apportionment of Representatives*, Quart. Pub. Amer. Statistical Assoc., September 1921, pp. 859–870; or a brief abstract in the *Proceedings of the National Academy of Sciences*, vol. 7, pp. 123–127, April 1921. On the origin of the method of major fractions, see Willcox, *House Report No. 12*, 62nd Congress, 1st Session, April 15, 1911; or his presidential address as president of the American Economic Association in the *American Economic Review*, vol. 6, no. 1, supplement, pp. 1–16, March, 1916, or in the 1927 *Hearings*, pp. 79–86. See also F. W. Owens' paper *On the apportionment of representatives*, Quart. Pub. Amer. Statistical Assoc., December, 1921, pp. 958–968. On Dr. Hill's method of alternate ratios, see J. A. Hill, *House Report No. 12*, *loc. cit.*, 1911, and his statement accepting the method of equal proportions, on p. 16 of the 1927 *Hearings*.

An interchange of papers by Huntington and Willcox may be found in *Science*: Huntington, vol. 67, pp. 509–510, May 18, 1928; vol. 68, pp. 579–582, December 14, 1928; Willcox, vol. 69, pp. 163–165, February 8, 1929; Huntington, vol. 69, p. 272, March 8, 1929; Willcox, vol. 69, pp. 357–358, March 29, 1929; Huntington, vol. 69, pp. 471–473, May 3, 1929. See also Huntington, *American Political Science Review*, vol. 25, pp. 961–965, November, 1931. Also C. L. Dedrick's paper in the 1941 *Hearings*, pp. 42–44, and a short statement by Huntington, printed in the Appendix of the *Congressional Record* for April 28, 1941, page A 2053.

The controversial aspect of the situation has culminated in a tripartite article published in *Sociometry*, vol. 4, August, 1941: *The role of mathematics in*

congressional apportionment, pp. 278–282, by Huntington; a *Reply*, pp. 283–298, by Willcox; and a *Rejoinder*, pp. 299–301, by Huntington.

I DOUBT IT—A MATHEMATICAL CARD GAME

IRENE PRICE, Oshkosh State Teachers College

The cards for this game consist of a deck of 50 or more cards which may be made by cutting cardboard into small rectangles (or some other geometric figure) and placing a number from 0 to 7 on each one. The numbers are not written in the usual manner but in some symbolic way so that the players must do a little work before knowing the number. *Zero* may be written in the following ways: $1+\cos 180°$, $\log 1$, $\cos \pi/2$, $0-0$, $6°-1$, etc. *One* may be written i^4, $\cos 0°$, $\log_e e$, $\tan 225°$, etc. For *two* write $d(2x)/dx$, $4^{1/2}$, $\cos^2 45°$, ${}_2C_1$, $\sqrt{12}/\sqrt{3}$, $\sec \pi/3$, $2 \tan^2 225°$. For *three* write $\sqrt[4]{81}$, $3 \sin 90°$, $\sqrt{7+\sqrt{4}}$, $3(5)°$, $\cot^2 30°$, $6/\sqrt[3]{8}$, etc.

Choose the numbers in such a way that they will be within the ability of the group; that is, if students have not studied trigonometry, then omit those forms which contain trigonometric functions; if the students know calculus, add cards containing integrals and derivatives.

This game may be played by any number of players from 2 to 8. The cards are distributed to the players, one at a time, until all cards are given out. The dealer starts the game by placing a card face down on the table and calling it *zero*. The player to his left places a card face down on the table in front of him and calls "one"; the next player does likewise and says "two," and so on around the table. If at any time a person doubts that the player put down the number he declares, the challenger says, "I doubt it" at which the player exposes the card played. If it was the card declared, the challenger takes all of the cards the player has on the table in front of him; if it was not the card declared the player must take all of the cards the challenger has before him. The object of the game is to see who can get out of cards first. When all cards in the hand have been played, the player must pick up all the cards he has placed upon the table and play them again. After counting to seven the next player may lay down any number he wishes but the next players to his left must then lay down the next consecutive numbers until 7 is reached.

BOOKS FOR CLUBS

(*Continued from December 1938*)

46. *What is Mathematics? An Elementary Approach to Ideas and Methods*, by R. Courant and H. Robbins. New York, Oxford, 1941. 521 pages. \$3.75. Contains a wealth of material for new topics for discussion at club meetings. A few samples are: repeated reflections, drawing with mechanical instruments, Schwarz's triangle problem, Steiner's problem, Appollonius' problem, an instrument for doubling the cube.

47. *Fundamental Mathematics*, by D. Harkin. New York, Prentice-Hall, 1941.

434 pages. $3.00. Secondary school as well as college clubs will find many worthwhile subjects for consideration, such as: development of the number concept, the decimal system, finite modular arithmetic, stories about fractions, symbolism, continued fractions, tesselation, stellated polyhedra, and many others.

48. *Fundamentals of Mathematics*, by M. Richardson. New York, Macmillan, 1941. 525 pages. $3.25. One chapter entitled "Impossibilities and unsolved problems" discusses a number of popular subjects. Other interesting topics as well as references for further study can be found.

49. *Mathematics—Its Magic and Mastery*, by A. Bakst. New York, Van Nostrand, 1941. 790 pages. $3.95. Program committees interested in attractive topics for club meetings will find interest aroused in such chapter headings as: how to multiply and like it, Napier's escape from drudgery, every number has its fingerprint, algebra hits the jackpot, the perils of flatland, railroading among the stars. The chapter on ballistics contains an unusually simple introduction to the subject.

KAPPA MU EPSILON

FIFTH BIENNIAL CONVENTION

The fifth biennial convention of *Kappa Mu Epsilon* was held at Central Missouri State Teachers College, Warrensburg, Missouri, on April 18 and 19, 1941, with the *Missouri Beta* chapter as host. Twenty-three chapters were represented by one hundred and fifty-nine delegates and visiting members. The convention opened with a general assembly on Friday evening which included an address by Dr. W. C. Morris of Warrensburg entitled *Playing a little with electricity*, and a demonstration of a reed organ converted into an electronic organ by George W. Wood, a student at Warrensburg. The educational program on Saturday morning consisted of two student papers, *Projective measurements* by William Wallis of *Texas Alpha* and *Some interesting facts about the cycloid* by Jerry Falvey of *Louisiana Alpha*, and discussions by Professor O. J. Peterson of *Kansas Beta* and Professor E. R. Sleight of *Michigan Alpha* on *ABCD* and *Early English arithmetics*, respectively. Professor C. V. Newsom, national president, presided at all meetings. Professor Emmett Ellis of Warrensburg was the convention chairman.

A number of revisions of the constitution were made at the business meeting on Saturday afternoon. The fraternity voted to publish an official journal at least once a year. Sectional meetings were encouraged but mathematical contests were not favored. The convention concluded with a banquet on Saturday evening at which Albro F. Stepp, student at *Missouri Beta*, served as toastmaster.

The following were elected to serve as the officers for the next biennium:

President Pythagoras, Professor O. J. Peterson, *Kansas Beta*, State Teachers College, Emporia, Kansas

Vice-President Euclid, Professor E. D. Mouzon, *Texas Beta*, Southern Methodist University, Dallas, Texas

Secretary Diophantus, Professor E. Marie Hove, *Nebraska Alpha*, State Teachers College, Wayne, Nebraska

Treasurer Newton, Professor H. Van Engen, *Iowa Alpha*, State Teachers College, Cedar Falls, Iowa

Historian Hypatia, Miss Orpha Ann Culmer, *Alabama Beta*, State Teachers College, Florence, Alabama

Past President Zeno, Professor C. V. Newsom, *New Mexico Alpha*, University of New Mexico, Albuquerque, New Mexico.

UNDERGRADUATE PUBLICATIONS

The Pentagon, official journal for *Kappa Mu Epsilon*, national honorary fraternity in mathematics, made its initial appearance in the fall of 1941. Edited by Professor C. V. Newsom of the University of New Mexico, its sixty-eight pages contain articles on *Mathematics and national defense* by W. L. Hart, *Dyadic arithmetic* by H. D. Larsen, and *Robert Record's Whetstone of Witte* by E. R. Sleight as well as a section devoted to *The mathematical scrapbook* and a summary of activities of the twenty-six chapters. Professor E. R. Sleight of Albion College, Albion, Michigan, will edit all student papers submitted for publication. Professor E. A. Hazlewood, Texas Technological College, Lubbock, Texas, is making a collection of items for *The mathematical scrapbook.* News notes are submitted to Miss Orpha Ann Culmer, Alabama State Teachers College, Florence, Alabama or to Miss E. Marie Hove, Nebraska State Teachers College, Wayne, Nebraska. Clubs or individuals not members of the national fraternity may subscribe for the magazine at thirty cents a copy or one dollar for two years. Subscriptions and business communications should be sent to C. B. Barker, University of New Mexico, Albuquerque, New Mexico.

CLUB REPORTS, 1940–41

Pi Mu Epsilon, University of Arizona

The 40th chapter of *Pi Mu Epsilon* was installed on April 7, 1941 at the University of Arizona at Tucson. The installation ceremony was conducted by W. E. Milne, Director-General of *Pi Mu Epsilon*, with the assistance of Professor James B. Shaw. Thirty-one of the thirty-three charter members were initiated into membership during the ceremony. At a banquet following the installation, Professor Milne gave an address on *Primitive number systems.*

Oberlin Mathematics Club

References used in preparing papers for club meetings are always welcomed by this department. The report of the Oberlin Mathematics Club lists a number which were used. John Hammerle spoke on *Boolean algebra* and concluded his talk with applications given in the paper *Relations and reason* by W. V. Quine in the *Technology Review*, vol. 41, 1939, p. 299. Allen Strehler gave a demonstration of soap films with wire loops based on *Soap film experiments with minimal surfaces* by R. Courant, this MONTHLY, vol. 47, pp. 167–174. Roselyn Siegel found the article on map making in the *Encyclopedia Britannica*, 14th Edition, an excellent guide. Other speakers were Dr. R. W. Wagner on *Topology*, Bolton Strauch on *Applications of life and interest tables to the problems of life insurance and annuities*, Miss Orpah Clark on *The regular solids and related polyhedrons and crystals*, and William Hosier on *Continued fractions* (see Hall and Knight's *Higher Algebra*). In addition short biographical sketches of mathematicians chosen from Bell's *Men of Mathematics* were given at each meeting and a student contest held was based on twenty-five trick questions and puzzles and modeled on the plan of the radio program *Information Please.*

Regis College Mathematical Club

A typical meeting includes talks and demonstrations on popular mathematical subjects and some time devoted to recreational mathematics. Mary Kelly gave a demonstration of the ruler and compasses construction of a Clifford chain and Margaret McCarthy at a later meeting explained the analytical approach to the Clifford chain problem. Sister Thomas á Kempis gave a digest of *The walking polygot* which appeared in *Scripta Mathematica*, December 1939. Teresa Launie discussed *Mathematics and defense* in which she stressed the importance of training in mathematics and discussed applications in ballistics and used for reference the field and coast artillery manuals published by the U. S. Government Printing Office for the Army and Navy. Evelyn Carrellas described *The Snellius approximation and its construction and application to the three famous problems of antiquity*, from Dantzig's *Number, the Language of Science. The fourth dimension* was discussed by Eleanor Fleming, *Primitive counting* by Marjorie Sullivan, *Evariste Galois* by Gertrude McDonald, *The life of Pascal* by Cecelia Reininger, and *The trisection problem* by Helen Cleary.

Harold Young, superintendent of personnel of the Employers Insurance Group of Boston, spoke on *Insurance as a future for mathematics majors.* The club also participated in the activities of the Greater Boston Mathematics Clubs Association, acting as host for the April 1941 meeting. Entertainments at meetings were based on Ball's *Mathematical Recreations and Essays*, Jones' *Mathematical Wrinkles*, and Steinhaus' *Mathematical Snapshots.* Officers were: President, Irene Thomas; Vice-President, Cornelia Dinneen; Secretary, Eleanor Fleming; Treasurer, Marjorie Sullivan; Faculty Adviser, Sister Leonarda.

Mathematics-Physics Club, Haverford College

Topics discussed at meetings included: *The problem of apportionment of representatives* by R. B. Dickson, *Problems in television* by T. H. Chambers, *Fermat's last theorem* by R. Strohl, *Some figures in topology* by Dr. E. E. Betz, *Development of Wilson cloud chambers* by K. A. Wright, and *Linear Diophantine equations* by P. R. O'Connor. Guest speakers were Professor Arnold Dresden of Swarthmore College who gave a talk entitled *Intuition in mathematics*, and Dr. T. F. Anderson, RCA research fellow, who gave an illustrated lecture at a joint meeting of the Mathematics-Physics, Biology, Engineering, and Chemistry Clubs of the college on *The electron microscope.* Winners of the freshman prize examination were R. Day and E. C. Alvord, Jr. Officers were: President, G. R. Strohl, Jr.; Secretary, W. Franzen.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR. AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 506. *Proposed by R. V. Heath, Wall St., New York.*

Show that, for every positive integer n, the last $n+9$ digits of

$$90625^{2^n}$$

form an automorphic number. [See 1941, 407.]

E 507. *Proposed by V. Thébault, San Sebastián, Spain.*

In an orthocentric tetrahedron with orthocenter H and circumcenter O, show that the radical planes of the circumsphere with the respective spheres whose diameters are the four medians, meet the Euler lines of the corresponding faces in four points lying in a plane perpendicular to OH. (Cf. E 467. The *medians* of a tetrahedron join its vertices to the centroids of the opposite faces.)

E 508. *Proposed by R. K. Allen, Montpelier, Vermont.*

How many bridge hands are there where all thirteen tricks can be taken at no trump regardless of the distribution of the cards? It is assumed that declarer

will always play his highest cards and not intentionally lose any tricks. (Cf. E 448.)

E 509. *Proposed by the late J. E. Trevor, Cornell University.*

An architect is designing a house for his client's seventy-five foot lot. The "square bedroom" is to have a square floor, and it will contain an ordinary double bed and other furniture. The owner specifies that the music room shall be two feet longer than it is wide, and that its floor-area in square feet shall be three times that of the square bedroom. It is also specified that the widths of the two rooms shall be integer numbers of feet. Find these widths.

E 510. *Proposed by S. H. Gould, University of Toronto.*

Given $a_{p,q}=a_{p,q-1}+qa_{p-1,q}$, $a_{p,1}=a_{1,q}=1$, $(p, q=1, 2, 3, \cdots)$, prove

$$(p-1)a_{p,q}=\sum_{k=1}^{p-1}a_{p-k,q}\left\{(q+1)^{k+1}-\binom{p+q}{k+1}\right\}.$$

SOLUTIONS

E 464 [1941, 210]. *Proposed by Emma Lehmer, Berkeley, Calif.*

Prove that, for any prime $p>3$,

$$\binom{kp^{\alpha}}{np^{\beta}}\equiv\binom{kp^{\alpha-\gamma}}{np^{\beta-\gamma}}\pmod{p^{\alpha-\gamma+3}}.$$

Solution by the Proposer.

This result follows from repeated applications of the corresponding theorem in which $\gamma=1$, namely,

$$\binom{kp^{\alpha}}{np^{\beta}}\equiv\binom{kp^{\alpha-1}}{np^{\beta-1}}\pmod{p^{\alpha+2}}.$$

We first consider those factors of the binomial coefficient

$$\binom{kp^{\alpha}}{np^{\beta}}=\frac{kp^{\alpha}(kp^{\alpha}-1)(kp^{\alpha}-2)\cdots(kp^{\alpha}-np^{\beta}+1)}{1\cdot 2\cdots(np^{\beta}-1)np^{\beta}}$$

which are multiples of p; and after dividing out p from each factor of the numerator and denominator, we obtain

$$\frac{kp^{\alpha-1}(kp^{\alpha-1}-1)\cdots(kp^{\alpha-1}-np^{\beta-1}+1)}{1\cdot 2\cdots(np^{\beta-1}-1)np^{\beta-1}}=\binom{kp^{\alpha-1}}{np^{\beta-1}}.$$

The remaining factors can be grouped into batches of $p-1$ ratios each of factors lying between any two multiples of p, as follows:

$$\frac{\{kp^\alpha-(tp+1)\}\{kp^\alpha-(tp+2)\}\cdots\{kp^\alpha-(tp+p-1)\}}{(tp+1)(tp+2)\cdots(tp+p-1)}$$
$$=1-kp^\alpha\sum_{r=1}^{p-1}\frac{1}{tp+r}+k^2p^{2\alpha}\sum_{r\neq s=1}^{p-1}\frac{1}{(tp+r)(tp+s)}-+\cdots\equiv 1 \pmod{p^{\alpha+2}}.$$

For, it is well known that any elementary symmetric function of the first $p-1$ integers (or of any $p-1$ integers lying between two multiples of p) is divisible by p; moreover, the expression

$$\sum_{r=1}^{p-1}\frac{1}{tp+r}\equiv\sum_{r=1}^{p-1}\frac{1}{r}-pt\sum_{r=1}^{p-1}\frac{1}{r^2}$$

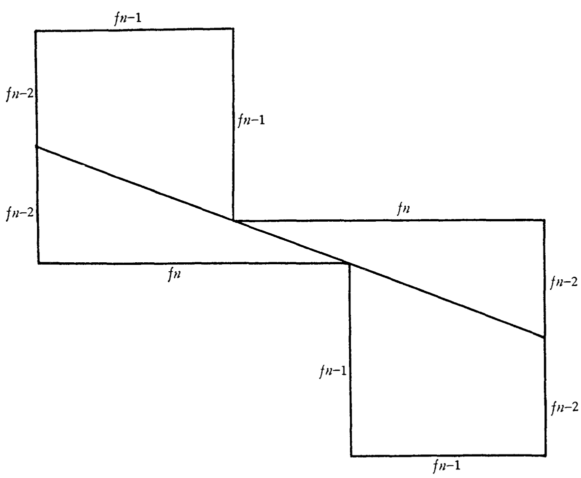
is divisible by p^2, since $\sum(1/r)$ is divisible by p^2 (for $p>3$) by Wolstenholme's theorem [1941, 269], and $\sum(1/r^2)$ is divisible by p.

E 468 [1941, 266]. *Proposed by W. R. Ransom, Tufts College.*

The Fibonacci numbers, defined by $f_1=f_2=1$, $f_{j+1}=f_{j-1}+f_j$, are known to yield a puzzle in which a square of side f_n is cut into four pieces which can apparently be rearranged to form a rectangle $f_{n-1}\times f_{n+1}$. Show that the same four pieces can be rearranged to form a figure which appears to consist of two rectangles $f_{n-1}\times 2f_{n-2}$ connected by a rectangle $f_{n-4}\times f_{n-2}$, the error being again one unit of area.

Solution by H. W. Eves, Pittsburgh, Pa.

We divide the given square into four pieces and rearrange these as in W. W. Rouse Ball's *Mathematical Recreations and Essays*, London, 1940, p. 85. To



show that the error in area is one unit, we observe that

$$(1)\qquad f_{n+1}f_{n-1}=f_n^2+(-1)^n.$$

This is readily proved by induction as follows. Assume that (1) holds for $n=k$; then we have

$$\begin{aligned} f_{k+2}f_k &= (f_{k+1}+f_k)f_k = f_{k+1}f_k + f_k^2 \\ &= f_{k+1}f_k + f_{k+1}f_{k-1} - (-1)^k \\ &= f_{k+1}(f_k + f_{k-1}) - (-1)^k \\ &= f_{k+1}^2 + (-1)^{k+1}. \end{aligned}$$

The second rearrangement, depicted above, is possible since

$$f_n - f_{n-1} = f_{n-2}$$

and

$$2f_{n-2} - f_{n-1} = f_{n-2} - (f_{n-1} - f_{n-2}) = f_{n-2} - f_{n-3} = f_{n-4}.$$

To show that the error in area is again one unit, we notice that

$$\begin{aligned} 4f_{n-1}f_{n-2} + f_{n-2}f_{n-4} &= 4f_{n-1}f_{n-2} + f_{n-2}(2f_{n-2} - f_{n-1}) \\ &= f_{n-2}(4f_{n-1} + 2f_{n-2} - f_{n-1}) \\ &= f_{n-2}(3f_{n-1} + 2f_{n-2}) \\ &= f_{n-2}(f_n + f_{n-2} + 2f_{n-1}) \\ &= f_nf_{n-2} + f_{n-2}^2 + 2f_{n-1}f_{n-2} \\ &= f_{n-1}^2 - (-1)^n + f_{n-2}^2 + 2f_{n-1}f_{n-2} \\ &= (f_{n-1} + f_{n-2})^2 - (-1)^n \\ &= f_n^2 - (-1)^n. \end{aligned}$$

Also solved by the proposer.

E 469 [1941, 266]. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Show that the exradii and circumradius of a triangle satisfy the identity

$$\sum \frac{a^2(b^2-c^2)}{r_a(r_b^2-r_c^2)} = 4R.$$

Solution by C. W. Trigg, Los Angeles City College.

From the well known relations

$$r = \frac{\Delta}{s}, \quad r_a = \frac{\Delta}{s-a}, \quad \frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}, \quad \text{and} \quad \Delta^2 = rr_ar_br_c,$$

it may be shown that

$$(r_b + r_c)(r_a - r) = a^2 \quad \text{and} \quad (r_b - r_c)(r_a + r) = b^2 - c^2.$$

It follows that

$$\sum \frac{a^2(b^2-c^2)}{r_a(r_b^2-r_c^2)} = \sum \frac{r_a^2-r^2}{r_a} = \sum r_a - r^2 \sum \frac{1}{r_a} = r_a + r_b + r_c - r = 4R.$$

(See Altshiller-Court, *College Geometry*, p. 73.)

Also solved by F. A. Alfieri, H. W. Bailey, W. E. Buker, H. W. Eves, Albert Furman, Edward Smith, E. P. Starke, and P. D. Thomas.

E 470 [1941, 266]. *Proposed by W. E. Buker, Pittsburgh Public Schools.*

Circle I has its center on another circle J. They intersect at A and C. From any point B on J, draw BC, intersecting I again at D. Prove that $BD=BA$.

Solution by D. H. Browne, Buffalo, N. Y.

Let O be the center of I. Since the arcs AO and CO are equal, BO bisects $\angle ABC$ (internally or externally). Hence D is the image of A by reflection in BO, and $BD=BA$.

Also solved by F. A. Alfieri, W. B. Clarke, H. A. DoBell, William Douglas, H. W. Eves, Albert Furman, F. W. Morgan, E. P. Starke, P. D. Thomas, C. W. Trigg, and the proposer.

E 471 [1941, 337]. *Proposed by L. G. Johnson, Ann Arbor, Michigan.*

A watch attached to a chain is swung around in a circle with the same angular velocity as that of its second hand. Show that the path traced by the tip of the second hand is a limaçon or a circle according as the sense of motion is clockwise or counterclockwise.

Solution by E. P. Starke, Rutgers University.

Let T be the tip of the second hand, and P its axis. Choose the fixed point of the chain (around which the watch is swung) as the origin, and let the positive x-axis be the direction in which PT points directly away from O. Let the lengths OP and PT be designated by a and b, respectively. Then at any instant the coördinates of P are $(a \cos \theta, a \sin \theta)$, and those of T are

$$(1) \qquad x = a \cos \theta + b \cos (\theta + \phi), \quad y = a \sin \theta + b \sin (\theta + \phi),$$

where θ is the angle POX and ϕ is the angle which PT makes with OP produced. The two cases of the problem are given by $\phi=\theta$ and $\phi=-\theta$. For $\phi=\theta$, (1) becomes

$$x = a \cos \theta + b \cos 2\theta, \qquad y = a \sin \theta + b \sin 2\theta,$$

or

$$x + b = (a + 2b \cos \theta) \cos \theta, \qquad y = (a + 2b \cos \theta) \sin \theta,$$

which is a limaçon, having the polar equation $r=a+2b \cos \theta$ if the pole is taken

at $(-b, 0)$. On the other hand, for $\phi = -\theta$, (1) becomes the circle

$$x = a \cos \theta + b, \qquad y = a \sin \theta,$$

with center $(b, 0)$ and radius a.

E 451 [1941, 65]. *Proposed by W. E. Buker, Pittsburgh Public Schools.*

There are three containers, having capacities of a, b, c quarts, where $a > b > c$ (positive integers). With the largest container full and the others empty, it is desired to divide the liquid into two equal portions, using these containers and no others. For what values of a, b, c is a solution possible?

Partial Solution by D. H. Browne, Buffalo, N. Y.

It is implied, of course, that a is even. The successive operations consist, essentially, in increasing or decreasing the amount of liquid in the largest container by b or c quarts. There is no loss of generality in assuming b and c to be relatively prime; for, any common divisor d must divide $\frac{1}{2}a$ also, and we merely have the problem for a/d, b/d, c/d, in terms of a new unit equal to d quarts. It is found that solutions are always possible when

$$b + c - 2 \leqq a \leqq 2(b + c).$$

(See Uspensky and Heaslet, *Elementary Number Theory*, p. 184.) The latter inequality is clearly necessary in order to be able to reduce the liquid in the largest container by as much as $\frac{1}{2}a$ quarts. There are certain "adventitious" solutions with $a < b + c - 2$; e.g. when $a = 22$, $b = 16$, $c = 9$, or $a = 40$, $b = 27$, $c = 17$. In these instances the successive amounts in the three containers are as follows:

22	0	0	40	0	0
13	0	9	13	27	0
13	9	0	13	10	17
4	9	9	30	10	0
4	16	2	30	0	10
20	0	2	3	27	10
20	2	0	3	20	17
11	2	9	20	20	0

Also solved to this extent by the proposer.

Editorial Note. The actual solution for given values of a, b, c can be written concisely by giving the successive amounts in the two smaller containers together. This amount is initially 0 and finally $\frac{1}{2}a$. Thus, in the above instances we have the sequences

$$0, 9, 18, 2, 11; \quad 0, 27, 10, 37, 20.$$

Adopting the notation of H. D. Grossman's *Generalization of the water-fetching puzzle* [1940, 374], we see that such a sequence occurs as part of the cycle of s's or σ's, defined as follows:

$$s_0 = 0, \quad s_1 = b, \quad s_{i+1} = s_i + b \text{ or } s_i - c \text{ according as } s_i < c \text{ or } s_i \geqq c.$$

The definition for σ_k is the same with b and c interchanged. The complete cycle of σ's is just the cycle of s's written backwards. If b and c are relatively prime, there are $b+c$ distinct numbers in the cycle, namely $s_0, s_1, \cdots, s_{b+c-1}$, and these are a permutation of $0, 1, \cdots, b+c-1$. Thus we can solve the problem if and only if, in running round the cycle, either forwards or backwards, from 0 to $\frac{1}{2}a$, we do not pass any number greater than a.

In the case when $a=22$, $b=16$, $c=9$, the cycle of s's (with the relevant part emphasized) is as follows:

0, 16, 7, 23, 14, 5, 21, 12, 3, 19, 10, 1, 17, 8, 24, 15, 6, 22, 13, 4, 20, **11**, **2**, **18**, **9**.

Here we could not have $a=20$ instead of 22, since it is impossible to go from 0 to 10 without passing 21 or 22.

Since the cycle contains no number greater than $b+c-1$, the forward and backward routes from 0 to $\frac{1}{2}a$ are *both* available whenever $a \geqq b+c-1$ (giving different solutions). When $a=b+c-2$, one of these routes is blocked by the number $b+c-1$, but the other is still open. The following theorem provides a sufficient condition for the existence of solutions with $a<b+c-2$:

If $c \equiv \pm r \pmod{b-c}$, *where* $0<r<\frac{1}{2}(b-c)-1$, *there are solutions for* $a=2(c+nr)$ *with* $n=0, 1, 2, \cdots$.

Proof. If $c=m(b-c)-r$, we have

$$s_{2h-1} = hb - (h-1)c \quad \text{and} \quad s_{2h} = hb - hc$$

for $h \leqq m$, but $s_{2m+1}=mb-(m+1)c=r$. The greatest of the first $2m+1$ s's is $s_{2m-1}=2c+r$. The next batch of s's is given by

$$s_{2m+1+k} = s_k + r \tag{1}$$

for $k \leqq 2m+1$, and the greatest of these is $s_{2m-1}+r=2c+2r$. The sequence continues to satisfy (1) so long as the "maxima"

$$s_{n(2m+1)-2} = 2c + nr$$

remain less than $b+c$. Hence we can proceed from $s_0=0$ to

$$s_{n(2m+1)-1} = c + nr = \tfrac{1}{2}a$$

without passing any number greater than $2c+nr=a-nr$.

If $c=m(b-c)+r$, we may suppose $m>0$ (since the solution is trivial when $\frac{1}{2}a$ is a multiple of c). Thus we have

$$\sigma_{2h-1} = hc - (h-1)b \quad \text{and} \quad \sigma_{2h} = (h+1)c - (h-1)b$$

for $h \leqq m+1$, including $\sigma_{2m+1}=(m+1)c-mb=r$ and $\sigma_{2m+2}=c+r$. The greatest of the first $2m+1$ σ's is $\sigma_2=2c$. Thereafter, we have $\sigma_{2m+1+k}=\sigma_k+r$ so long as the maxima

$$\sigma_{n(2m+1)+2} = 2c + nr$$

remain less than $b+c$. Hence we can proceed from $\sigma_0=0$ to $\sigma_{n(2m+1)+1}=c+nr$ without passing any number greater than $2c+nr$.

A proof that the above condition is necessary as well as sufficient, or a counter-example, will be welcomed.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis. Mo. All manuscripts should be typewritten, wiih double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4021. *Proposed by Orrin Frink, Jr., Pennsylvania State College.*

The differential operator D^2+1 may be factored in many ways; for example, it may be written $(D+\cot x)(D-\cot x)$, or $(\sec x\cdot D)(\cos x\cdot D+\sin x)$, or $(\sin x\cdot D+2\cos x)(D\csc x)$. Show that the most general method of expressing the differential operator $(D+a)^2+b^2$ as the product of two real first order differential operators is given by the formula

$$(D + a)^2 + b^2 = r^{-1}[D + a - b\tan(bx + c) - r'/r]\cdot r[D + a + b\tan(bx + c)],$$

where a, b, and c are real numbers, and r is a differentiable function of x.

4022. *Proposed by H. K. Humphrey, Winnetka, Ill.*

Derive a formula for the number of sets of p distinct integers taken from the first n positive integers, each set of p integers having the given sum S. Thus, for $n=16$, $p=4$, $S=34$, the number is 86.

4023. *Proposed by J. A. Greenwood, Duke University.*

Find an expression for the determinant of order $2n$

$$\begin{vmatrix} \theta I_n, & A_n \\ A_n, & \theta I_n \end{vmatrix},$$

where θI_n is a square matrix of order n having the variable θ in the principal diagonal and zeros elsewhere, and A_n is a square symmetric matrix of order n with a for each principal diagonal element, unity for the elements in the two parallels immediately above and below this principal diagonal, and zeros elsewhere.

4024. *Proposed by N. A. Court, University of Oklahoma.*

Given two twin tetrahedrons $(T) \equiv ABCD$, $(T') \equiv A'B'C'D'$ (see the proposer's *Modern Pure Solid Geometry*, p. 58, art. 191), consider the tetrahedron (A') formed by the face BCD of (T) and the three planes forming the trihedral angle A' of (T'); let (B'), (C'), (D') be the analogous tetrahedrons for the vertices B', C', D' of (T'). Show that the twelve-point spheres of the tetrahedrons (A'), (B'), (C'), (D') (*ibid.*, p. 251, art. 764) are tangent to the twelve-point sphere of (T).

4025. *Proposed by V. Thébault, San Sebastián, Spain.*

Let A_1', A_2', $\cdots$, A_{2n}' be the vertices of equilateral triangles constructed externally (or internally) on the sides A_1A_2, A_2A_3, $\cdots$, $A_{2n}A_1$ of a plane polygon of $2n$ sides $(P) \equiv A_1A_2 \cdots A_{2n}$, and M_1, M_2, $\cdots$, M_n be the midpoints of the principal diagonals A_1A_{n+1}, A_2A_{n+2}, $\cdots$, A_nA_{2n} of (P). The midpoints M_1', M_2', $\cdots$, M_n' of the principal diagonals $A_1'A_{n+1}'$, $A_2'A_{n+2}'$, $\cdots$, $A_n'A_{2n}'$ of the polygon $(P') \equiv A_1'A_2'$, $\cdots$, A_{2n}' are the vertices of equilateral triangles constructed upon the sides of the polygon $(p) \equiv M_1M_2 \cdots M_n$.

Generalize by replacing the equilateral triangles by similar isosceles triangles.

4026. *Proposed by V. Thébault, San Sebastián, Spain.*

(1) Construct a triangle ABC knowing a, A and given that the median and symmedian from A are perpendicular and parallel to two given directions. (2) Indicate the properties of this special triangle. (3) Let B' and C' be the projections of B and C on a variable straight line AP which cuts BC in P. The locus of the harmonic conjugate of P with respect to B' and C' is a right strophoid having the vertex A for a double point and tangent to the bisectors of angle A.

SOLUTIONS

3961 [1940, 323]. *Proposed by V. Thébault, San Sebastián, Spain.*

Each face angle of a given trihedral angle $O-XYZ$ is $\pi/3$, and on the respective edges the points A, B, C are located. Show that the Monge point of the tetrahedron $OABC$ describes a sphere as A, B, C vary on the edges so that $OA^2+OB^2+OC^2$ remains constant.

Solution by the Proposer.

This interesting problem is an application of a known formula, but seldom used, which expresses the circumradius R of a tetrahedron $OABC$ in terms of the lengths of the edges a', b', c' through O and the angles α, β, γ between these edges, namely,

$$4V^2R^2 = \sum a'^2 \sin^2 \alpha + 2\sum b'c'(\cos \beta \cos \gamma - \cos \alpha), \tag{1}$$

where $V^2 = 1+2 \cos \alpha \cos \beta \cos \gamma - (\cos^2\alpha+\cos^2\beta+\cos^2\gamma)$. Setting each of the

three angles equal to 60°, we have

$$8R^2 = 3\sum a'^2 - 2\sum b'c'. \tag{2}$$

After introducing the edges $BC=a$, $CA=b$, $AB=c$, we find that

$$8R^2 = 2\sum a^2 - \sum a'^2. \tag{3}$$

Then using the formulas (6) in this MONTHLY [1935, 430], we find that

$$8R^2 - \sum a'^2 = 2\left[\sum a^2 - \sum a'^2\right] = 8\left[R^2 - (O\Omega)^2\right],$$

where Ω is the Monge point. Finally we have

$$(O\Omega)^2 = \tfrac{1}{8}\sum a'^2,$$

and this justifies the proposition of the problem.

Editorial Note. A solution may be obtained without the use of the circumradius R; and for convenience we alter the notation. Let the three points be A_i, $(i=1, 2, 3)$, determined by the vectors $OA_i=a_i\mathbf{e}_i$, where $\mathbf{e}_i$ is the unit vector OE_i, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ forming a right-hand system. Also, let $\mathbf{e}_k'$ be a unit vector orthogonal to $\mathbf{e}_i$ and $\mathbf{e}_j$ so that $\mathbf{e}_i$, $\mathbf{e}_j$, $\mathbf{e}_k'$ is a right-hand system. Since $OE_1E_2E_3$ is a regular tetrahedron, we have at once

$$\mathbf{e}_i \times \mathbf{e}_j = \frac{\sqrt{3}}{2}\mathbf{e}_k', \quad \mathbf{e}_i\cdot(\mathbf{e}_j \times \mathbf{e}_k) = \frac{\sqrt{2}}{2}, \quad \mathbf{e}_i'\cdot\mathbf{e}_j' = -\tfrac{1}{3}, \quad \mathbf{e}_i\cdot\mathbf{e}_i' = \sqrt{\tfrac{2}{3}}. \tag{1}$$

We now use the well known theorem that the plane through the midpoint of A_iA_j and perpendicular to the opposite edge OA_k passes through M, the Monge point of $OA_1A_2A_3$, with the vector $\mathbf{m}$. Then we have $2\mathbf{m}\cdot\mathbf{e}_k=a_i(\mathbf{e}_i\cdot\mathbf{e}_k)+a_j(\mathbf{e}_j\cdot\mathbf{e}_k)$, or

$$4\mathbf{m}\cdot\mathbf{e}_k = a_i + a_j. \tag{2}$$

The vector $\mathbf{m}$ may be written

$$\begin{aligned} &\sqrt{\tfrac{2}{3}}\mathbf{m} = \sum(\mathbf{e}_k\cdot\mathbf{m})\mathbf{e}_k', \qquad 4\sqrt{\tfrac{2}{3}}\mathbf{m} = \sum(a_i + a_j)\mathbf{e}_k', \\ &\tfrac{32}{3}\mathbf{m}^2 = \sum(a_i + a_j)^2 - \tfrac{2}{3}\sum(a_i + a_j)(a_j + a_k), \qquad \mathbf{m}^2 = \tfrac{1}{8}\sum a_i^2. \end{aligned} \tag{3}$$

Since in this problem the right side is constant, the Monge point M lies on a fixed sphere with center O.

The circumradius R of $OA_1A_2A_3$, where the angle between the edges $\mathbf{e}_i$, $\mathbf{e}_j$ is now α_k, may be found in a similar way. Since the circumcenter C, with the vector $\mathbf{c}$, lies on the plane perpendicular to OA_i at its midpoint, we have $2\mathbf{c}\cdot\mathbf{e}_i=a_i$, $(i=1, 2, 3)$. We now define the associated vectors by the equations $\mathbf{e}_i \times \mathbf{e}_j=\sin\alpha_k\mathbf{e}_k'$, $(\mathbf{e}_i \times \mathbf{e}_j)\cdot\mathbf{e}_k=V$, the volume of the parallelepiped with the three edges OE_1, OE_2, OE_3. Then we have

$$V\mathbf{c} = \sum (\mathbf{c}\cdot\mathbf{e}_i) \sin \alpha_i \mathbf{e}_i', \qquad 2V\mathbf{c} = \sum a_i \sin \alpha_i \mathbf{e}_i',$$

$$(4) \qquad \begin{aligned} 4V^2R^2 &= \sum a_i^2 \sin^2 \alpha_i + 2\sum a_j a_k \sin \alpha_j \sin \alpha_k \, \mathbf{e}_j' \cdot \mathbf{e}_k', \\ &= \sum a_i^2 \sin^2 \alpha_i + 2\sum a_j a_k (\cos \alpha_j \cos \alpha_k - \cos \alpha_i). \end{aligned}$$

The determinant formula for V^2 has been given several times in this MONTHLY and here it reduces to

$$V^2 = 1 + 2 \cos \alpha_1 \cos \alpha_2 \cos \alpha_3 - (\cos^2 \alpha_1 + \cos^2 \alpha_2 + \cos^2 \alpha_3).$$

In the case of this problem where $\alpha_i = 60°$, we get

$$8R^2 = 3\sum a_i^2 - 2\sum a_j a_k.$$

3963 [1940, 399]. *Proposed by V. Thébault, San Sebastián, Spain.*

In an orthocentric tetrahedron the first sphere of twelve points is the locus of points the sum of whose powers, with respect to the spheres having as diameters the edges (or bimedians), is zero. Generalize.

Note. The bimedians are the straight line segments joining the midpoints of opposite edges.

Solution by Frank Ayres, Jr., Dickinson College.

In n-space, let $P_i \equiv (x_i^{(')}, x_i^{('')}, \cdots, x_i^{(n)}) \equiv (x_i)$, $(i = 1, 2, \cdots, n+1)$ be the vertices of an orthocentric $(n+1)$-point and $P \equiv (x)$ be its orthocenter. Let the radii of the $n+2$ mutually orthogonal hyperspheres having these points as centers be r_i and r, respectively. When the origin is taken at the center of the hypersphere circumscribing the $(n+1)$-point, it may be shown (Ayres, *On $n+2$ mutually orthogonal hyperspheres*, National Mathematics Magazine, vol. 10, 1936) that

$$\sum x_i^2 = R^2, \qquad \sum_{i \neq j} x_i x_j = R^2 - \tfrac{1}{2}(r_i^2 + r_j^2),$$

$$\sum x^2 = R^2 + nr^2, \qquad 2\sum x_i x = 2R^2 - r_i^2 + (n-1)r^2,$$

where $\sum$ indicates summation with respect to the superscripts and

$$R^2 = \frac{(n-1)^2}{4} r^2 + \tfrac{1}{4} \sum_i r_i^2$$

is the square of the radius of the circumscribing hypersphere.

The power of a point (X) with respect to the hypersphere whose diameter is the edge P_iP_j is given by

$$\sum \left(X - \frac{x_i + x_j}{2} \right)^2 - \sum \left(\frac{x_i - x_j}{2} \right)^2 = \sum X^2 - \sum X(x_i + x_j) + \sum x_i x_j;$$

hence, summing for all the edges, we have

$$(1)\qquad \binom{n+1}{2}(\sum X^2 + R^2) - n\sum X(x_1 + \cdots + x_{n+1}) - \frac{n}{2}\sum_i r_i^2.$$

The power of (X) with respect to the hypersphere whose diameter is the bimedian joining the midpoints of P_iP_j and P_sP_t is given by

$$\sum X^2 - \tfrac{1}{2}\sum X(x_i + x_j + x_s + x_t) + \tfrac{1}{4}\sum (x_i + x_j)(x_s + x_t);$$

hence, summing for all the bimedians, we have

$$(2)\quad \binom{n+1}{4}(\sum X^2 + R^2) - \frac{1}{2}\binom{n}{3}\sum X(x_1 + \cdots + x_{n+1}) - \frac{1}{4}\binom{n}{3}\sum_i r_i^2.$$

Consider now the hypersphere G, whose center is the center of gravity of the orthocentric $(n+1)$-point, the square of whose radius is $\sum r_i^2/(n+1)^2$, and whose equation is

$$(3)\qquad \sum X^2 - \frac{2(n-1)}{n+1}\sum Xx + R^2 - \frac{1}{n+1}\sum_i r_i^2 = 0.$$

Since both (1) and (2) reduce to the left member of (3), we have the following:

The hypersphere G is the locus of points the sum of whose powers, with respect to the hyperspheres having the edges (or bimedians) of an orthocentric $(n+1)$-point as diameters, is zero.

A generalization may also be made as follows:

The first $n(n+1)$-point hypersphere of an orthocentric $(n+1)$-point is the locus of points the sum of whose powers, with respect to the hyperspheres centered at

$$\frac{n+1}{2(n-1)}\cdot\frac{x_i + x_j}{2} \text{ and with rad.}^2 = \frac{(n-3)^2}{16}\cdot r^2 + \frac{(n+1)^2}{32(n-1)}(r_i^2 + r_j^2),$$

is zero.

The first $n(n+1)$-point hypersphere is the locus of points the sum of whose powers, with respect to the hyperspheres centered at

$$\frac{n+1}{2(n-1)}\cdot\frac{x_i + x_j + x_s + x_t}{4} \text{ and with rad.}^2 = \frac{(n-3)^2}{16}r^2 + \frac{(n+1)^2(n-2)}{64(n-1)^2}(r_i^2 + r_j^2 + r_s^2 + r_t^2),$$

is zero.

In 3-space, these are the theorems of the problem.

Solved also by L. M. Kelly.

Editorial Note. Kelly considered only the tetrahedron, and his solution is based on two theorems, one by Thébault in his article, *On spheres associated with the tetrahedron*, in this MONTHLY, 1935, p. 433:

The sum of the powers of any point in space with respect to the spheres described on the edges of a tetrahedron as diameters is twice the sum of the powers of the same point with respect to the spheres described on the bimedians.

For the orthocentric tetrahedron, the three spheres with the bimedians as diameters coincide. The other theorem is due to Lagrange and states that, if masses m_i are at points A_i, the center of mass is denoted by G, the lengths A_iA_j are denoted by e_{ij}, and the distances of any point X from the points A_i are denoted by x_i, then

$$[XG]^2 = \frac{\sum m_i x_i^2}{\sum m_i} - \frac{\sum m_i m_j e_{ij}^2}{(\sum m_i)^2}, \qquad i \neq j. \tag{1}$$

For the present problem a unit mass is considered at each of the six midpoints of the edges of the orthocentric tetrahedron. His generalization is the above Thébault theorem.

The solution by Ayres can be formally simplified by using vectors $\mathbf{a}_i$ from the orthocenter H as origin to the vertices A_i of the simplex $A_1A_2, \cdots, A_{n+1}$, denoting by $\mathbf{g}$ and $\mathbf{c}$ the vectors of the centroid G and the circumcenter C of the simplex, and by R the circumradius. It is easily shown that $\mathbf{a}_i \cdot \mathbf{a}_j$ is a constant m, $i \neq j$. If we set $m = r^2$, the sphere with center H and radius r is sometimes called the polar sphere, or conjugate sphere, of the simplex. We shall set down some results which are reasily deduced:

$$\begin{gathered} \mathbf{a}_i \cdot \mathbf{a}_j = m; \qquad 2\mathbf{c} = (n+1)\mathbf{g} = \sum \mathbf{a}_i; \qquad \mathbf{c}^2 = R^2 + nm; \\ \sum \mathbf{a}_i^2 = (n+1)^2\mathbf{g}^2 - n(n+1)m = 4R^2 - n(n-3)m. \end{gathered} \tag{2}$$

The power of the point with the vector $\mathbf{x}$ with respect to the sphere on A_iA_j as diameter is $\mathbf{x}^2 - (\mathbf{a}_i + \mathbf{a}_j)\cdot\mathbf{x} + m$, and the sum of all such powers is

$$\binom{n+1}{2}[\mathbf{x}^2 - 2\mathbf{g}\cdot\mathbf{x} + m]. \tag{3}$$

The expression in the brackets set equal to zero is the equation of a sphere with center G, the square of whose radius is $\mathbf{g}^2 - m$; hence this sphere (G) is orthogonal to the polar sphere (H).

We now consider the special case where n is odd. Let $\mathbf{g}_{i_1}$ be the vector of the centroid of $(n+1)/2$ vertices A_{i_1}, and $\mathbf{g}_{i_2}$ that of the centroid of the remaining vertices A_{i_2}, and let b_{i_1} be the length of the segment joining the two centroids. Then $b_{i_1}^2 = (\mathbf{g}_{i_1} - \mathbf{g}_{i_2})^2$, $4\mathbf{g}^2 = (\mathbf{g}_{i_1} + \mathbf{g}_{i_2})^2$, and we then find that $b_{i_1}^2 = 4(\mathbf{g}^2 - m)$.

Hence these ${}_nC_{(n+1)/2}$ bimedians have equal lengths and bisect each other at G. They determine a single sphere (G) with center G which is the same as the one mentioned above. Other theorems may be obtained for the general simplex, replacing H by the Monge point M. The proof of the Lagrange theorem is quite easy, taking the origin of vectors at the point X.

The system of $n+2$ spheres used by Ayres contains the polar sphere (H) whose radius may be zero, real, or a pure imaginary number; the spheres (A_i) with center A_i are fixed by making each orthogonal to (H); hence we must have $r_i^2=\mathbf{a}_i^2-m$. Then (A_i) and (A_j) are orthogonal, since $r_i^2+r_j^2=\mathbf{a}_i^2+\mathbf{a}_j^2-2m=(\mathbf{a}_i-\mathbf{a}_j)^2$; and we now have a system of $n+2$ spheres orthogonal in pairs. Using (2) we easily find that $\sum r_i^2=4R^2-(n-1)^2m$, which is the same as an equation in the above solution.

The inverse of a point with vector $\mathbf{x}$ with respect to (H) has the vector $m\mathbf{x}/\mathbf{x}^2$. The sphere (C') which is the inverse of the circumsphere (C) is important and its equation can be easily obtained. The equations of the four spheres mentioned above and of their common radical plane $[P]$ are given below for reference:

$$
\begin{aligned}
&(H):\ \mathbf{x}^2 - m = 0; \qquad (C'):\ n\mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} + m = 0;\\
&(G):\ \mathbf{x}^2 - 2\mathbf{g}\cdot\mathbf{x} + m = 0; \qquad (C):\ \mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} + nm = 0;\\
&[P]\ \ \mathbf{g}\cdot\mathbf{x} - m = 0.
\end{aligned}
\tag{4}
$$

The isogonal conjugate of H is H' with vector $2\mathbf{c}/n$, and the projections of H, H' on the face opposite to A_i have the vectors $m\mathbf{a}_i/\mathbf{a}_i^2$, $[(n+1)\mathbf{g}-\mathbf{a}_i]/n$; these points are, respectively, the foot of the altitude from vertex A_i and the centroid of the face opposite to A_i. These last two points together with the point $\mathbf{a}_i/n$ lie on (C'), so that we have $3(n+1)$ of its points.

If H does not fall on a vertex, then m is not zero; if it does, say $H=A_{n+1}$, then $m=0$. In this latter case the equation of the face opposite to A_{n+1} is

$$\mathbf{x}\cdot \sum \mathbf{a}_i/\mathbf{a}_i^2 = 1;$$

whereas, if $m\neq 0$, the equation of the face opposite to A_i is $\mathbf{a}_i\cdot\mathbf{x}=m$, $(i=1, 2, \cdots, n+1)$; that is, it is the polar plane of A_i with respect to the polar sphere (H).

3964 [1940, 399]. *Proposed by V. Thébault, San Sebastián, Spain.*

The sum of the powers of the vertices of a tetrahedron, with respect to the Monge sphere of the circumscribed ellipsoid of Steiner, is equal to the negative of half the sum of the squares of the edges.

Editorial Note. If we take the centroid as origin of vectors $\mathbf{a}_i$ to the vertices of the tetrahedron $A_1A_2A_3A_4$ and denote by e_{ij} the length of the edge A_iA_j, then we have

$$0 = (\sum \mathbf{a}_i)^2 = \sum \mathbf{a}_i^2 + 2\sum \mathbf{a}_i\cdot\mathbf{a}_j,$$
$$\sum e_{ij}^2 = \sum (\mathbf{a}_i - \mathbf{a}_j)^2 = 3\sum \mathbf{a}_i^2 - 2\sum \mathbf{a}_i\cdot\mathbf{a}_j = 4\sum \mathbf{a}_i^2.$$

Let (D) be the sphere with center D at the end of vector $\mathbf{d}$, and radius r. In order for the sum of the powers of the vertices with respect to (D) to have the required value, we must have

$$\sum (\mathbf{a}_i^2 - 2\mathbf{d}\cdot\mathbf{a}_i + \mathbf{d}^2 - r^2) = \tfrac{1}{4}\sum e_{ij}^2 + 4(\mathbf{d}^2 - r^2) = -\tfrac{1}{2}\sum e_{ij}^2,$$
$$r^2 = \mathbf{d}^2 + \tfrac{3}{16}\sum e_{ij}^2.$$

The proposer stated that the sphere designated in the problem has its center at the centroid and the square of its radius is $(3/16)\sum e_{ij}^2$; hence, for this sphere $\mathbf{d}=0$.

3967 [1940, 491]. *Proposed by V. Thébault, San Sebastián, Spain.*

For a given triangle ABC a second triangle $A'B'C'$ is formed, where AA', BB', CC' are segments of altitudes and $AA'/BC=BB'/CA=CC'/AB=k$. (1) Show that the two triangles have the same centroid. (2) Examine the variation of the area of $A'B'C'$. (3) For what value of k do the two triangles have the same angle of Brocard? (4) If $k=\pm 1$, show that the centers of squares constructed exteriorly, or interiorly, on the sides of $A'B'C'$ are the vertices of ABC.

Solution by F. Underwood, University College, Nottingham, England.

The required results can be obtained by elementary coördinate geometry· Taking B as origin and the positive x-axis along BC, we find for the coördinates of the vertices of ABC and $A'B'C'$ the following:

$$\begin{aligned} &A(c\cos B, \sin B);\quad B(0,0);\quad C(a,0);\\ (1)\quad &A'(c\cos B, c\sin B - ka);\quad B'(kb\sin C, kb\cos C);\quad C'(a - kc\sin B, kc\cos B);\\ &X_1 + X_2 + X_3 = c\cos B + a + k(b\sin C - c\sin B) = c\cos B + a;\\ &Y_1 + Y_2 + Y_3 = c\sin B - k(a - b\cos C - c\cos B) = c\sin B;\end{aligned}$$

where X_1, Y_1 are coördinates of A', *etc.* The last two results prove that ABC and $A'B'C'$ have the same controid.

Let Δ, Δ_1 denote the areas of ABC, $A'B'C'$; then, writing the determinant for $2\Delta_1$ and using reductions, we find that

$$(2)\qquad 2\Delta_1 = 2\Delta(1 + 3k^2) - kt,\quad t = a^2 + b^2 + c^2.$$

The algebraic minimum for Δ_1 is $\Delta - t^2/48\Delta$, when $k=t/12\Delta$. The area Δ_1 increases without limit as k takes on large positive or negative values.

Let the Brocard angles for the two triangles be ω, ω_1. Then $\cot\omega=\cot A+\cot B+\cot C=Rt/abc=t/4\Delta$. Denote the sides of $A'B'C'$ by a_1, b_1, c_1; then we have

$$a_1^2 = (B'C')^2 = (a - 2kb \sin C)^2 + k^2(c \cos B - b \cos C)^2$$
$$= (a - 4k\Delta/a)^2 + k^2(c^2 - b^2)^2/a^2.$$

Now $16\Delta^2+(c^2-b^2)^2=a^2(2b^2+2c^2-a^2)$; and hence

$$a_1^2 = a^2 - 8k\Delta + k^2(2b^2 + 2c^2 - a^2), \tag{3}$$

with similar expressions for b_1 and c_1. Then, setting $t_1=a_1+b_1+c_1$, we shall have

$$\cot \omega_1 = t_1/4\Delta = \frac{t(1 + 3k^2) - 24k\Delta}{4\Delta(1 + 3k^2) - 2kt}. \tag{4}$$

If $\omega=\omega_1$, we find that after reductions, $kt^2=48k\Delta^2$, and there are two cases; either $k=0$, or $t^2=48\Delta$. The second case requires that $(a^2+b^2+c^2)^2=3(2\sum b^2c^2-\sum a^4)$, or $\sum a^4-\sum b^2c^2=0$, or

$$(b^2 - c^2)^2 + (c^2 - a^2)^2 + (a^2 - b^2)^2 = 0.$$

Hence ABC and $A'B'C'$ have the same Brocard angle if and only if the two triangles coincide or ABC is equilateral.

When $k=\pm 1$, the equation (3) gives

$$a_1^2 = \mp 8\Delta + (2b^2 + 2c^2) = 2(b^2 + c^2 \mp 2bc \sin A).$$

For $k=1$, we have

$$(AB')^2 = (c \cos B - b \sin C)^2 + (c \sin B - b \cos C)^2 = b^2 + c^2 - 2bc \sin A;$$
$$(AC')^2 = (a - c \sin B - c \cos B)^2 + (c \sin B - c \cos B)^2$$
$$= (b \cos C - c \sin B)^2 + (b \sin C - c \cos B)^2 = b^2 + c^2 - 2bc \sin A.$$

Hence A is the center of one of the squares with side $B'C'$. For $k=-1$, similar computations give for a_1^2, $(AB')^2$, $(AC')^2$ the above results with the minus sign changed to plus. Hence, in this case also, A is the center of one of the squares with side $B'C'$. This completes the proof of (4).

Editorial Note. We may also write

$$\Delta_1 = \Delta(3k^2 - 2k \cot \omega + 1),$$

so that $\Delta_1=0$ for $3k=\cot \omega \pm\sqrt{\cot^2\omega-3}$. Hence, if $\cot \omega=2$ and $k=1/3$ or 1, we have $\Delta_1=0$.

The proposer gave the above expressions for cot ω, and considered k as positive when AA' is directed toward the opposite side, *etc.* He said that part (1) follows easily on observing that the points A', B', C' are the centers of similar rectangles constructed interiorly (or exteriorly) on the sides of $A_2B_2C_2$, the anticomplementary triangle of ABC. (2) The area of $A'B'C'$ vanishes under certain conditions: thus for $k=1$ and $\cot \omega=2$, the points A', B', C' are collinear.

(3) It is necessary and sufficient that cot $\omega=3$. (4) The points A', B', C' are centers of squares constructed interiorly (or exteriorly) on B_2C_2, C_2A_2, A_2B_2, and triangle $A'B'C'$ is the anticomplement of the one whose vertices are centers of squares on BC, CA, AB. The desired results follow from these facts.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to C. O. Oakley, Haverford College, Haverford, Pennsylvania.

A few more copies of the MONTHLY for October 1940 are needed to replace losses to members and libraries in the British Isles. Copies for this purpose sent to the Secretary of the Association will be forwarded by him.

The editors of the Annals of Mathematics have taken an action similar to that of the Duke University Press and have agreed to continue to allow the half rate after 1942 to those whose membership in the Association and whose subscription to the Annals of Mathematics are unbroken from 1942 to the year in question. This arrangement is expected to be permanent. Those members of the Association who contemplate subscribing for either of these journals should, therefore, do so before the end of this year at the rate of \$2.50 a year for the Annals of Mathematics and \$2.00 a year for the Duke Mathematical Journal.

Professor G. D. Birkhoff of Harvard University has been elected an honorary member of the London Mathematical Society.

The honorary degree of doctor of laws has been conferred by Lehigh University upon Dean R. G. D. Richardson of Brown University.

Professor L. C. Bagby of Linsly Institute of Technology has been appointed educational director of the air school of the Hartung Aircraft Corporation, Detroit, with a special assignment to mathematics, drafting and course planning.

Dr. H. R. Branson of Dillard University has been appointed an assistant professor at Howard University.

Dr. W. B. Caton of Athens College, Alabama, has been appointed acting head of the department at Southwestern College, Winfield, Kansas.

Associate Professor J. E. Davis of Drexel Institute of Technology has been promoted to a professorship.

Dr. F. G. Dressel of Duke University has been promoted to an assistant professorship.

Professor Emeritus W. F. Durand of Stanford University has been appointed a member of the National Advisory Committee on Aeronautics.

Dr. A. S. Galbraith of the University of Rochester has been appointed an assistant professor at Colby College.

Professor I. M. Hostetter of Howard College, Alabama, has been appointed an assistant professor at Oregon State College.

Dr. L. F. Ollmann of the College of Wooster has been appointed an associate professor and head of the department at Hofstra College.

Dr. W. J. Schart of Ohio State University has been appointed professor of mathematics at the Aviation Cadet Replacement Center, Maxwell Field, Alabama.

Dr. C. E. Sealander of the State University of Iowa has been appointed an assistant professor at the University of South Dakota.

J. W. Sheedy of Michigan State College has been promoted to an assistant professorship.

Associate Professor E. C. Stopher of Ashland College has accepted a position at Brockport, New York, State Normal School.

Dr. W. F. Whitmore of the University of California has accepted a position at the Naval Ordnance Laboratory, Washington, D. C.

Assistant Professor Louise A. Wolf of the University of Wisconsin Extension Division has been appointed lecturer at the University of Wisconsin.

The following appointments to instructorships have been announced:

Colorado College: Dr. Margaret M. Hansman
Cornell University: Dr. Joseph Lehner, J. C. Smith
Frostburg, Maryland, State Teachers College: Dr. W. N. Hallett
Illinois Institute of Technology: Dr. W. S. Snyder
Indiana Technical College: H. S. Kieval
University of Maryland: Dr. L. C. Hutchinson
Mundelein College, Chicago: E. L. Godfrey
New Mexico College of A. and M. A.: J. A. Joseph
University of Oklahoma: Dr. Ingo Maddaus
Oregon State College: Dr. Rhoda Manning
Pennsylvania State College: Dr. Abraham Schwartz
University of Pennsylvania: E. K. Ritter
Purdue University: Dr. J. W. T. S. Youngs, Dr. G. W. Whitehead
United States Naval Academy: T. J. Benac, Dr. E. E. Betz
Wells College: Dr. Mary D. Clement
University of Wisconsin: H. N. Laden
College of Wooster: H. L. Meyer, Jr.
Yale University: Dr. R. P. Dilworth

Professor R. L. Charles of Franklin and Marshall College died December 13, 1941, at the age of fifty-six. He was a charter member of the Mathematical Association.

Dr. W. P. Durfee, who was head of the department of mathematics at Hobart College from 1884 until his retirement in 1929, and dean of the college from 1888 to 1925, died December 17, 1941, in his eighty-seventh year. He was a charter member of the Association.

Professor C. E. Magnusson of the University of Washington died July 10, 1941.

E. J. Maurus, professor of mathematics at the University of Notre Dame from 1897 to 1939, died November 26, 1941, at the age of sixty-nine. He had been a member of the Mathematical Association for twenty-one years.

Dr. U. G. Mitchell, professor of mathematics and until last September chairman of the department at the University of Kansas, died early January 1, 1942, at the age of sixty-nine years. He was a charter member of the Association and served as an associate editor of the MONTHLY from 1915 through 1921.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-seventh Annual Meeting, New York, N. Y., December 30–31, 1942.

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Decatur, May 8–9, 1942
INDIANA, Crawfordsville, May 1–2, 1942
IOWA, Mt. Pleasant, April 17–18, 1942
KANSAS, Hays, March 27–28, 1942
KENTUCKY, Lexington, April 11, 1942
LOUISIANA-MISSISSIPPI, Jackson, Miss., March 6–7, 1942
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Ashland, Va., May 1942
METROPOLITAN NEW YORK, New York, April 18, 1942
MICHIGAN
MINNESOTA
MISSOURI, Kansas City, April 17, 1942
NEBRASKA, Omaha, May 9, 1942
NORTHERN CALIFORNIA, San Francisco, Jan 30, 1943
OHIO, Columbus, April 2, 1942
OKLAHOMA, Oklahoma City, Feb. 13, 1942
PHILADELPHIA, Philadelphia, Nov. 28, 1942
ROCKY MOUNTAIN, Golden Colo., April 17–18, 1942
SOUTHEASTERN, Emory University, Ga., March 26–27, 1942
SOUTHERN CALIFORNIA, Los Angeles, March 14, 1942
SOUTHWESTERN, State College, N. M., April 27–28, 1942
TEXAS, Lubbock, April 3–4, 1942
UPPER NEW YORK STATE, Rochester, May 2, 1942
WISCONSIN, Oshkosh, May 2, 1942

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 450 Ahnaip Street, Menasha, Wisconsin, or 97 Elm Street, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, 97 Elm Street, Oberlin, Ohio, to reach him five weeks before the change becomes effective.

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THE TWENTY-SIXTH ANNUAL MEETING OF THE ASSOCIATION

The twenty-sixth annual meeting of the Mathematical Association of America was held at Bethlehem, Pennsylvania, on Wednesday and Thursday, December 31, 1941, and January 1, 1942, in conjunction with the meetings of the American Mathematical Society, the Association for Symbolic Logic, and the National Council of Teachers of Mathematics. About five hundred ten were in attendance at the meetings, including the following one hundred ninety-five members of the Association:

C. R. ADAMS, Brown University
R. P. AGNEW, Cornell University
C. B. ALLENDOERFER, Haverford College
J. E. ALMAN, Lincoln School, Columbia University
H. A. ARNOLD, Princeton University
LAURA M. ASHBAUGH, Moravian College for Women
H. P. ATKINS, University of Rochester
ROBERT ATKINSON, The Shipley School
FRANK AYRES, Jr., Dickinson College

H. M. BACON, Stanford University
N. H. BALL, U. S. Naval Academy
I. A. BARNETT, University of Cincinnati
E. G. BEGLE, University of Michigan
J. A. BENNER, Lafayette College
A. A. BENNETT, Brown University
Brother BERNARD ALFRED, Manhattan College
WILLIAM BETZ, Rochester, New York, Public Schools
A. H. BLACK, Lebanon Valley College
H. F. BOHNENBLUST, Princeton University
JULIA W. BOWER, Connecticut College
L. E. BOYER, State Teachers College, Millersville, Pa.
J. W. BRADSHAW, University of Michigan
C. C. BRAMBLE, U. S. Naval Academy
R. W. BRINK, University of Minnesota
H. W. BRINKMANN, Swarthmore College
FOSTER BROOKS, Kent State University
N. R. BRYAN, University of Maine

S. S. CAIRNS, Queens College
W. D. CAIRNS, Oberlin College
P. A. CARIS, University of Pennsylvania
MILDRED E. CARLEN, Brown University
I. S. CARROLL, Syracuse University
W. F. CHENEY, Jr., University of Connecticut
RANDOLPH CHURCH, U. S. Naval Academy
L. W. COHEN, University of Kentucky
NANCY COLE, Sweet Briar College
J. B. COLEMAN, University of South Carolina
RICHARD COURANT, New York University
H. B. CURRY, Pennsylvania State College
E. H. CUTLER, Lehigh University

D. R. DAVIS, State Teachers College, Montclair, N. J.
J. E. DAVIS, Drexel Institute
L. J. DECK, Muhlenberg College
F. F. DECKER, Syracuse University
CARL DENBOW, Ohio University
L. L. DINES, Carnegie Institute of Technology
ARNOLD DRESDEN, Swarthmore College
JANET C. DURAND, Beaver College

W. E. FERGUSON, University of Missouri
F. A. FICKEN, Cornell University
W. W. FLEXNER, Cornell University
M. M. FLOOD, Princeton University
L. R. FORD, Illinois Institute of Technology
TOMLINSON FORT, Lehigh University
R. M. FOSTER, Bell Telephone Laboratories
J. W. FOUST, Central Michigan College of Education
J. S. FRAME, Brown University

R. E. GASKELL, University of Alabama
H. M. GEHMAN, University of Buffalo
Rev. F. J. GERST, Loyola University, Chicago
B. P. GILL, College of the City of New York
J. S. GOLD, Bucknell University
MICHAEL GOLDBERG, Bureau of Ordnance, Navy Department
H. H. GOLDSTINE, University of Michigan
CORNELIUS GOUWENS, Iowa State College
H. S. GRANT, Rutgers University
C. H. GRAVES, Pennsylvania State College
L. J. GREEN, Georgia School of Technology
F. L. GRIFFIN, Reed College

D. W. HALL, Brown University
H. H. HARTZLER, Goshen College

L. A. HAZELTINE, Stevens Institute of Technology
E. R. HEDRICK, University of California at Los Angeles
C. E. HEILMAN, Elizabethtown College
COLEMAN HERPEL, Hazleton Undergraduate Center, Pennsylvania State College
H. C. HICKS, Carnegie Institute of Technology
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P. V. KUNKEL, Cedar Crest College

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R. E. LANGER, University of Wisconsin
V. V. LATSHAW, Lehigh University
SOLOMON LEFSCHETZ, Princeton University
A. J. LEWIS, University of Denver
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H. F. MAC NEISH, Brooklyn College
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MARSTON MORSE, Institute for Advanced Study
E. J. MOULTON, Northwestern University
F. D. MURNAGHAN, Johns Hopkins University

C. A. NELSON, New Jersey College for Women
M. A. NORDGAARD, Upsala College

C. O. OAKLEY, Haverford College
E. G. OLDS, Carnegie Institute of Technology
OYSTEIN ORE, Yale University
F. W. OWENS, Pennsylvania State College
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C. C. RICHTMEYER, Central Michigan College of Education
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F. L. WREN, George Peabody College

R. C. YATES, Louisiana State University

OSCAR ZARISKI, Johns Hopkins University

Those attending the meetings had rooms in the dormitories of Lehigh University and took their meals at the cafeteria nearby. The parlor and committee rooms of Drown Hall served as a social center with very convenient access from the lecture hall and dormitories. The ladies of the department of mathematics gave a delightful tea Tuesday afternoon to the large number of mathematicians and guests. The streets and homes of Bethlehem were elaborately lighted in Christmas fashion and were greatly enjoyed by all, including visits to the older parts of Bethlehem and to the Christmas festivities.

The joint dinner was held Wednesday evening at Hotel Bethlehem. Following the dinner Professor W. M. Smith acted as toastmaster, and President Williams gave a cordial welcome to the group, speaking of his concern over the problems of the new year and giving voice to his confident hope and expectation that in the new year the developments in social security and similar policies might not affect our colleges adversely. To the great delight of the audience a quartet from Bethlehem sang Lisa Lehmann's "Persian Garden." Following this Professor Snyder exhibited a beautiful parchment scroll, presented on behalf of many members of the Society, giving appropriate tribute to Professor Raymond Clare Archibald for his able services as librarian of the Society the past twenty-one years. Professor Kline spoke eloquently of the experiences of an organization secretary and of the important work in defense and war activities that must be done by mathematicians. The quartet and members of the department of mathe-

matics then presented a light opera "Confusion Overthrown," written by Dean Fort; it need not be said that this had a great deal of mathematical as well as romantic content. A resolution of thanks was adopted unanimously recognizing the hospitality of the administration of Lehigh University, the efficient work of the local committee, and the hospitality of the ladies of the department. The evening's novel celebration closed only with the coming of the New Year.

The American Mathematical Society held sessions from Monday afternoon through Wednesday afternoon. A symposium was held Tuesday afternoon consisting of two addresses, "The mathematical theory of traveling waves" by Professor L. V. Bewley, and "Some new methods of solution of two-dimensional problems in elasticity" by Professor I. S. Sokolnikoff, with discussions by Alan Hazeltine, Ernst Weber, D. L. Holl, J. L. Synge, and J. N. Goodier. An invited address was given by Dr. Oscar Zariski on "Normal varieties and birational correspondences."

The Association for Symbolic Logic held two sessions on Wednesday afternoon. Professor H. B. Curry gave the retiring presidential address, and following this a joint session was held with the Society at which five papers were read.

The National Council of Teachers of Mathematics held morning and afternoon sessions on Wednesday and Thursday with numerous addresses by teachers of college and secondary school mathematics. An unusual feature of the meetings was a session on multi-sensory aids, organized under the supervision of Professor E. H. C. Hildebrandt; various moving pictures of a mathematical nature, drawings and trivision films were exhibited. A very gratifying attendance was observed at these sessions and the members of the National Council joined with the other mathematicians on all the social occasions.

The Mathematical Association held sessions Thursday morning and afternoon, the notable program having been arranged by Professors Arnold Dresden and L. L. Dines, chairman. The program follows, together with abstracts of some of the papers numbered in accordance with their place on the program:

First Session of the Association

1. "On semi-continuity" by Professor Tibor Radó, Ohio State University.
2. "The role of function spaces in function theory" by Professor C. R. Adams, Brown University.
3. "Remarks on divisors of zero" by Professor N. H. McCoy, Smith College.

1. The paper of Professor Radó will appear in an early issue of the Monthly.

2. In the case of each of several familiar classes of functions Professor Adams described the procedure of introducing a metric, or definition of distance from one function of the class to another. For the metric space thus obtained he defined the fundamental notions of Cauchy sequence of elements of the space, density of one set of elements with respect to another, and category of a set of elements, as well as completeness, separability, and compactness of the space. Emphasis was laid on the fact that, since a set of second category is never empty, these general ideas may sometimes profitably be employed to establish the exist-

ence, in a given class of functions, of functions with certain specified properties. In particular and by way of illustration, the speaker sketched Banach's proof that in the class of continuous functions most (in a sense determined by category) of them have at no point finite upper and lower derivative numbers on either side of the point.

3. The paper of Professor McCoy will appear in an early issue of the MONTHLY.

SECOND SESSION OF THE ASSOCIATION

1. "On a proof of Eisenstein for the law of quadratic reciprocity" by Professor H. A. RADEMACHER, University of Pennsylvania.

2. "What mathematics has done for me" by Professor B. L. NEWKIRK, Rensselaer Polytechnic Institute.

3. "Operational calculus" by Professor T. L. SMITH, Carnegie Institute of Technology.

2. The paper by Professor Newkirk was a case history of the effect of a mathematical background on a period of activity in science and engineering. The author's undergraduate course was classical. It was followed by three years of graduate work and teaching, and three years' study in Germany, all with astronomy (celestial mechanics) as the major and physics and mathematics as minors.

After a short period of work in astronomy he followed a developing interest in engineering in accepting an invitation to teach mathematics and mechanics in an engineering college. His mathematical background enabled him to handle courses in mechanics of materials and hydraulics in which he had no previous training or experience.

He developed an interest in gyroscopic phenomena which led to a study (theoretical and experimental) of dynamic balancing. This resulted in a patentable improvement in the art of balancing, and the development of a balancing machine that was well received especially in the automotive field.

A study was made of critical speeds of rotating shafts and of other features of behavior of rapidly rotating shafts. This study was both theoretical and experimental and it explained a number of points of considerable economic importance.

Details of the study of dynamic balancing and of shaft behavior were given in this paper to illustrate the parts played respectively by the mathematical theory and by experimental studies.

Facts were presented to speak for themselves but the author's feeling is that mathematical theory furnished the approach through the literature, suggested experimental studies, facilitated interpretation of experiments and led to important results through coordination of the two angles of attack. It is to be noted also that the mathematical background enabled the author to follow developing interests in a rapidly changing world.

3. Professor Smith gave an account of the operational calculus of Oliver Heaviside (1850–1925) who made extensive use of operational methods in solving differential equations, mainly in electrical circuit problems. His method was to replace the operator d/dt by p, whenever it operated on a function which vanished initially.

The Heaviside method combines a differential equation and the initial conditions (which are ordinarily used last to evaluate the integration constants) in a single equation. Then this equation is solved for the unknown function algebraically and simplified by separation into partial fractions or expansion in series in $1/p$; when p is now again treated as an operator, the solution corresponding to the given initial conditions is obtained.

The use of the Laplace transform as an operator is equivalent to the Heaviside method. It is just as convenient and fast, but allows everything to be made rigorous. It also increases the power of the method by making use of the Bromwich contour integral which is the inverse of the Laplace transformation. The Bromwich contour can be frequently deformed into a closed curve, so that the integral is evaluated by the Cauchy residue theory.

An electrical circuit problem serves to illustrate another advantage of the operational method when applied to simultaneous differential equations. Instead of first finding the general solution, inserting the initial conditions and solving the resulting simultaneous equations for the integration constants, the operational method makes it possible to solve immediately for the unknown current in any mesh produced by the given initial conditions.

Boundary value problems in partial differential equations may be often solved expeditiously entirely by the use of operator methods. For example, consider the motion of a stretched string with fixed ends and given initial displacement and velocity. Using the Laplace transform and then the finite sine transform, the differential equation together with the end-point and initial conditions are all combined into a single equation. After solving this for the unknown function, the solution is found by taking the inverses of both transforms.

The first boundary value problem for an infinite strip is easily solved by taking successively the Laplace transform and the Fourier transform (obtained, with its inverse, by splitting the Fourier integral theorem into two transformations). The result defines a function over a half-plane, which coincides over the strip with the solution of the problem.

Meetings of the Board of Governors

Members of the outgoing and of the incoming Board met on Wednesday afternoon and Thursday noon, respectively.

The following five institutions and forty-seven persons were elected to membership on applications duly certified:

To Institutional Membership

LOYOLA UNIVERSITY, New Orleans, Louisiana

LOYOLA COLLEGE, Baltimore, Maryland

COLLEGE OF THE HOLY CROSS, Worcester, Massachusetts

AUGUSTANA COLLEGE, Sioux Falls, South Dakota

ESCUELA DE INGENIEROS INDUSTRIALES, Santiago de Chile

To Individual Membership

H. C. ARNOLD, A.M. (Ohio State) Tech. Director, Federal Enamel & Stamping Co., Pittsburgh, Pa.

J. E. BELLARDO, M.S. (St. Bonaventure) Instr., Aquinas Coll., Grand Rapids, Mich.

S. U. BENSCOTER, M.S., C.E. (Illinois) Asst. Engr., U. S. Eng. Dept., U. S. Eng. Office, Vicksburg, Miss.

W. H. BRADFORD, M.S. (Louisiana) Instr., McNeese Jr. Coll. of L.S.U., Lake Charles, La.

B. H. BUIKSTRA, M.S. (Kansas State Coll.) Instr., Kansas State Coll., Manhattan, Kans.

F. A. BUTTER, Jr., Ph.D. (Stanford) Asst. Prof., Univ. of Southern California, Los Angeles, Calif.

W. S. H. CRAWFORD, B.A. (Mount Allison) Teaching Asst., Univ. of Minnesota, Minneapolis, Minn.

L. E. CURFMAN, M.S. (Colorado) Prof., Kansas State Teachers Coll., Pittsburg, Kans.

MYRTLE F. DICKINSON, M.S. (Catholic Univ.) Teacher, Orleans Parish School Board, New Orleans, La.

W. H. DURFEE, A.M. (Harvard) Instr., Cornell Univ., Ithaca, N. Y.

PAUL EBERHART, A.M. (Kansas) Asst. Prof., Head of Dept. of Math. and Eng., Washburn Mun. Univ., Topeka, Kans.

ISLÂM EROKAN, M.C.E. (Cornell) Lt., Turkish Army, Turkish Embassy, Washington, D. C.

F. D. FAULKNER, B.S. in Educ. (Kansas State T.C., Emporia) Instr., Kansas State Coll., Manhattan, Kan.

R. A. FETTERER, M.S. (Catholic Univ.) Instr., St. Francis Seminary, St. Francis, Wis.

E. L. FITZGERALD, B.S. (California), A.B. (Gonzaga Univ.) Instr., Univ. of Santa Clara, Santa Clara, Calif.

J. W. FOUST, Ph.D. (Michigan) Asst. Prof., Central Michigan Coll. of Educ., Mt. Pleasant, Mich.

R. W. GARDNER, A.M. (Boston Univ.) Vice Pres., Head of Dept., Northwest Nazarene Coll., Nampa, Idaho

W. J. HARRINGTON, Ph.D. (Cornell) Instr., Pennsylvania State Coll., State College, Pa.

W. R. HARVEY, M.S. (Michigan) Memb. Tech. Staff, Bell Telephone Labs., New York, N. Y.

J. R. HEVERLY, M.S. (Pennsylvania State College) Instr., Math. and Physics, Pennsylvania State Coll., Mont Alto Undergrad. Center, Chambersburg, Pa.

RAYMOND HUCK, M.S. (Illinois) Instr., Alabama Poly. Inst., Auburn, Ala.

EVAN JOHNSON, Jr., Ph.D. (Chicago) Asst. Prof., Pennsylvania State Coll., State College, Pa.

J. S. KNAPPER, M.S. (Pennsylvania State) Asst. Supervisor of Math., Pennsylvania State Coll., Central Exten., Lemont, Pa.

D. M. KRABILL, Ph.D. (Ohio State). Instr., Potomac State School, Keyser, W. Va.

EDNA E. KRAMER-LASSER, Ph.D. (Columbia) Chm. of Dept., Thomas Jefferson High School, New York, N. Y.

R. L. KRUEGER, Ph.D. (Marquette) Prof., Wittenberg Coll., Springfield, Ohio

RACHAEL A. LAROE, A.M. (Tennessee) Instr., Mary Hardin-Baylor Coll., Belton, Texas.

W. T. MACCREADIE, Ph.D. (Cornell) Asst. Prof., Bucknell Univ., Lewisburg, Pa.

Sister MARY ROBERDETTE, A.M. (Illinois) Instr., Clarke Coll., Dubuque, Iowa

Mrs. EDITH J. MCKISSOCK, M.E. (Akron) Prof., Youngstown Coll., Youngstown, Ohio

H. L. MEYER, Jr., M.S. (Chicago) Instr., Coll. of Wooster, Wooster, Ohio

Q. E. NELSON. Student, Eastern New Mexico Coll., Portales, N. M.

ADRIEN POULIOT. Dir. of Dept., Dean of Faculty of Sciences, Laval Univ., Quebec, P.Q., Canada

L. M. REAGAN, A.M. (Kansas) Asst. Prof., Poly. Inst. of Brooklyn, Brooklyn, N. Y.

E. C. REX, M.S. (Washington) Lecturer, Univ. of Southern California, Los Angeles, Calif.

R. B. RICE, A.B. (Wooster) Grad. asst., Ohio State Univ., Columbus, Ohio

J. B. ROSSER, Ph.D. (Princeton) Asso. Prof., Cornell Univ., Ithaca, N. Y.

ABRAHAM SCHWARTZ, Ph.D. (Mass. Inst. of Tech.) Instr., Pennsylvania State Coll., State College, Pa.

C. E. SEALANDER, Ph.D. (Iowa) Asst. Prof., Univ. of South Dakota, Vermillion, S. D.

R. S. SEAMONS, A.B. (Utah) Instr., High School, Pocatello, Idaho

C. V. L. SMITH, Ph.D. (Harvard) Asst. Prof., Lafayette Coll., Easton, Pa.

RUTH C. (Mrs. C. C.) SORRELLS, A.M. (Columbia) Teacher, Highland Park High School, Dallas, Texas

M. F. STILLWELL, A.M. (Syracuse) Instr., Rensselaer Poly. Inst., Troy, N. Y.

H. L. STUART, A.B. (Dickinson) 402 S. Hanover St., Carlisle, Pa. *In war service.*

Mrs. LOLA B. TAYLOR, A.M. (Peabody) Head of Dept., Clifton Jr. Coll., Clifton, Texas

Brother LADISLAUS WALBERT, B.S. (St. Mary's) Instr., Christian Brothers Coll., Memphis, Tenn.

J. E. WOOD, A.M. (Colo. State Coll. of Educ.) Instr., Math. and Phys. Sci., Jr. Coll., Scottsbluff, Nebr.

The Secretary announced the names of the Regional Governors elected by the membership of various regions for a two-year term:

Region 1 (New England), W. F. CHENEY, Jr.

Region 2 (New York and Eastern Canada), R. P. AGNEW

Region 5 (Alabama, Georgia, Florida, North Carolina and South Carolina), H. A. ROBINSON

Region 7 (Kentucky, Ohio and Tennessee), C. G. LATIMER

Region 10 (Kansas, Missouri and Nebraska), L. M. BLUMENTHAL

Region 12 (Arizona, Colorado, New Mexico, Utah and Wyoming), O. H. RECHARD

Region 13 (Northwestern States and Western Canada), W. E. MILNE

The financial report for the year 1941 was presented and accepted. It had been previously examined by Professor Langer for the Finance Committee and by President Brink. The securities of the Association have now been transferred to the custody of the Cleveland Trust Company which is to serve as financial adviser for the year 1942.

At the invitation of the Board, Professor S. T. Sanders reported that, due to the policy of economy instituted by the Governor of Louisiana, the administration of Louisiana State University has recommended to the State Board of Supervisors a reduction in the subsidy of the National Mathematics Magazine from approximately $2,700 to $600. As the result of the discussion, the President and Secretary were requested and instructed to send letters to President C. B. Hodges of the University and to J. E. Smitherman, chairman of the Louisiana State Board of Supervisors, expressing our appreciation of the importance of the magazine and of the contribution which the University has thereby made in recent years to the cause of mathematics, and of general education, in the South and in the whole country, our great regret at the loss of influence that a discontinuance of the Magazine would entail, and our hope that the proposed reduction in the subsidy will be annulled.

The Board voted (1) to nominate G. B. Price as the representative of the Association in the American Documentation Institute; (2) to appropriate fifty dollars for expenses of the joint War Preparedness Committee of the Association and Society; (3) to appoint D. R. Curtiss member of the Finance Committee for a four-year term.

The following associate editors of the MONTHLY were elected for the year 1942, as recommended by Professor L. R. Ford:

L. M. Blumenthal
W. B. Carver
N. B. Conkwright
H. S. M. Coxeter
W. M. Davis
Otto Dunkel
B. F. Finkel
J. S. Frame
Orrin Frink, Jr.
M. R. Hestenes
R. E. Langer
E. J. Moulton
J. R. Musselman
C. O. Oakley
Virgil Snyder
R. J. Walker
Marie J. Weiss

ANNUAL BUSINESS MEETING

The annual business meeting and election of officers was held Thursday morning, January 1, 1942. The Secretary announced the names of those who had been elected to membership at the meeting of the Board. He reports here the deaths of the following members:

ELI ALLISON, Head of department of mathematics, Franklin School, New York, N. Y. (November 11, 1941)
C. S. ATCHISON, Head of department of mathematics, Washington and Jefferson College. (November 21, 1941)
W. C. BOYRER, Accounting engineer, Consolidated Edison Co., Charleston, S. C. (September 28, 1940)
R. L. CHARLES, Professor of physics and electricity, Franklin and Marshall College. (December 13, 1941)
S. DICKSTEIN, Professor of mathematics and history, University of Warsaw. (1939)
O. C. EDWARDS, Assistant Professor of mechanical engineering, University of Minnesota. (July 17, 1941)
REV. F. J. FEINLER, Pastor, St. Ann's Church, Grants Pass, Oregon. (August 1941)
J. W. GLOVER, Professor emeritus of mathematics, University of Michigan. (July 15, 1941)
W. C. GRAUSTEIN, Professor of mathematics, Harvard University. (January 22, 1941)
E. J. HIRSCHLER, Professor of mathematics and astronomy, Bluffton College. (May 22, 1941)
D. D. LEIB, Professor of mathematics, Connecticut College. (June 15, 1941)
E. J. MAURUS, Professor of mathematics and surveying, University of Notre Dame. (November 26, 1941)
U. G. MITCHELL, Professor of mathematics, University of Kansas. (January 1, 1942)
ETHEL I. MOODY, Instructor in mathematics, Pennsylvania State College. (April 11, 1941)
R. E. MORITZ, Chairman of department of mathematics, University of Washington. (December 28, 1940)
C. M. SPARROW, Professor of physics, University of Virginia. (August 30, 1941)
H. L. SWEET, Mathematics teacher, Phillips Exeter Academy. (March 27, 1941)
T. H. TALIAFERRO, Dean of the Faculty, Chairman of department of mathematics, University of Maryland. (September 25, 1941)
J. E. TREVOR, Professor emeritus of thermodynamics, Cornell University. (May 4, 1941)
ALICE WINBIGLER, Professor emeritus of mathematics, Monmouth College. (May 27, 1941)

The results of the election of officers were as follows:

First Vice-President for a two-year term: TOMLINSON FORT, Lehigh University.

Governors at large for a three-year term: W. L. AYRES, Purdue University, and R. L. WILDER, University of Michigan.

The Association adopted the two amendments proposed for action at the meeting, one according to which two nominations are made by the Executive Committee only in the case of the Second Vice-President and members of the Finance Committee, the other whereby the By-Law offering life membership was cancelled, the Association voting at the same time that the contracts already made under this By-Law are to be fulfilled. The Association also voted blanket approval to the draft of the By-Laws as printed in the Register of October 1941, subject to the two amendments just adopted.

The following resolution, offered by Professor Langer, was unanimously adopted:

> The functioning of an organization, such as this Mathematical Association of America, is at all times dependent upon the appeal through which it enlists in its service, without the promise of any material recompense, men of energy, vision and good will. That the Association has unfailingly succeeded in doing this is eloquent of the support which its purposes and ideals have enjoyed. Of crucial importance to the Association is the editorial direction of its organ, the AMERICAN MATHEMATICAL MONTHLY. The Editor-in-Chief, the officer entrusted with this, fulfills a task which is incessantly demanding upon his time and thought. At this meeting Professor Moulton lays down this office. For over five years he has given steadily and in large measure of his care and energy, to the end that he has maintained for the MONTHLY an editorial policy guided by a continual striving after an optimum of usefulness, without ever a recession from the highest of scholarly standards.
>
> Be it, therefore, resolved: That the Mathematical Association of America acknowledges that Professor Elton James Moulton has served it with singular ability and generosity, and assures him of its sincere thanks and genuine appreciation.

On recommendation of a committee appointed by the President, the Chauvenet Prize of one hundred dollars was awarded to Professor Saunders Mac Lane for his two papers, "Modular fields," AMERICAN MATHEMATICAL MONTHLY, vol. 47 (1940), pp. 259–274 and "Some recent advances in algebra," AMERICAN MATHEMATICAL MONTHLY, vol. 46 (1939), pp. 3–19. In view of the fact that the Chauvenet Fund does not now produce the amount hitherto given each three years, the recommendation of the committee was adopted that the award continue to be made at three-year intervals but that hereafter the amount of the prize be re-

duced to $50, with the further provision that only such papers be considered for the prize as come within the range of profitable reading of Association members.

Report of the Secretary-Treasurer as Treasurer, December 20, 1941

Receipts

Balance Dec. 12, 1940		$6,176.09
1940 indiv. dues	$ 381.98	
1940 instit. dues	52.50	
1941 indiv. dues	7,982.70	
1941 instit. dues	660.00	
1941 subscriptions	1,069.10	
Initiation fees	342.00	
Advertising	573.50	
Reprints	297.53	
Sale copies of MONTHLY	724.13	
Sale First Carus Mon.	21.25	
Sale Second Carus Mon.	21.25	
Sale Third Carus Mon.	25.00	
Sale Fourth Carus Mon.	15.00	
Sale Fifth Carus Mon.	16.25	
Sale Sixth Carus Mon.	296.25	
Sale Archibald's *Outline of Hist. of Math.*	188.47	
Life membership fee	56.03	
Annals subscriptions	5.00	
Duke Journal subscriptions	4.00	
Math. Reviews subscriptions	13.00	
Sale Rhind Papyrus	20.00	
Drury Coll. int. Hardy fund	120.00	
Refund Chicago meeting	21.47	
Profit sale Youngstown Bonds	78.78	
Profit sale Firestone Bond	34.17	
Int. Genl. End. Fund	634.76	
Int. Carus Fund	164.37	
Int. Chace Fund	271.40	
Int. Chauvenet Fund	15.00	
Int. current funds	60.99	
Payment from restricted Carus Fund	49.70	
Payment from restricted Chace Fund	2.20	14,217.78
Total 1941 receipts to date		$20,393.87

Expenditures

Publisher's bills (Nov. '40–Oct. '41)		$ 5,784.42
Supplement "Math. in Industry"		108.33
Reprints		359.61
Preparing *Register*		209.52
Printing *Register*		555.23
President's office		32.37
Editor-in-chief's office		875.66
Expense Executive Committee		89.09
Expense Finance Committee		35.92
Membership campaign		369.54
Conference Committee on Education		50.00
Secretary-Treasurer's office		
Postage	$ 477.43	
Bond	11.26	
Office expense	143.58	
Express, tel., etc.	85.50	
Clerical work	2,361.47	
Printing	282.40	
Bank charge	17.03	3,378.67
Annals subvention		200.00
Duke Journal subvention		200.00
Math. Reviews subvention		1,000.00
Expense of sections		300.21
Expense acct. regional governors		8.75
Baton Rouge meeting		149.80
Chicago meeting		73.62
Lehigh meeting		90.00
Paid *Annals* subscriptions		10.00
Forwarded *Annals* subscriptions		5.00
Forwarded *Duke Journal* subscriptions		4.00
Forwarded *Math. Reviews* subscriptions		26.00
Sust. memb. in Amer. Math. Society		100.00
Insurance back copies MONTHLY		2.32
Storage back copies MONTHLY		30.00
Paid B. F. Finkel int. Hardy Fund		120.00
Refund subscriptions		8.55
Paid back copies MONTHLY		76.75
Library expense		17.38

RECEIPTS (continued)		EXPENDITURES (continued)	
		Paid *Eudemus* int. from Chace Fund	215.71
		Printing Archibald's *Outline*	323.87
		Expense account *Outline*	12.35
		Expense account Carus Com.	12.36
		War Preparedness Com.	96.67
		Distributing War Preparedness report	99.76
		Printing Sixth Carus Mon.	1,250.29
		Honorarium Sixth Carus Mon.	300.00
Total expenditures	16,581.75	Total expenditures	$16,581.75
Balance to end of 1941 business	$ 3,812.12	Checking account	217.15
Received on 1942 business	2,512.86	Oberlin Savgs. Bk. acct. restricted	583.10
		Peoples Banking Co. savgs. acct.	1,835.76
		Cleveland Trust Co. savgs. acct.	1,688.97
		U.S. Defense Bonds	2,000.00
Book Balance Dec. 20, 1941	$ 6,324.98	Bank Balance Dec. 20, 1941	$ 6,324.98

EXHIBIT OF THE FUNDS OF THE ASSOCIATION

CARUS MONOGRAPH FUND

Balance, December 12, 1940			$7,320.68
Receipts: Sales		$ 395.00	
Interest		205.62	
Profit sale Firestone Bonds		34.17	634.79
			$7,955.47
Expenditures: Expense acct. Carus Fund		$ 12.36	
Printing Sixth Carus Monograph		1,250.29	
Honorarium Sixth Carus Monograph		300.00	1,562.65
			$6,392.82
	Market Value Dec. 31, 1941	Cost or Face	
Certificate of deposit	$1,681.51	$1,681.51	
C. & O. $3\frac{1}{2}\%$ Ref. Mort. Bonds Ser. D, 1996	2,030.00	2,000.00	
U. S. Treasury $3\frac{1}{8}\%$ Bond of 1946–49	1,082.50	1,000.00	
HOLC 3% Bond 1944–52	1,046.25	1,000.00	
U. S. Savings Bonds	164.00	150.00	
U. S. Defense Bond	1,000.00	1,000.00	
Cash in bank, restr., certif. of participation	347.90	347.90	7,179.41
Due to the general treasury			786.59
Balance December 20, 1941			$6,392.82

Arnold Buffum Chace Fund

Balance December 12, 1940		\$8,088.29
Receipts: Sale Papyrus	\$ 20.00	
Interest	271.40	291.40
		\$8,379.69
Expenditures: Paid *Eudemus* int. from Chace Fund		215.71
		\$8,163.98

	Market Value Dec. 31, 1941	Cost or Face
U. S. Treasury $3\frac{1}{8}\%$ Bonds 1946–49	\$2,165.00	\$2,000.00
HOLC 3% Bond 1944–52	1,360.12	1,300.00
U. S. Savings Bonds	1,270.00	1,125.00
Montana Power Co. $3\frac{3}{4}\%$ First Mort. Bonds 1966	1,045.00	1,000.00
North American Co. 4% Deb. Bond 1959	1,040.00	1,000.00
$\frac{1}{2}$ Shawinigan W. & P. Co. $4\frac{1}{2}\%$ First Mort. Bond 1970	436.25	500.00
N. Y. Steam Corp. $3\frac{1}{2}\%$ First Mort. 1963	1,053.75	1,000.00
Cash in bank, restr., certif. of participation	15.40	15.40
Cash in bank, unrestricted	223.58	223.58

Balance December 20, 1941 ... \$8,163.98

Jacob Houck Memorial Fund

Balance December 12, 1940	\$7,928.94
Receipts: Interest	252.23
	\$8,181.17

	Market Value Dec. 31, 1941	Cost or Face
U. S. Treasury Bonds, registered	\$4,375.00	\$4,000.00
$1\frac{1}{2}$ Shawinigan W. & P. Co. $4\frac{1}{2}\%$ First Mort. Bonds 1970	1,308.75	1,500.00
Gatineau Power Co. $3\frac{3}{4}\%$ First Mort. Bond 1969	800.00	1,000.00
Peoples Banking Co. savings acct.	1,681.17	1,681.17

Balance December 20, 1941 ... \$8,181.17

Chauvenet Prize Fund

Balance December 12, 1940	\$ 617.94
Interest	15.00
	\$ 632.94

	Market Value Dec. 31, 1941	Cost or Face
HOLC 3% Bond 1944–52	\$ 523.12	\$ 500.00
Cash in bank	132.94	132.94

Balance December 20, 1941 ... \$ 632.94

Life Membership Fund

Liability on life memberships as of January 1, 1941	$ 867.39
Received on life membership	56.03
	$ 923.42
To be transferred to current funds, surplus	23.25
Liability on life memberships as of January 1, 1942	$ 900.17

General Endowment Fund

	Market Value Dec. 31, 1941	Cost or Face	
Balance December 12, 1940			$18,200.00
U. S. Treasury $3\frac{1}{4}\%$ Bonds 1944–46	$1,050.00	$1,000.00	
U. S. Treasury $3\frac{1}{4}\%$ Bonds 1943–45	1,042.50	1,000.00	
HOLC 3% Bonds 1944–52	5,754.37	5,500.00	
Land Trust Certificate, Hotel Cleveland Site	550.00	700.00	
Montana Power Co. $3\frac{3}{4}\%$ First Mort. Bonds 1966	2,090.00	2,000.00	
Texas Power & Light Co. 5% First Mort. Bond 1956	1,070.00	1,000.00	
C. & O. $3\frac{1}{2}\%$ Ref. Mort. Bond Ser. D 1996	1,015.00	1,000.00	
Penn. R. R. Co. $3\frac{3}{4}\%$ Genl. Mort. Bonds Ser. C 1970	1,782.50	2,000.00	
Cols. & So. Ohio Elec. $3\frac{1}{4}\%$ First Mort. Bonds 1970	2,145.00	2,000.00	
Oberlin Savings Bank savings account	2,000.00	2,000.00	
Balance December 20, 1941			$18,200.00

Of the funds on hand, indicated in the first division of this financial report, $786.59 is due the general treasury from the Carus Monograph Fund, $223.58 is due from the general treasury to the Chace Fund and $132.94 to the Chauvenet Prize Fund, while $900.17 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date January 1, 1942.

When the accounts were closed December 20, 1941, in order to make a complete report, there remained on the total business for 1941 the following items:

Bills Receivable

1941 individual dues	$ 200.00
Advertising	150.00
From Carus Mon. Fund	786.59
	$1,136.59

Bills Payable

Publisher's bills (Nov., Dec. '41)	$1,250.00
Editor-in-chief's office	80.00
Secretary-Treasurer's office	150.00
Subsidy *Duke Journal*	50.00
Chace Fund	223.58
Chauvenet Fund	132.94
Life membership fund	900.17
	$2,786.69

If to the balance on 1941 business shown in this report, $3,812.12, there be added the estimated bills receivable, $1,136.59, and there be subtracted the estimated bills payable, $2,786.69, there results an estimated final balance at the close of 1941 business of approximately $2,160.

W. D. Cairns, *Secretary-Treasurer*

THE TWENTY-FIFTH ANNUAL MEETING OF THE KENTUCKY SECTION

The twenty-fifth annual meeting of the Kentucky Section of the Mathematical Association of America was held at Eastern Kentucky State Teachers College on Saturday, April 26, 1941, in conjunction with the annual meeting of the Kentucky Academy of Science. Professor H. A. Wright, chairman of the Section, presided.

There were fifty-one in attendance, including the following twenty members of the Association: N. B. Allison, J. G. Black, M. C. Brown, L. W. Cohen, H. H. Downing, L. A. Fair, Charles Hatfield, W. R. Hutcherson, E. D. Jenkins, Fritz John, C. G. Latimer, F. Elizabeth LeStourgeon, R. S. Park, Sallie E. Pence, D. W. Pugsley, D. E. South, Guy Stevenson, S. Helen Taylor, Mary E. Williams, H. A. Wright.

A luncheon meeting was held in the Student Union Building at which the following officers were elected for the next year: Chairman, L. A. Fair, Morehead State Teachers College; Secretary, D. E. South, University of Kentucky.

The following papers were presented:

1. "On regular polyhedra" by Charles Hatfield, Jr., University of Kentucky, introduced by Professor Cohen.

2. "Reading ability as a factor in solving problems in algebra" by Dr. S. R. Hemphill, Berea College, introduced by Professor Hutcherson.

3. "Amateur research in astronomy" by Professor W. L. Moore, University of Louisville.

4. "Dimensionality" by Professor L. W. Cohen, University of Kentucky.

5. "Graphical representation of complex roots" by Frances Hamilton, Transylvania College, introduced by Professor Wright.

6. "Biological problem solved by difference equations" by Professor N. B. Allison, Kentucky Wesleyan College.

7. "Suggestions for the teaching of mathematics from American Council on Education tests" by Dean S. Helen Taylor, Ashland Junior College.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Mr. Hatfield used the five regular polyhedra of solid geometry as the basis for a set of five polyhedra which have interesting properties. A given one of this "derived" set is had by replacing the faces of a given polyhedron by pyramids, whose bases are faces of the original polyhedra. If these pyramids for a given polyhedron are the same size, then the exterior vertices of the "derived" one become vertices of one of the five regular polyhedra. Under such a "transformation" the tetrahedron transforms into a tetrahedron, the hexahedron into the octahedron, and vice versa, the dodecahedron into the icosahedron, and vice versa. As a special case of the above pyramid construction, we have the type of polyhedron whose faces are rhombi by making the pyramids of such size that the

triangular faces of a given pyramid form rhombi with the faces of other pyramids.

2. Dr. Hemphill reported on an experimental study to improve the ability of pupils to solve verbal algebra problems. More than five hundred students in ten schools in seven cities in eastern Kansas participated in the experiment. Special efforts were made to master the vocabulary of the text, extensive use was made of reading exercises involving mathematical material, and the study of verbal problems was distributed throughout the semester. Achievement was measured by a test prepared for that purpose. The findings of the study indicate that most pupils in beginning algebra will improve in ability to solve verbal problems if, under the supervision of their classroom teachers, they effectively study the vocabulary of their text and selected reading exercises that involve mathematical material. Pupils of superior mental ability seem more likely to benefit by such procedures than pupils of average or low mental ability.

3. The fields of research discussed by Professor Moore were those by which cooperation by large groups of amateurs are desirable. These include observations of variable stars, meteors, occultations by the moon, observations of the moon to determine if there are any changes in its surface features, comet seeking and the search for novae. Professor Moore also described the work of the Louisville Astronomical Society in the observation of the transit of Mercury November 11, 1940. This work was in cooperation with the Naval Observatory of Washington.

4. Some points in the history of the notion of dimension were indicated by Professor Cohen and the work of Vrysohn and Menger was discussed.

5. Miss Hamilton's paper was an elaboration of an article of the same title by J. A. Ward in the National Mathematics Magazine, vol. 11, no. 7, page 297, (Apr. 1937). Carefully prepared large-scale graphs of cubic and quartic equations were used to illustrate the theory and to indicate the range of practical use.

6. Professor Allison discussed a difference equation with constant coefficients derived from biological reproduction. The solution involves the sum of like powers of roots of an algebraic equation and its discriminant.

7. Dr. Taylor made a study of the Thurstone Psychology Tests, the Achievement Tests in Mathematics and the Literary Comprehension Tests (a test on reading ability). Correlation coefficients were: Mathematics and Psychology, $r=.61$; Mathematics and Literary Comprehension, $r=.44$; Literary Comprehension and Psychology, $r=.59$. These coefficients were computed from data of 306 students who took the tests in three successive years, 1938, 1939, 1940 as a part of the Freshman orientation program at Ashland Junior College.

It is suggested that the accrediting agencies such as the State University and the Southern Association should be in agreement as in other sections of the country and that the University should not accredit schools for college entrance unless they are able to be admitted to the Southern Association. The students in the group above were from five Southern Association high schools and from a large group of unaccredited schools. There should be improvement of instruction

in intermediate algebra and solid geometry; there should be more intelligent counselling of students as to the amount and kind of mathematics courses taken; up-to-date texts should replace many of those now in use; and constructive curriculum study should be undertaken by the teachers in each school in their own department groups.

D. E. SOUTH, *Secretary*

THE SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the United States Naval Academy, Annapolis, Maryland, on Saturday May 10, 1941, with a morning session, luncheon, and afternoon session. Dean T. McN. Simpson, Jr., chairman of the Section, presided.

The attendance was sixty-five, including the following thirty-six members of the Association: H. A. Arnold, A. A. Aucoin, M. W. Aylor, N. H. Ball, G. A. Bingley, Archie Blake, W. E. Bleick, C. C. Bramble, Randolph Church, G. R. Clements, L. S. Dederick, J. A. Duerksen, W. C. Flaherty, Michael Goldberg, Bertha I. Hart, G. A. Hedlund, L. M. Kells, Solomon Kullback, A. E. Landry, C. L. Leiper, Florence P. Lewis, S. B. Littauer, E. J. McShane, Sister Thomas Marie Maloney, T. W. Moore, C. H. Rawlins, Jr., J. N. Rice, Irwin Roman, R. E. Root, J. B. Scarborough, T. McN. Simpson, Jr., F. W. Sohon, G. C. Vedova, C. H. Wheeler III, G. T. Whyburn, R. H. Wilson, Jr.

At the business meeting the following officers of the Section were elected for 1941–1942: Chairman, E. J. McShane, University of Virginia; Secretary, C. H. Wheeler III, University of Richmond; to the Executive Committee, Archie Blake, U. S. Coast and Geodetic Survey, and G. C. Vedova, University of Maryland. The Section accepted invitations from Georgetown University for the fall meeting of 1941, and from Randolph-Macon College for the spring meeting of 1942. A motion was passed expressing the appreciation of the Section to the authorities of the United States Naval Academy for their generous hospitality.

At the invitation of the Section, Professor E. J. McShane of the University of Virginia delivered an address on "Computation of flat trajectories."

After an address of welcome by Captain G. H. Forte of the United States Naval Academy, the following papers were read:

1. "The basic equations of interior ballistics" by Professor C. C. Bramble, United States Naval Academy.
2. "Some mathematical tools for the young scientist" by Dr. Irwin Roman, United States Geological Survey.
3. "Symmetric recursion formulas for the reversion of power series" by Dr. W. E. Bleick, United States Naval Academy.
4. "Rocketry" by Professor W. A. Conrad, United States Naval Academy, introduced by Professor Kells.

5. "Computation of flat trajectories" by Professor E. J. McShane, University of Virginia.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor Bramble pointed out that most interior ballistic systems are based on (1) the gas law, (2) the energy equation, or (3) the law of burning. The procedure of investigators using the latter method has differed, but in many cases has led to a set of auxiliary tables determined by some form of approximate integration. In other attempts to solve the problem, the law of burning has been replaced by an empirical relation connecting the distance the projectile has moved with the velocity of the projectile. Professor Bramble also stated that there are formulas which use the results obtained with a known gun as a "model experiment" for the purpose of making predictions as to the behavior of a new design.

2. Dr. Roman stated that many mathematical tools are available to young scientists who have had a course in calculus, but are usually lost in the intricacies of specialized fields. He gave typical examples illustrating numerical roots, polynomial evaluation, Lagrangean multipliers for linear systems, interpolation, differences, least squares, successive approximations, recursion formulas, departures from standard conditions and the use of computation aids.

3. Dr. Bleick gave the relations between the coefficient b_n of the series $y=x+\sum_1^\infty b_i x^{i+1}$ and the coefficient c_n of the reversed series $x=y+\sum_1^\infty c_i y^{i+1}$ which were in terms of $a_j=\sum_1^{i-1} b_i c_{j-1}$, where $j=2, 3, \cdots, n$. He pointed out that these formulas were considerably simpler than those which have appeared in the literature.

4. Professor Conrad discussed the problem: Given the solar system as it is, to place a man on the planet Mars. He pointed out the many difficulties of travelling from the Earth to Mars by rocket, but stated how all of these, with the exception of the fuel problem, could be satisfactorily overcome. Professor Conrad suggested the expensive step rocket or the equally costly artificial satellite as a solution for the fuel problem.

5. Professor McShane stated that if we ignore the small motion perpendicular to the vertical plane containing the initial direction, the motion of a projectile is given by two second order differential equations. These can be solved to an arbitrarily high degree of approximation by numerical integration. But sometimes adequate accuracy can be obtained by approximation methods involving less computational labor. Professor McShane discussed four such methods, of varying degrees of accuracy and correspondingly varying degrees of computational difficulty. One of these was the Hitchcock-Kent method; another was a modification of that method with more accurate estimate of air density. The other two methods are presented in detail in a paper which will be offered to this MONTHLY.

C. H. WHEELER III, *Secretary*

ON EQUAL SUMS OF SQUARES

I. A. BARNETT, University of Cincinnati
C. W. MENDEL, University of Illinois

Many special cases of the Diophantine equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_m^2$$

have been discussed* by various writers. The methods used apply for the most part to the special equations considered by each writer. In this paper we shall give a complete parametric solution of the above equation. In section 1 we first develop a new treatment of the linear homogeneous Diophantine equation in n variables. In section 2 we consider the above equation for $m=n$ and give a complete parametric solution in terms of an arbitrary integral vector and an arbitrary integral skew-symmetric matrix. The number of parameters in this solution may be reduced if we use the results of section 1, though a certain simplicity of form is lost in this way.

In section 3 the system of equations

$$\begin{cases} x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2, \\ x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n, \end{cases}$$

is considered and it is shown how the results of section 2 permit us to give a complete solution of this problem in a form which is more symmetrical than the one usually presented.

In sections 4 and 5 we return to the equation first mentioned with unequal m and n and apply the results of section 2 to give a complete solution to this problem. Finally, in section 6 we solve the problem of the sum of n squares equal to a square.

1. Linear Diophantine equation. Let small Roman letters represent n-partite vectors whose components are integers. We shall denote by $\Delta(a)$ or $\Delta(a_1, a_2, \cdots, a_n)$ the greatest common divisor of these n integers. As usual $a \cdot x$ will stand for the inner product, $a_1x_1+a_2x_2+ \cdots +a_nx_n$.

We consider first the homogeneous Diophantine equation

$$a \cdot x = 0 \tag{1}$$

and suppose $\Delta(a)=1$. It is readily seen that equation (1) is satisfied by $x=Ca$ where C is a skew-symmetric matrix with arbitrary integers for elements. This solution was given by Betti† but it was first shown by Giudice‡ who used a method of induction, that every integral solution of (1) is given by this expression.

* Dickson, History of the Theory of Numbers, vol. 2, chapters VI, VII, VIII.

† Cf. J. Bertrand, Traité elem. d'algebre, 1850, translated by G. E. Betti, Florence, 1862, p. 285.

‡ Giornale di Mat., 36, 1898, p. 226.

The number of arbitrary parameters entering in the solution $x=Ca$ is clearly $n(n-1)/2$ and Giudice showed* further that the general solution of equation (1) can be expressed in terms of $n-1$ parameters. The formulas which he obtained are rather involved and we shall give in this section a simpler expression for x in terms of a and $n-1$ integral parameters.

Let $\epsilon_i=\Delta(a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$. Since $\Delta(a)=1$ it follows at once that $\Delta(\epsilon_i, \epsilon_j)=1$, $(i\neq j)$, and $\Delta(a_i, \epsilon_i)=1$. Hence the ith component a_i of a will be divisible by $\epsilon_1\epsilon_2\cdots\epsilon_{i-1}\epsilon_{i+1}\cdots\epsilon_n$ so that, if we let $\epsilon=\epsilon_1\epsilon_2\cdots\epsilon_n$ there will exist an integral vector α such that $\alpha_i=a_i\epsilon_i/\epsilon$, $(i=1, 2, \cdots, n)$. From the definitions of the ϵ_i and the vector α we see that the n components α_i of the vector α are relatively prime in sets of $n-1$. We also observe from equation (1) that each component x_i must be divisible by the corresponding ϵ_i so that, by setting $x_i=\epsilon_i t_i$ and dividing by ϵ, equation (1) becomes

$$\alpha\cdot t = 0. \tag{2}$$

Equation (2) is of the same form as (1) but with the added restriction that $\Delta(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_n)=1$. Since $\Delta(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})=1$ there exist† sets of integers $r_1, r_2, \cdots, r_{n-1}$, *fixed* once for all, such that

$$r_1\alpha_1 + r_2\alpha_2 + \cdots + r_{n-1}\alpha_{n-1} = 1 - \alpha_n.$$

Hence, if we define $r_n=1$ the last equation may be written as

$$r\cdot\alpha = 1, \qquad (r_n = 1). \tag{3}$$

In equation (2) let us set

$$t = \sigma r - \theta,$$

where θ is a parameter vector and σ is a scalar to be determined. We see at once that

$$\alpha\cdot t = \sigma(\alpha\cdot r) - \alpha\cdot\theta,$$

so that, by (2) and (3) we find

$$\sigma = \alpha\cdot\theta. \tag{4}$$

Thus, for every integral vector θ,

$$t_i = (\alpha\cdot\theta)r_i - \theta_i, \qquad (i = 1, 2, \cdots, n), \tag{5}$$

furnishes a solution of (2) and

$$x_i = \epsilon_i\{(\alpha\cdot\theta)r_i - \theta_i\}, \qquad (i = 1, 2, \cdots, n), \tag{6}$$

yields a solution of equation (1). This solution involves n parameters θ_i but we shall show presently that one of these may be taken to be zero without loss of any generality of the solution.

* *Ibid.*, p. 230.

† Dickson, Introduction to the Theory of Numbers, Theorem 2, page 2.

It remains to show that all solutions of equation (1) are given by the expressions (6). Corresponding to a given solution x of equation (1) there is a unique t of equation (2), defined by $x_i=\epsilon_i t_i$. It will suffice to show that every such solution t may be obtained from (5) by a suitable choice of the integral parameters θ_i.

Considering first the nth equation of (5) we see that (since $r_n=1$)

$$\alpha\cdot\theta = t_n + \theta_n,$$

and using this last expression for $\alpha\cdot\theta$ in the remaining equations of (5), we find

$$\theta_i = r_i(t_n + \theta_n) - t_i, \qquad (i = 1, 2, \cdots, n-1).$$

Now θ_n may be assigned arbitrarily (say, $\theta_n=0$), and we thus determine a unique set of integral values for the remaining parameters $\theta_1, \theta_2, \cdots, \theta_{n-1}$. Thus it is clear that every integral solution of equation (2) is given by the expressions (5) with $\theta_n=0$, and, hence, every integral solution of equation (1) is given by (6) with $\theta_n=0$.

We may summarize the procedure as follows: *Compute* $\epsilon_i=\Delta(a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$, $\epsilon=\epsilon_1\epsilon_2\cdots\epsilon_n$, $\alpha_i=a_i\epsilon_i/\epsilon$. *Find a particular solution of* $r\cdot\alpha=1$ $(r_n=1)$. *Then the general solution of* $a\cdot x=0$ *is given by* $x_i=\epsilon_i\{(\alpha\cdot\theta)r_i-\theta_i\}$.

Example. To solve the equation*

$$(7) \qquad 3x_1 + 6x_2 + 16x_3 - 80x_4 + 360x_5 = 0,$$

we observe that $a=(3, 6, 16, -80, 360)$, $\epsilon_1=2$, $\epsilon_2=\epsilon_3=\epsilon_4=\epsilon_5=1$, $\epsilon=2$, $\alpha=(3, 3, 8, -40, 180)$. The equation (3) becomes

$$3r_1 + 3r_2 + 8r_3 - 40r_4 = -179 \qquad (r_5 = 1),$$

of which a solution is $r=(0, -1, -2, 4, 1)$ and by (4)

$$\sigma = \alpha\cdot\theta = 3\theta_1 + 3\theta_2 + 8\theta_3 - 40\theta_4, \qquad (\theta_5 = 0)$$

Hence the general solution of (7) is

$$x_1 = -2\theta_1, \quad x_2 = -(\sigma + \theta_2), \quad x_3 = -(2\sigma + \theta_3), \quad x_4 = 4\sigma - \theta_4, \quad x_5 = \sigma.$$

2. Equal sums of n squares. We consider next the Diophantine equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2,$$

which may be written as $x\cdot x=y\cdot y$, or

$$(8) \qquad (x - y)\cdot(x + y) = 0.$$

Define a positive integer γ and an integral vector a by the relations

$$(9) \qquad \gamma = \Delta(x - y), \quad x - y = \gamma a, \quad \Delta(a) = 1,$$

so that equation (8) becomes

* Giudice, page 231.

$$a \cdot (x + y) = 0. \tag{10}$$

By Betti's formula we know that

$$x + y = Ca, \tag{11}$$

where C is an arbitrary skew-symmetric integral matrix, gives all integral solutions of (10). From $x-y=\gamma a$ and (11) we obtain

$$2x = (C + \gamma I)a, \quad 2y = (C - \gamma I)a, \tag{12}$$

where I is the nth order identity matrix. Thus we see that every solution of (8) is given by (12), but, while all vectors x, y obtained from (12) will satisfy equation (8), it is not true that all these vectors will be integral. We could, of course, select the parameters a, C, γ to make x and y integral vectors, but we would still have the problem of giving the precise description of these parameters in order to obtain all integral solutions of the equation (8) and no others.*

We shall find it convenient to introduce certain vectors and matrices for the purposes of the exposition that follows. Let

$$\omega = (-1)^{n+1}, \; J = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ \omega & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The element in the ith row and jth column of the n by n matrix J may also be written as

$$\delta_{i+1}^{j} + \omega\delta_{i+1}^{j+n},$$

where δ_β^α is the Kronecker delta with $\alpha, \beta = 1, 2, \cdots, 2n$. Further, let $u=(1, 1, \cdots, 1)$ be the n-partite vector each of whose components is unity and let U be the n by n matrix, each of whose elements is unity.

We may readily show that

$$J^r = \begin{pmatrix} 0 & I_{n-r} \\ \omega I_r & 0 \end{pmatrix},$$

where I_r is the rth order identity matrix, and also that

$$J^n = \omega I, \quad (J^r)' = \omega J^{n-r}, \tag{13}$$

where the prime indicates the transpose matrix. Finally, let

$$K = J - J^2 + J^3 - \cdots - \omega J^{n-1},$$

and we find

$$(I + J)(I - K) = 2I, \quad I - K \equiv U \pmod{2}. \tag{14}$$

* Cf. Dickson, Introduction to the Theory of Numbers, page 46, first paragraph.

Returning to equation (8) we find, on substituting $x-\gamma a$ for y that

$$2a\cdot x = \gamma a\cdot a. \tag{15}$$

From this last equation we see that $\gamma a\cdot a$ must be even, and we let $\rho=1$ or 2, according as $a\cdot a$ is odd or even. By the first equation of (14) we see that $I+J$ is non-singular so that the vector equation

$$2a = \rho(I+J)b \tag{16}$$

defines a unique vector b. The solution of (16) for b yields, by the first of (14)

$$b = (I-K)a/\rho. \tag{17}$$

In case $\rho=2$, we must show that the components of b are integers. In any case ($\rho=1$ or 2) we note that $a\cdot a\equiv\rho \pmod 2$ and hence

$$u\cdot a \equiv a\cdot a \equiv \rho \pmod{2}. \tag{18}$$

Also, by the second of (14) we have $I-K\equiv U \pmod 2$, so that

$$\rho b = (I-K)a \equiv Ua \equiv \rho u \pmod{2},$$

and hence $\rho b\equiv\rho u \pmod 2$. Thus every component of b is an integer and in case $\rho=1$, every component of b is odd, since $b\equiv u \pmod 2$.

Using (16) we see that the condition $\Delta(a)=1$ requires $\Delta(b)=1$ or 2. However, if $\Delta(b)=2$, the case $\rho=2$ would make $\Delta(a)=2$; and in case $\rho=1$, we have just seen that the components of b must be odd. Hence in both cases $\Delta(b)=1$. The two conditions $\Delta(b)=1$, $b\equiv u \pmod 2$ when $\rho=1$, together are sufficient to insure that the components of the vector a obtained from the vector b by means of (16) will be integers with $\Delta(a)=1$.

Returning now to equation (16) we find

$$4a\cdot a = \rho^2(I+J)b\cdot(I+J)b = \rho^2(b\cdot b + Jb\cdot b + b\cdot Jb + Jb\cdot Jb).$$

Since Jb is a vector we have $Jb=(Jb)'=bJ'$ and by the second of (13), $J' =\omega J^{n-1}$. Also, $b\cdot Jb=Jb\cdot b$ so that

$$2a\cdot a = \rho^2(b\cdot b + Jb\cdot b) = \rho^2(b+Jb)\cdot b = \rho^2(I+J)b\cdot b.$$

Hence, equation (15) becomes

$$2\rho(I+J)b\cdot x = \rho^2\gamma(I+J)b\cdot b,$$

or

$$(I+J)b\cdot(2x-\gamma\rho b) = 0. \tag{19}$$

Now let $\rho\gamma=2\lambda$. This determines an integer λ since $\gamma a\cdot a\equiv\gamma\rho \pmod 2$ from which follows that γ must be even when $\rho=1$. Equation (19) becomes

$$(I+J)b\cdot(x-\lambda b) = 0. \tag{20}$$

If this is multiplied throughout by $\rho/2$ so that, by equation (16), $\Delta[\rho(I+J)b/2]=1$, we find by Betti's formula that

$$
\begin{aligned}
x &= \lambda b + \rho C(I + J)b/2, \\
y &= -\lambda Jb + \rho C(I + J)b/2,
\end{aligned}
\tag{21}
$$

give all integral solutions of equation (8). Thus we see that *all integral solutions of* $x\cdot x=y\cdot y$ *are given by* (21) *where* b, C, λ *are parameters satisfying* $\Delta(b)=1$, $C'=-C$, $\rho=1$ *or* 2, *and in case* $\rho=1$, *every* b_i *is odd.*

If, in formulas (21) we replace b by its expression in terms of a as given by (17), and make use of the identity $J(I-K)=I+K$ (which is readily derived from the first of (14)), we obtain

$$
\begin{aligned}
\rho x &= (D + \lambda I)a, \\
\rho y &= (D - \lambda I)a,
\end{aligned}
\tag{22}
$$

where

$$D = \rho C - \lambda K. \tag{23}$$

Formulas (22) *give the complete solution of the equation* (8) *provided*

(i) D *is an arbitrary integral skew-symmetric matrix,*

(ii) a *is an arbitrary integral vector with* $\Delta(a)=1$,

(iii) λ *is an arbitrary positive integer,*

(iv) $\rho=1$ *or* 2 *but when* $\rho=2$ *all the elements of* $D-\lambda I$ *must have the same parity.*

The truth of (iv) follows if we observe that, in order that C as given by (23), be an integral matrix when $\rho=2$, we must have

$$D - \lambda I = 2C - \lambda(I + K) \equiv \lambda U \pmod 2.$$

Also, in case $\rho=2$, and λ odd, it is necessary that we require $a\cdot u\equiv 0 \pmod 2$. The earlier requirement (18), viz., $a\cdot u\equiv\rho \pmod 2$ need not be retained in the other cases, though, dropping it may lead to a duplication of solutions.

Formulas (22) bear a striking resemblance to formulas (12) but we have now accomplished our purpose, namely, to give the precise descriptions of C, γ, a for obtaining all integral solutions.

If we observe that $Jb=(b_2, b_3 \cdots, b_n, \omega b_1)$ where $\omega=(-1)^{n+1}$, we may write the solutions (21) in the form

$$x_i = \lambda b_i + \rho\sum_{j=1}^{n} c_{ij}(b_j + b_{j+1})/2, \quad y_i = -\lambda b_{i+1} + \rho\sum_{j=1}^{n} c_{ij}(b_j + b_{j+1})/2, \tag{24}$$

where $c_{ij}=-c_{ji}$ and $b_{i+n}=b_i$, (n odd); $b_{i+n}=-b_i$, (n even).

Instead of employing the formulas of Betti for the solutions of equation (10), we may use the expressions (6) to put the solution of equation (8) in a form involving fewer parameters. We readily see that, by the use of (6) and $x-y=\gamma a$, the solution of (8) is

$$x_i = \lambda b_i + \epsilon_i\{(\alpha\cdot\theta)r_i - \theta_i\}, \quad y_i = -\lambda b_{i+1} + \epsilon_i\{(\alpha\cdot\theta)r_i - \theta_i\}, \tag{25}$$

where the vector b is related to the vector a by equation (16), $b_{i+n}=b_i$ (n odd), and $b_{i+n}=-b_i$ (n even), and ϵ_i, α, θ, r_i, are defined as in section 1. Formulas (25) may readily be written in the matrix form. If we denote by $D(a)$ the diagonal matrix having $a_1, a_2, \cdots, a_n$ as the elements of the diagonal, the solution (25) becomes

$$\begin{aligned} x &= \lambda b + D(\epsilon)[D(r)UD(\alpha) - I]\theta, \\ y &= -\lambda Jb + D(\epsilon)[D(r)UD(\alpha) - I]\theta. \end{aligned} \tag{26}$$

3. A simultaneous set of Diophantine equations. As an application of formulas (24) we shall now consider the problem of finding all integral solutions of the system of equations

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= y_1 + y_2 + \cdots + y_n, \\ x_1^2 + x_2^2 + \cdots + x_n^2 &= y_1^2 + y_2^2 + \cdots + y_n^2. \end{aligned} \tag{27}$$

In his Introduction to the Theory of Numbers, Dickson gives* the complete solution of the problem for $n=3$ and $n=4$ and indicates in an exercise how the problem could be solved for general n. We see at once from formulas (24) that $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i$ implies $\sum_{i=1}^n(b_i+b_{i+1})=0$. In case n is odd, this reduces to $\sum_{i=1}^n b_i=0$ and in case n is even, the condition becomes $\sum_{i=2}^n 2b_i=0$ (since $b_{n+1} = -b_1$. In either case these relations could be used to eliminate b_n in formulas (24) and the general solution of the system (27) would result.

As an illustration, let us consider the case $n=3$. We shall use formulas (25) instead of (24), since this will reduce the number of parameters. Using equation (16), we have $a_1+a_2+a_3=\rho(b_1+b_2+b_3)$, and since we require $b_1+b_2+b_3$ to vanish, we see that $a_1+a_2+a_3=0$. In view of this last condition and $\Delta(a_1, a_2, a_3)=1$, we see that the a's are relatively prime in pairs, so that, in the notation of section 1, $\epsilon_1=\epsilon_2=\epsilon_3=1$. Also, since $a_1+a_2+a_3=0$ it follows that $\rho=2$, so that

$$a_1 = b_1 + b_2, \quad a_2 = b_2 + b_3, \quad a_3 = b_3 + b_1.$$

Recalling that $\theta_3=0$, we see that the solution (25) becomes

$$\begin{aligned} x_1 &= \lambda b_1 + \sigma r_1 - \theta_1, & y_1 &= -\lambda b_2 + \sigma r_1 - \theta_1, \\ x_2 &= \lambda b_2 + \sigma r_2 - \theta_2, & y_2 &= \lambda(b_1 + b_2) + \sigma r_2 - \theta_2, \\ x_3 &= -\lambda(b_1 + b_2) + \sigma, & y_3 &= -\lambda b_1 + \sigma, \end{aligned} \tag{28}$$

where

$$\sigma = \theta_1(b_1 + b_2) - \theta_2 b_1, \quad r_1(b_1 + b_2) - r_2 b_1 - b_2 = 1.$$

Rewrite the last equation in the form

$$(r_1 - 1)(b_1 + b_2) - (r_2 - 1)b_1 = 1, \tag{29}$$

* Pages 52, 54, and page 58, ex. 3.

and set

$$\theta_1 = \phi_1(r_1 - 1) + \phi_2 b_1, \quad \theta_2 = \phi_1(r_2 - 1) + \phi_2(b_1 + b_2).$$

These last equations may be solved for integers ϕ_1 and ϕ_2 in view of (29), and we find

$$\sigma = \phi_1, \quad \sigma r_1 - \theta_1 = -\phi_2 b_1 + \phi_1, \quad \sigma r_2 - \theta_2 = -\phi_2(b_1 + b_2) + \phi_1.$$

The solution (28) of the system (27) now takes the form

$$\begin{aligned} x_1 &= (\lambda - \phi_2)b_1 + \phi_1, & y_1 &= -\lambda b_2 - \phi_2 b_1 + \phi_1, \\ x_2 &= (\lambda - \phi_2)b_2 - \phi_2 b_1 + \phi_1, & y_2 &= (\lambda - \phi_2)(b_1 + b_2) + \phi_1, \\ x_3 &= -\lambda(b_1 + b_2) + \phi_1, & y_3 &= -\lambda b_1 + \phi_1. \end{aligned} \tag{30}$$

These are precisely the solutions given by Dickson (page 52), if we set $\lambda = G$, $b_1 = -A$, $b_2 = A - B$, $\phi_2 = D + G$, and observe that ϕ_1 is the arbitrary constant which he mentions.

4. Sum of n squares equal to sum of m squares. Let us now consider the equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_m^2, \qquad (m < n). \tag{31}$$

This problem has been treated by Ballantine and Brown* who gave an algorithm which enables them to go from one solution to another, and, by repeated application of this method, to reach eventually any prescribed solution.

In this section we shall show that, by relating the choice of the vector a entering in the formulas (22) to the choice of the skew-symmetric matrix D, we are able to find all integral solutions of equation (31).

Let λ be a given positive integer and D a given integral skew-symmetric matrix. We wish to determine those vectors a which, when used in formula (22), will cause the last $n-m$ components of the vector y to vanish. Let the adjoint of the matrix $D - \lambda I$ be denoted by $R = \|r_{ij}\|$, and let E stand for the matrix consisting of the first m columns of R, so that

$$E = \|r_{i\alpha}\|, \quad (i = 1, 2, \cdots, n; \alpha = 1, 2, \cdots, m)$$

Let us introduce in the customary fashion, the determinantal factors D_i and the invariant factors d_i for the matrix E. As is well known $d_1 = D_1$, $d_i = D_i/D_{i-1}$ ($i = 2, 3, \cdots, m$), and for each value of i, d_{i-1} is a divisor of d_i. Furthermore, there exist† two integral matrices A and B with dimensions n by n and m by m respectively, and each of determinant unity, such that $AEB = E^*$ where E^* is the normal form of E and is given by the matrix

* Pythagorean sets of numbers, this MONTHLY, 1938, vol. 45, p. 298.

† Veblen and Franklin, On matrices whose elements are integers. Annals of Mathematics, Sec. ser., vol. 23, p. 6.

$$E^* = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & d_m \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since the matrix A is of determinant unity, its reciprocal A^{-1} is also an integral matrix. Now, let

$$a = A^{-1}\theta, \tag{32}$$

where θ is an arbitrary integral n-partite vector. Since $\theta = Aa$ every integral vector a can be represented in the form (32). Also, it is clear that the condition $\Delta(a) = 1$ is equivalent to $\Delta(\theta) = 1$.

Substituting the expression a as given by (32) in the second of the formulas (22), we obtain

$$\rho y = (D - \lambda I)A^{-1}\theta.$$

Multiplying both sides of this last equation on the left by the matrix AR, we find

$$\rho ARy = AR(D - \lambda I)A^{-1}\theta = A \left| D - \lambda I \right| A^{-1}\theta = \left| D - \lambda I \right| \theta, \tag{33}$$

since $R(D-\lambda I) = |D-\lambda I| I$. Denoting by F the matrix consisting of the last $n-m$ columns of the adjoint R, we may write $R = (E, F)$. Hence we have

$$AR = A(E, F) = (AE, AF) = (E^*B^{-1}, AF),$$

since $AEB = E^*$. From the form of the matrix, E^* we see that the matrix AR may be written

$$AR = \begin{pmatrix} G & V \\ O & W \end{pmatrix},$$

where the dimensions of G, V, and W are m by m, m by $n-m$, and $n-m$ by $n-m$, respectively. Since the matrices A and R are non-singular,* the same is also true for the matrices G and W. Hence, by (33) we obtain

$$\left| D - \lambda I \right| \theta = \rho \begin{pmatrix} G & V \\ 0 & W \end{pmatrix} y. \tag{34}$$

Denoting by $\bar{\theta}$ and $\bar{y}$ the vectors composed of the last $n-m$ components of θ and y respectively, we have from (34)

* As is well known, the characteristic equation of a skew-symmetric matrix has no non-zero real root. Consequently, the matrix, $D-\lambda I$, $\lambda > 0$, is non-singular as is also its adjoint R.

(35) $$|D-\lambda I|\,\bar{\theta} = \rho W\bar{y},$$

and, since W is non-singular and $|D-\lambda I|\neq 0$ we see that the condition $\bar{y}=0$ is equivalent to $\bar{\theta}=0$.

Finally, it remains to be seen that for the case $\rho=2$ integral values of y, and not merely halves of integers, are obtained from (34). We know in this case that all the elements of the matrix $D-\lambda I$ are of the same parity and, in fact, that $D-\lambda I\equiv\lambda U \pmod 2$, from which it is seen that

$$\rho y = (D-\lambda I)A^{-1}\theta \equiv \lambda U A^{-1}\theta \pmod 2.$$

Clearly, all the components of the vector $\lambda UA^{-1}\theta$ have the same value, say, c, so that $\rho y_i\equiv c \pmod 2$. But for $i=m+1, m+2, \cdots, n$, we know that each $y_i=0$. Hence c is even and all the y_i's are integers.

Thus we see that, *if the prescribed* λ *and* D *and the associated vector* a, *given by* (32) *with* $\theta_{m+1}=\theta_{m+2}=\cdots=\theta_n=0$ *and* $\Delta(\theta)=1$, *are substituted in the formulas* (22), *we obtain solutions of equations* (31) *and all solutions are found in this way.*

In applications it is usually simpler not to compute the matrices A and B but rather choose an m by m integral matrix Q, such that every element of the matrix EQ is divisible by d_m. That there exist such matrices Q is readily seen by considering the diagonal matrix P having the integers $d_m/d_1, d_m/d_2, \cdots, d_m/d_{m-1}, 1$ for diagonal elements. One easily observes that BP is one such Q, since $EBP=A^{-1}E^*P$ and the right side is readily found to be divisible by d_m. Having found such a matrix Q, we set

$$a = \frac{1}{d_m}EQ\theta,$$

where θ is now an m-partite integral vector with $\Delta(\theta)=1$. With this value of a substituted in formulas (22), we again obtain all solutions of equation (31).

An Illustration. Let us start with $\lambda=3$,

$$D=\begin{pmatrix} 0 & 5 & 5\\ -5 & 0 & 1\\ -5 & -1 & 0\end{pmatrix}\qquad D-\lambda I=\begin{pmatrix} -3 & 5 & 5\\ -5 & -3 & 1\\ -5 & -1 & -3\end{pmatrix}.$$

We may take $\rho=1$ or 2 but shall consider only the case $\rho=2$, so that $\gamma=2\lambda/\rho=3$. The determinant $|D-\lambda I|$ has the value -180. The adjoint of $D-\lambda I$ is

$$R=\begin{pmatrix} 10 & 10 & 20\\ -20 & 34 & -22\\ -10 & -28 & 34\end{pmatrix},$$

so that

$$E = \begin{pmatrix} 10 & 10 \\ -20 & 34 \\ -10 & -28 \end{pmatrix}, \quad D_1 = 2, \quad D_2 = 180, \quad d_1 = 2, \quad d_2 = 90.$$

We may take

$$Q = \begin{pmatrix} 9 & 4 \\ 0 & 5 \end{pmatrix}$$

and

$$a = \frac{EQ\theta}{d_2} = \frac{1}{90} \begin{pmatrix} 10 & 10 \\ -20 & 34 \\ -10 & -28 \end{pmatrix} \begin{pmatrix} 9 & 4 \\ 0 & 5 \end{pmatrix} \theta = (\theta_1 + \theta_2, \ -2\theta_1 + \theta_2, \ -\theta_1 - 2\theta_2).$$

When this vector a is substituted in

$$\rho x = (D + \lambda I)a, \quad \rho y = (D - \lambda I)a,$$

we obtain

$$x = (-6\theta_1 - \theta_2, \ -6\theta_1 - 2\theta_2, \ -3\theta_1 - 6\theta_2),$$
$$y = (-9\theta_1 - 4\theta_2, \ -5\theta_2, \ 0)$$

as the solution corresponding to the given D and λ for $\rho = 2$.

If, however, we wish to use the first method given in this section, we should have to find matrices A and B which, in this case, may be taken to be

$$A = \begin{pmatrix} 12 & 0 & 1 \\ -19 & 0 & -10 \\ 5 & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix},$$

and

$$E^* = AEB = \begin{pmatrix} 2 & 0 \\ 0 & 90 \\ 0 & 0 \end{pmatrix}.$$

We find

$$A^{-1} = \begin{pmatrix} 10 & 1 & 0 \\ 7 & 1 & 1 \\ -19 & -2 & 0 \end{pmatrix},$$

and, since $\theta_3 = 0$,

$$a = A^{-1}\theta = (10\theta_1 + \theta_2, \ 7\theta_1 + \theta_2, \ -19\theta_1 - 2\theta_2).$$

On introducing this a in formulas (22), we obtain

$$x = (-15\theta_1 - \theta_2, \ -24\theta_1 - 2\theta_2, \ -57\theta_1 - 6\theta_2),$$
$$y = (-45\theta_1 - 4\theta_2, \ -45\theta_1 - 5\theta_2, \ 0).$$

It may readily be shown that this solution is equivalent to the one given above.

5. Parametric solution of equation (31). **An illustration.** The reader will observe that, strictly speaking, the method given in the previous section does not give a complete solution of equation (31) in parameters, inasmuch as, for a matrix of indeterminates D, one could not exhibit the matrices A and B. However, the complete solution in parameters could be given by the use of the notations of tensor analysis. We shall illustrate the method by finding the general solution of the problem

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2, \tag{36}$$

and point out as we go along what the modifications would be for the general problem.

Let the matrix $D-\lambda I$ be given by

$$D - \lambda I = \begin{pmatrix} -\lambda & -\zeta & -\eta \\ \zeta & -\lambda & -\xi \\ \eta & \xi & -\lambda \end{pmatrix},$$

where ξ, η, ζ, λ are arbitrary integers and λ is positive. Consider the matrix

$$Z = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ \theta_1 & \theta_2 & \theta_3 \\ \eta & \xi & -\lambda \end{pmatrix}.$$

For the general equation (31) the matrix Z is obtained by replacing the first row of $D-\lambda I$ by $\phi_1, \phi_2, \cdots, \phi_n$, the next $m-1$ rows by a set of $n(m-1)$ parameters* $\theta_{\alpha\beta}(\alpha=1, 2, \cdots, m-1;\ \beta=1, 2, \cdots, n)$, and leaving the remaining $n-m$ rows of $D-\lambda I$ unaltered. Let μ be the g. c. d. of all the $(n-m)$th order determinants formed from the last $n-m$ rows of the matrix Z. Then *if the vector a, defined by setting μa_i equal to the cofactor of ϕ_i in the matrix Z, is substituted in the formulas* (22), *the general parametric solution of equation* (31) *will be obtained.*

Returning to our special equation (36), we have $\mu=\Delta(\xi, \eta, \lambda)$ and

$$\mu a_1 = -\theta_2\lambda + \theta_3\xi, \quad \mu a_2 = \theta_1\lambda - \theta_3\eta, \quad \mu a_3 = -\theta_1\xi + \eta\theta_2.$$

Using $\mu\rho x=(D+\lambda I)\mu a$, $\mu\rho y=(D-\lambda I)\mu a$, we find

$$\begin{aligned} \mu\rho x_1 &= -\theta_1(\xi\eta + \lambda\zeta) + \theta_2(\eta^2 - \lambda^2) - \theta_3(\xi\lambda + \eta\zeta), \\ \mu\rho x_2 &= \theta_1(\lambda^2 - \xi^2) - \theta_2(\lambda\zeta - \xi\eta) + \theta_3(\lambda\eta - \xi\zeta) \\ \mu\rho x_3 &= 2\lambda(\xi\theta_1 - \eta\theta_2), \end{aligned}$$

* The number of parameters introduced here will, in general, be larger than the m parameters θ employed in the solution of this problem in section 4. This indicates that in certain special situations not all of these parameters would be essential, but the relative simplicity of the solution given in this section, compensates for this loss.

$$\mu\rho y_1 = -\theta_1(\xi\eta + \lambda\zeta) + \theta_2(\lambda^2 + \eta^2) + \theta_3(\xi\lambda - \eta\zeta),$$
$$\mu\rho y_2 = -\theta_1(\lambda^2 + \xi^2) - \theta_2(\lambda\zeta - \xi\eta) - \theta_3(\xi\zeta + \lambda\eta),$$
$$\mu\rho y_3 = 0.$$

where $\mu = \Delta(\xi, \eta, \lambda)$, $\Delta(\theta_1, \theta_2, \theta_3) = 1$, λ is a positive integer, $\rho = 1$ unless λ, ξ, η, ζ are all of the same parity, in which case $\rho = 1$ or 2. Thus we have a general parametric solution of equation (36)

6. On the sum of n squares equal to a square. We are now in a position to give a complete parametric solution of the Diophantine equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2. \tag{37}$$

In this case ($m=1$) the formula $a = EQ\theta/d_m$ of section 4 takes on a particularly simple form. Here E consists of one column whose elements are $r_{11}, r_{21}, \cdots, r_{n1}$, $D_1 = d_1$ is the g. c. d. of these elements, and the matrix Q has only one element which would, in fact be unity. The single arbitrary parameter θ would be restricted to be ± 1 by the condition $\Delta(\theta) = 1$. Hence, if we represent by μ either of the values $\pm\Delta(r_{11}, r_{21}, \cdots, r_{n1})$ we have $a_i = r_{i1}/\mu$. Using these components of the vector a in formulas (22), we see that a complete parametric solution of equation (37) is given by

$$y_1 = \frac{1}{\rho\mu}\left| D - \lambda I\right|, \qquad x_1 = y_1 + \frac{2\lambda}{\rho\mu} r_{11}, \qquad x_i = \frac{2\lambda}{\rho\mu} r_{i1}, \quad (i = 2, 3, \cdots, n), \tag{38}$$

where we should keep in mind that the n quantities r_{i1} must be expressed in terms of the elements of the matrix $D - \lambda I$ which are the parameters for our solution. We recall that $\rho = 1$ unless all of the elements of $D - \lambda I$ have the same parity, in which case ρ may be 1 or 2.

We know, of course, that the solution (39) satisfies the equation (37). If one attempts to verify directly that this is the case, he is led to an identity involving the elements of the adjoint of the characteristic matrix of a skew-symmetric matrix. This identity is believed to be new and may be derived as follows.

Since R is the adjoint of the matrix $D - \lambda I$ we have

$$(D - \lambda I)R = \left| D - \lambda I\right| I,$$

from which we get

$$DR = \left| D - \lambda I\right| + \lambda R. \tag{39}$$

Taking the transpose of both sides of this last equation, we obtain

$$-R'D = \left|D - \lambda I\right| + \lambda R', \tag{40}$$

where we have denoted by R' the transpose of R and have used the defining property of D, viz., $D' = -D$. Multiply the equation (39) by R' on the left, and

the equation (40) by R on the right and add. We obtain

$$(41) \qquad |D - \lambda I| (R + R') + 2\lambda R'R = 0,$$

the desired relation, an identity in λ and the elements of the matrix D. One may obtain a similar relationship with R and R' interchanged, so that R is commutative with R'.

In particular, the identity (41) yields

$$|D - \lambda I| r_{11} + \lambda \sum_{i=1}^{n} r_{i1}^2 = 0,$$

the relation which one needs in order to verify directly that (38) is a solution of equation (37).

THE ADAMS SPHERE*

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1. Introduction. Among the analogies existing between the triangle and the tetrahedron are some remarkable circles and spheres. Corresponding to the nine-point circle of a triangle there has long been pointed out two twelve-point spheres for an orthocentric tetrahedron and more recently a twelve-point sphere for any tetrahedron [1]. Likewise, for the isodynamic tetrahedron, spheres are known corresponding to the two Lemoine circles of a triangle [2]; more recently a system of Lemoine spheres has been given for any tetrahedron [3]. Finally, for the general tetrahedron, we have pointed out the Longchamps sphere, previously obtained only for the orthocentric tetrahedron [4] and having many analogies to the circle of the same name for the triangle [5]. The present article concerns the synthetic study of a sphere of a general tetrahedron corresponding to the Adams circle [6] of a triangle. In collaboration with R. Bouvaist (Nantes, France) we first pointed out this sphere without giving any derivation [7].

2. Adams Circles. Given a triangle ABC, its incircle (I) with center I and touching BC, CA, AB at A', B', C', consider a concentric circle (I, ρ) of radius ρ cutting BC, CA, AB in X and X', Y and Y', Z and Z'. The lines XZ', ZY', YX' form a triangle $A_1B_1C_1$ homothetic to ABC, the homothetic center being the point P, the intersection of the lines AA', BB', CC', i.e., the Gergonne point of ABC or the Lemoine point of $A'B'C'$ and of $A_1B_1C_1$.

The sides BC, CA, AB being perpendicular to the radii IA', IB', IC' of (I)

* Translated from the French by J. E. LaFon, University of Oklahoma.

at A', B', C', the equal segments XX', YY', ZZ' are antiparallel to B_1C_1, C_1A_1, A_1B_1 in the angles A_1, B_1, C_1 of $A_1B_1C_1$. Thus (I, ρ) *is a Tucker circle of the triangle* $A_1B_1C_1$.

The circles (ω_a), (ω_b), (ω_c) of radii ρ_a, ρ_b, ρ_c tangent respectively to AB and AC at Z and Y', to BA and BC at Z' and X, to CB and CA at X' and Y are orthogonal to a circle (P) of center P, for the radical center of these circles is P since A', B', C' bisect XX', YY', ZZ' and $AY=AZ'$, $BZ=BX'$, $CX=CY'$. Conversely, given a circle (P) the circles (O_a), (O_b), (O_c) orthogonal to it and tangent to AB and AC, BA and BC, CB and CA meet these sides in six points of a circle concentric with (I).

When (P) is of zero radius, the circles $(O_a)\equiv(\omega_a')$, $(O_b)\equiv(\omega_b')$, $(O_c)\equiv(\omega_c')$ whose centers are within the segments AI, BI, CI touch the sides of ABC at Z'' and Y''', Z''' and X'', X''' and Y'' in such a manner that the lines $Y''Z'''$, $Z''X'''$, $X''Y'''$ meet in P and are perpendicular to the bisectors AI, BI, CI. *The circle* (I, ρ') passing through X'', X''', Y'', Y''', Z'', Z''' *is the Adams circle of the triangle* ABC as well as *the first Lemoine circle of the triangle* A_1', B_1', C_1', *formed by the lines* $X''Z'''$, $Z''Y'''$, $Y''X'''$.

It should also be observed that $X''Y'''$, $Y''Z'''$, $Z''X'''$ are the radical axes of (ω_a'), (ω_b'), (ω_c') of radii ρ_a', ρ_b', ρ_c' associated with the circles (ω_a''), (ω_b''), (ω_c'') of radii ρ_a'', ρ_b'', ρ_c'' which pass through P and its symmetrics with respect to AI, BI, CI and then touch AB and AC, BA and BC, CB and CA at Z_1' and Y_2', Z_2' and X_1', X_2' and Y_1' in such a manner that $A'X_1'=A'X_2'=B'Y_1'=B'Y_2'=C'Z_1'=C'Z_2'=3X'X''/2$. Thus *the six points* X_1', X_2', Y_1', Y_2', Z_1', Z_2' *lie on a circle* (I, ρ'') *concentric with the inscribed circle of the triangle* ABC which may be called *the second Adams circle*. In addition (I, ρ'') *is a Tucker circle of the triangle* $A_2'B_2'C_2'$ *determined by the lines* $X_2'Z_1'$, $Y_1'Z_2'$, $X_1'Y_2'$.

Using the customary notations for the triangle ABC and putting $4R+r=\delta$, simple calculations give successively

$$X''X''' = Y''Y''' = Z''Z''' = 8Rr\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\Big/\delta = 2S/\delta,$$

$$\rho_a' = 4Rr\cos^2\frac{A}{2}\Big/\delta, \quad \rho_a'' = 4r\left(1-3R\cos^2\frac{A}{2}\Big/\delta\right),$$

$$\rho_a' + \rho_b' + \rho_c' = 4Rr\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\Big/\delta = 2r,$$

$$\rho_a'' +' \rho_b'' + \rho_c'' = 4r\left[3 - 3R\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\Big/\delta\right] = 6r,$$

$$\rho' = r[1+(p/\delta)^2]^{1/2}, \qquad \rho'' = r[1+(3p/\delta)^2]^{1/2}.$$

The *associates* of the Gergonne point P of the triangle ABC give similar results and there are four *first circles* and four *second circles* of Adams.

3. Adams Spheres. In a tetrahedron $T \equiv ABCD$, the inscribed sphere (I) with center I touches the faces BCD, CDA, DAB, ABC at A', B', C', D'. A sphere (I, ρ) of arbitrary radius ρ concentric with (I) cuts these same faces in equal circles (A'), (B'), (C'), (D') which meet the lines $(A'B, A'C, A'D)$, $(B'C, B'D, B'A)$, $(C'D, C'A, C'B)$, $(D'A, D'B, D'C)$ respectively between B and A', C and A', D and A', $\cdots$, A and D', B and D', C and D' at (X_b, X_c, X_d), (Y_c, Y_d, Y_a), (Z_d, Z_a, Z_b), (V_a, V_b, V_c). The triangles $X_bX_cX_d$, $Y_cY_dY_a$, $Z_dZ_aZ_b$, $V_aV_bV_c$ are equal for their circumcircles are equal and the angles $BA'C$, $CA'D$, $DA'B$ formed by the lines joining the vertices of the tetrahedron T to the points of contact A', B', C', D' of the insphere (I) with the faces are equal for the four faces [8].

The planes $Y_aZ_aV_a$, $X_bZ_bV_b$, $X_cY_cV_c$, $X_dY_dZ_d$ form a tetrahedron $T_1 \equiv A_1B_1C_1D_1$ homothetic to $A'B'C'D'$. The faces of T being perpendicular to the radii of (I) which circumscribes the tetrahedron $T' \equiv A'B'C'D'$ are anti-parallel to the opposite faces of T_1 in the trihedrons (A_1), (B_1), (C_1), (D_1). They meet the edges of T_1 in the vertices of the tangential triangles $X_b'X_c'X_d'$, $Y_c'Y_d'Y_a'$, $Z_d'Z_a'Z_b'$, $V_a'V_b'V_c'$ of the equal triangles $X_bX_cX_d$, $Y_cY_dY_a$, $Z_dZ_aZ_b$, $V_aV_bV_c$. These tangential triangles are equal in turn and the homothetic center of T and the tangential tetrahedron T_1' of T_1 is the point L whose distances to the faces of T_1' are proportional to the circumradii of these faces [9]. But L is also the homothetic center of T' and T_1; thus, *in the tetrahedron T_1, the sphere (I, ρ) belongs to a Tucker system of axis O_1L, where O_1 is the circumcenter* [10].

If the spheres (ω_a), (ω_b), (ω_c), (ω_d) are inscribed in the trihedrons (A), (B), (C), (D) of T and touch respectively the three adjacent faces at (Y_a, Z_a, V_a), (X_b, Z_b, V_b), (X_c, Y_c, V_c), (X_d, Y_d, V_d), the points A' and A_1, B' and B_1, C' and C_1, D' and D_1 belong to the radical axes of the spheres $[(\omega_b), (\omega_c), (\omega_d)]$, $[(\omega_c), (\omega_d), (\omega_a)]$, $[(\omega_d), (\omega_a), (\omega_b)]$, $[(\omega_a), (\omega_b), (\omega_c)]$. *The radical center of the spheres (ω_a), (ω_b), (ω_c), (ω_d) is thus the point L which remains fixed when the sphere (I, ρ) varies; these spheres (ω_a), (ω_b), (ω_c), (ω_d) are orthogonal to a sphere (L), which reduces to a point sphere when the spheres meet in this point.* Conversely, a sphere (L) of center L being given, the spheres (O_a), (O_b), (O_c), (O_d) orthogonal to this sphere and inscribed in the trihedrons (A), (B), (C), (D) respectively, touch the adjacent faces in twelve points of a sphere concentric with the insphere (I) of T.

In particular, when (L) is a point sphere, the spheres (ω_a'), (ω_b'), (ω_c'), (ω_d') passing through L and having centers within the segments AI, BI, CI, DI are tangent to the faces of T *at twelve points of a sphere concentric with the insphere* (I). (*Adams sphere*) [7].

In the general case, we have fixed the positions of the points (X_b, X_c, X_d), $\cdots$, (V_a, V_b, V_c), but as BA', CA', DA' cut the circle (A') in six points, there are four groups of spheres like $[(\omega_a), (\omega_b), (\omega_c), (\omega_d)]$ to consider. Furthermore, to each of the tetrahedrons $A'B'C'D'$ corresponding to the eight centers of the spheres tangent to the four faces of T, corresponds a point L, there being in all eight positions of L, with each of which can be associated quadruples of spheres, like $[(\omega_a), (\omega_b), (\omega_c), (\omega_d)]$ and $[(\omega_a'), (\omega_b'), (\omega_c'), (\omega_d')]$, these latter spheres hav-

ing their centers on AI, BI, CI, DI, $\cdots$ at points different from $(\omega_a, \omega_b, \omega_c, \omega_d)$, $\cdots$. Here, we shall not continue this study which goes beyond the task which we undertook.

4. Isogonic Tetrahedrons [11]. In this particular tetrahedron $ABCD$ in which the lines AA', BB', CC', DD' meet in the previously mentioned point L of the tetrahedron $T'=A'B'C'D'$, the tangent planes of the spheres (ω_a'), (ω_b'), (ω_c'), (ω_d') inscribed in the trihedrons (A), (B), (C), (D) and passing through L are parallel to the faces of BCD, CDA, DAB, ABC of T. Consequently the points X_d', L, V_a', for example, are collinear with the external center of similitude of (ω_d') and (ω_a'). Likewise, the lines $X_b' Y_a'$, $X_c' Z_a'$, $Y_c' Z_b'$, $Y_d' Z_b'$, $Z_d' V_c'$ meet in L. The planes through L parallel to BCD, CDA, DAB, ABC determine in T four homothetic tetrahedrons. The planes $(X_d' V_a', Z_d' V_c')$, $(Z_d' V_c', Y_d' V_b')$, $(Y_d' V_b', X_d' V_c')$, $\cdots$ are thus perpendicular to axes BI, CI, DI, AI of T.

Accordingly, the intersections $X_d' V_a'$, $Y_d' V_b'$, $Z_d' V_c'$, $X_b' Y_a'$, $X_c' Z_a'$, $Y_c' Z_b'$ of the planes are parallel to the edges of the tetrahedrons T' and T_1 associated with T. In the present case T' and T_1 being isodynamic, the Adams sphere of center I, containing the twelve points of contact of the spheres (ω_a'), (ω_b'), (ω_c'), (ω_d') with the faces of T and passing through L, is for T' and T_1 a known Lemoine sphere [2].

If the configurations consisting of the isogonic tetrahedron T and the spheres (ω_a'), (ω_b'), (ω_c'), (ω_d') are transformed by a homography leaving the plane at infinity invariant, the following proposition is obtained: [7] *Let a quadric (Q) be tangent to the faces $[A]$, $[B]$, $[C]$, $[D]$ of a tetrahedron $ABCD$ at A', B', C', D'. Suppose the lines AA', BB', CC', DD' meet at a point L. Let T_a, T_b, T_c, T_d be cones with vertices A, B, C, D circumscribing (Q). The planes through L parallel to the planes $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$ cut T_a, T_b, T_c, T_d in four conics which lie on a quadric (Q') homothetic and concentric to (Q).*

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A STUDY OF A QUADRILATERAL INSCRIBED IN A CIRCLE

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1. Introduction. The analytic treatment, by means of complex coördinates, of the properties of a quadrilateral inscribed in a circle suggests several considerations, which do not seem to have been developed before; particularly, the introduction of the generalized Simson line leads to remarkable properties, most of which are believed to be new, especially those on the envelope of the Simson lines of such a quadrilateral; similarly interesting properties will be obtained regarding orthopole-lines.

Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in the base-circle Γ, having as center 0 and as radius unity, Ω the point with coördinate 1; t_1, t_2, t_3, t_4 the turns* corresponding to the vertices; σ_1, σ_2, σ_3, σ_4 the symmetric functions of these turns

$$\sigma_1 = \Sigma t_1, \quad \sigma_2 = \Sigma t_1t_2, \quad \sigma_3 = \Sigma t_1t_2t_3, \quad \sigma_4 = t_1t_2t_3t_4.$$

If $\bar{a}$ is the conjugate of a, then

$$\bar{\sigma}_1 = \sigma_3/\sigma_4, \quad \bar{\sigma}_2 = \sigma_2/\sigma_4, \quad \bar{\sigma}_3 = \sigma_1/\sigma_4, \quad \bar{\sigma}_4 = 1/\sigma_4.$$

Further, τ being the coördinate of any point M on Γ, the Simson line Δ of M with respect to the quadrilateral is†

$$4x\tau^2 + 4\bar{x}\sigma_4 = 3\tau^3 + \sigma_1\tau^2 - \sigma_2\tau + \sigma_3 + 3\sigma_4/\tau. \tag{1}$$

It is also useful to mention here a few properties we have given‡ about the geometric meaning of the symmetric functions

$$s_1 = t_1 + t_2 + t_3, \quad s_2 = t_1t_2 + t_2t_3 + t_3t_1, \quad s_3 = t_1t_2t_3$$

of the turns corresponding to the vertices of a triangle $A_1A_2A_3$.

As is well known, s_1 is the orthocenter of $A_1A_2A_3$; s_2 is the image of s_1 in the circumdiameter parallel to the Simson line of Ω with respect to the triangle $A_1A_2A_3$ and s_3 is the point where Γ meets again the perpendicular drawn from Ω on that Simson line; the point $(s_1+s_3)/2$ is the orthopole§, in the triangle $A_1A_2A_3$, of the circumdiameter d perpendicular to $O\Omega$.

It may further be recalled that s_2/s_1 is the Feuerbach point of the tangential triangle of $A_1A_2A_3$,‖ or the point of Γ having its Simson line with respect to $A_1A_2A_3$ parallel to the Euler line of that triangle.

* Frank and F. V. Morley, Inversive Geometry, 1933, p. 15.

† The projections of M on its Simson lines with respect to the triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ are on a straight line, the Simson line of M with respect to the quadrilateral $A_1A_2A_3A_4$ (E. M. Langley, Educational Times, vol. LI; M. Kobayashi, Tôhoku Mathematical Journal, vol. 28, 1927, p. 46; see also my paper in this MONTHLY, vol. 47, 1940, pp. 466–468). Continuing the process, we find the definition of the Simson line of the point M with respect to any polygon inscribed in Γ.

‡ Mathesis, 1939, p. 72; this MONTHLY, vol. 46, p. 269.

§ See paragraph 6.

‖ P. Delens, Mathesis, 1937, p. 265; J. R. Musselman, this MONTHLY, vol. 45, 1938, p. 423.

2. The fundamental points. The points P_1, P_2, P_3, P_4 having as coordinates σ_1, σ_2, σ_3, σ_4 are important in the geometry of the quadrilateral.

The point P_1 is fixed for any given quadrilateral, whatever the position of Ω on Γ may be. The point G with coördinate $\sigma_1/4$ is the center of gravity of four equal masses placed at the points $A_i(t_i)$, and, as G lies on the join of O to $P_1(\sigma_1)$ and is such that $OP_1=4OG$, we find the following construction for P_1:

If G is the center of gravity of four equal masses placed at the vertices, P_1 is the point such that $\overline{O}P_1=4\overline{O}G$.

The point G is also the common mid-point of the joins of mid-points of the pairs of the opposite sides and diagonals.

For a given quadrilateral $A_1A_2A_3A_4$, the position of the point P_2 is variable and depends on the position of the unit-point on Γ. For, if a new unit-point Ω' were chosen, the new coördinate of the point corresponding to P_2, say P_2', would be σ_2/λ^2 and that of the primitive one σ_2/λ, so that the angles (OP_2', OP_2) and $(O\Omega, O\Omega')$ are equal.

When, for a given quadrilateral $A_1A_2A_3A_4$, Ω is a moving point on Γ, P_2 describes a concentric circle.

The construction of P_2 for a given unit-point Ω is very simple: if g is the center of gravity of equal masses placed at the points where Γ meets again the parallels drawn through Ω to the sides and the diagonals of the quadrilateral, then $\overline{O}P_2=6\overline{O}g$.

For the parallel to A_2A_3, for instance, drawn from Ω meets Γ at the point t_2t_3; the point g has for coördinate $\sigma_2/6$ and lies on the straight line joining O to the point with coördinate σ_2.

When Ω is at A_4, $\sigma_2/2$ is

$$\tfrac{1}{2}(t_1+t_2+t_3+t_1t_2+t_2t_3+t_3t_1)$$

and is therefore the projection of the orthocenter of $A_1A_2A_3$ on the circumdiameter parallel to the Simson line of A_4 with respect to $A_1A_2A_3$; so we find the following theorem:

If $A_1A_2A_3A_4$ is a quadrilateral inscribed in a circle with center O, the perpendiculars drawn from the orthocenter of each of the triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ on the respective Simson lines of A_1, A_2, A_3, A_4 with respect to these triangles are at the same distance from O.

The Simson line of Ω with respect to the quadrilateral being

$$4x+4\bar{x}\sigma_4=3+\sigma_1-\sigma_2+\sigma_3+3\sigma_4,$$

the point P_4 is the second intersection of the circle Γ with the parallel drawn through Ω to the Simson line of that point with respect to the quadrilateral.

As $\sigma_3=\bar{\sigma}_1\sigma_4$, P_3 will be derived from the image of P_1 in $O\Omega$ by a rotation $(O\Omega, OP_4)$.

Hence, for a given quadrilateral, when Ω describes Γ, P_3 describes a circle

having O as center. and OP_1 as radius.

The equations of OP_2 and P_1P_3 being

$$x - \bar{x}\sigma_4 = 0, \quad x + \bar{x}\sigma_4 = \sigma_1 + \sigma_3.$$

OP_2 is perpendicular and P_1P_3 parallel to the Simson line of Ω with respect to the quadrilateral, and, as further $OP_1 = OP_3$, P_1 and P_3 are symmetric with respect to OP_2.

When $\sigma_1 = 0$ or $\sigma_3 = 0$ the quadrilateral is a rectangle. But the case $\sigma_2 = 0$ is interesting: then $t_4 = s_2/s_1$, and we find the following theorem:

*When a quadrilateral inscribed in a circle is such that one of the vertices is the image in the circumcenter of the Feuerbach point of the tangential triangle of the triangle formed by the three others, the same property applies to the four vertices.**

3. A square associated with the quadrilateral. Let us consider the points M such that their Simson line is perpendicular to the radius OM, which has the equation $x = \bar{x}\tau^2$; then $\tau^4 = \sigma_4$, and it is easy to verify that the corresponding values of τ are fixed whatever may be the position of Ω on Γ.

There are four points L_1, L_2, L_3, L_4, forming a square, such that the Simson lines of the points L_i are perpendicular to the circumdiameters passing through these points.†

4. Envelope of the Simson lines. The values of τ corresponding to the Simson lines drawn from a point α are given by

$$3\tau^4 + (\sigma_1 - 4\alpha)\tau^3 - \sigma_2\tau^2 + (\sigma_3 - 4\bar{\alpha}\sigma_4)\tau + 3\sigma_4 = 0.$$

If $\pm\theta_1$ and $\pm\theta_2$ are the values of τ when α is the point G, i.e. the roots of

$$3\tau^4 - \sigma_2\tau^2 + 3\sigma_4 = 0, \tag{2}$$

the equations of the double tangents Δ_1, Δ_2 from G to the envelope E are

$$4x\theta_1^2 + 4\bar{x}\sigma_4 = \sigma_1\theta_1^2 + \sigma_3,$$
$$4x\theta_2^2 + 4\bar{x}\sigma_4 = \sigma_1\theta_2^2 + \sigma_3.$$

But

$$3(\tau_2 - \theta_1^2)(\tau' - \theta_2^2) = 3\tau^4 - \sigma_2\tau^2 + 3\sigma_4.$$

The straight line (1) cuts Δ_1 and Δ_2 at points $x_1(\xi_1)$ and $x_2(\xi_2)$ such that

$$4\xi_1 = \sigma_1 + 3(\tau^2 - \theta_2^2)/\tau, \quad 4\xi_2 = \sigma_1 + 3(\tau^2 - \theta_1^2)/\tau,$$

and the square of their distance is

* The Simson line of any of the vertices with respect to the triangle formed by the three others is then perpendicular to the Euler line of that triangle.

† A similar property applies to any polygon inscribed in a circle and having an *even* number of sides; when the number n of sides is *odd*, the points L_i having as coördinates the n values of $\sigma_n^{1/n}$ are such that their Simson lines are parallel to OL_i.

$$d^2 = (\xi_1 - \xi_2)(\bar{\xi}_1 - \xi_2) = -\frac{9}{16}\frac{(\theta_1^2 - \theta^2)^2}{\theta_1^2\theta_2^2};$$

hence, from (2),

$$4d = (36 - \sigma_2\bar{\sigma}_2)^{1/2} = (36 - OP_2^2)^{1/2}, \tag{3}$$

i.e. a constant, and the envelope of Δ is an astroid A.

The perpendiculars erected at X_1 and X_2 on Δ_1 and Δ_2 are

$$4x\theta_1^2 - 4\bar{x}\sigma_4 = \sigma_1\theta_1^2 + 3\theta_1^2(\tau^2 - \theta_2^2)/\tau - \sigma_3 + 3\sigma_4(\tau^2 - \theta_2^2)/\tau\theta_2^2,$$
$$4x\theta_2^2 - 4\bar{x}\sigma_4 = \sigma_1\theta_2^2 + 3\theta_2^2(\tau - \theta_1^2)/\tau - \sigma_3 + 3\sigma_4(\tau^2 - \theta_1^2)/\tau\theta_1^2,$$

and their intersection

$$x = \sigma_1/4 + 3\tau/2$$

describes a circle having G as center and $3/2$ as radius.

The envelope of the Simson lines of a quadrilateral inscribed in a circle Γ *is an astroid having as center the common mid-point of the distances between the mid-points of the pairs of opposite sides and diagonals; it is a parallel curve to a regular astroid inscribed in a circle having as radius one and a half times the radius of* Γ.

When $\sigma_2=0$ (see section 2), the astroid becomes a regular one.*

5. Determination of the astroid. The complete determination of A is very simple. Let M_1 and M_2 be the points on Γ having as Simson lines Δ_1 and Δ_2, and M_1' and M_2' the points on Γ such that, between arcs in the same rotation sense on Γ, $M_1M_1' = \Omega M_1$ and $M_2M_2' = \Omega M_2$; the coördinates of M_1' and M_2' are then θ_1^2 and θ_2^2.

As $\theta_1^2\theta_2^2 = \sigma_4$, $M_1'M_2'$ is parallel to ΩP_4. Further, as $\theta_1^2+\theta_2^2=\sigma_2/3$ we find the following construction for Δ_1 and Δ_2:

If J is the point on OP_2 such that $OJ=OP_2/6$ and if the perpendicular at J on OP_2 meets Γ at M_1', M_2', the lines Δ_1, Δ_2 are the Simson lines of the mid-points M_1, M_2 of the arcs $\Omega M_1'$, $\Omega M_2'$.

The astroid A is the envelope of the join of the projections X_1 and X_2 on Δ_1 and Δ_2 of a point moving on a circle having G as center and $3/2$ as radius.

The constant length X_1X_2 is given by (3) and easy to construct.

6. Orthopole and orthopole lines. The perpendiculars dropped on the sides of a triangle $A_1A_2A_3$ from the projections of the corresponding vertices on a straight line, δ,

$$x/a + \bar{x}/\bar{a} = 1$$

* Another regular astroid may be mentioned here in connection with any inscribed quadrilateral: *The line drawn through a moving point on the circumcircle, parallel to the Simson line of that point with respect to the quadrilateral, envelopes a regular astroid*; see my paper on "A theorem on a cyclic polygon," this MONTHLY, vol. 47, 1940, p. 466.

concur at the *orthopole* of δ with respect to the triangle.

The projection of A_1 on δ being

$$(a + t_1 - a/\bar{a}t_1)/2,$$

the orthopole is

$$(a + s_1 + \bar{a}s_3/a)/2.$$

Let us now consider the orthopoles K_1, K_2, K_3, K_4 of δ with respect to the four triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$.

The point K_4 is

$$x = (a + \sigma_1 - t_4 + \bar{a}\sigma_4/at_4)/2,$$

and

$$\bar{x} = (\bar{a} + \sigma_3/\sigma_4 - 1/t_4 + at_4/\bar{a}\sigma_4)/2.$$

If we eliminate t_4 from the above equations, we find that K_4 is on the line

$$2ax + 2\bar{a}\sigma_4\bar{x} = a\sigma_1 + \bar{a}\sigma_3 + a^2 + \bar{a}^2\sigma_4$$

and, the equation being symmetric in t_1, t_2, t_3, t_4, the four points K_1, K_2, K_3, K_4 are on that straight line.

The orthopoles of any straight line with respect to the triangles having as vertices three of the vertices of the quadrilateral are on a straight line.

When δ is the circumdiameter passing through the point τ on Γ, the equation of the line of orthopoles

$$2x\tau^2 - 2\bar{x}\sigma_4 = \sigma_1\tau^2 - \sigma_3$$

is satisfied by $x = \sigma_1/2$.

The line of orthopoles of a circumdiameter is perpendicular to the Simson lines of its extremities with respect to the quadrilateral and passes through the image of O in G.

When δ is the tangent to Γ at the point $M(\tau)$, the orthopole line is

$$2x\tau^2 + 2\bar{x}\sigma_4 = \sigma_1\tau^2 + \sigma_3 + 2\tau^3 + 2\sigma_4/\tau.$$

The orthopole line of the tangent at a point of the circumcircle is parallel to the Simson line of that point with respect to the quadrilateral.

We will now find the envelope E' of that orthopole line when $M(\tau)$ describes Γ.

As the tangents from a point α to E' correspond to the points of Γ having as turns the roots of the equation

$$\tau^4 - (\alpha - \sigma_1/2)\tau^3 - (\bar{\alpha}\sigma_4 - \sigma_3/2)\tau + \sigma_4 = 0, \tag{4}$$

the turns of the contact points of Γ with the tangents whose orthopole lines are the double tangents to E' are given by $\tau^4 + \sigma_4 = 0$.

The orthopole lines of the tangents to Γ parallel to the sides of the square $L_1L_2L_3L_4$ coincide two by two and are perpendicular to each other at the mid-point of OP_1.

If $\rho_1^2=(-\sigma_4)^{1/2}$ and $\rho_2^2=-(-\sigma_4)^{1/2}$, the equations of the double tangents are

$$2x\rho_1^2+2\bar{x}\sigma_4=\rho_1^2\sigma_1+\sigma_3,$$

$$2x\rho_2^2+2\bar{x}\sigma_4=\rho_2^2\sigma_1+\sigma_3,$$

and as

$$(\tau^2-\rho_1^2)(\tau^2-\rho_2^2)=\tau^3+\sigma_4,$$

(4) cuts these lines at the points

$$\sigma_1/2+(\tau^2-\rho_2^2)/\tau,\qquad \sigma_1/2+(\tau^2-\rho_1^2)/\tau;$$

the distance of these points being 4, we have the following theorem:

The orthopole lines of the tangents to the circumcircle envelope a regular astroid having its inscribed circle equal to the circumcircle of the quadrilateral.

7. Remarkable circles. As $s_1=\sigma_1-t_4$, the expression σ_1-t represents successively the orthocenters of $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ when t is replaced by t_1, t_2, t_3, t_4.

The orthocenters of the four triangles having as vertices three of the four vertices of the quadrilateral are on a circle equal to the circumcircle and having as center the point P_1.

Further the relation

$$s_1+s_3+t_4=\sigma_1-\sigma_4/t_4$$

shows that, if ω_1, ω_2, ω_3, ω_4 are the orthopoles of the circumdiameter α with respect to the triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$, the just mentioned orthocenter-circle, contains also the extremities ω_1', ω_2', ω_3', ω_4' of segments $A_1\omega_1'$, $A_2\omega_2'$, $A_3\omega_3'$, $A_4\omega_4'$ parallel to $O\omega_1$, $O\omega_2$, $O\omega_3$, $O\omega_4$ and equal to twice these respective segments.

It may also be noted that the projection on A_1A_2 of the point $2\sigma_4/\sigma_3$, which is the inverse, with respect to Γ, of the mid-point of OP_1, has for coördinate

$$\sigma_4/\sigma_3+(t_1+t_2)/2-t_1t_2/\sigma_1$$

and the centroid of the pedal triangle of the considered point with respect to $A_1A_2A_3$ is

$$\sigma_1/3-\sigma_2/3\sigma_1+\sigma_4/\sigma_3-t_4^2/3\sigma_1.$$

The centroids of the pedal triangles of the inverse of the mid-point of OP_1, with respect to Γ, of the triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ are on a circle.

8. Conics connected with the quadrilateral. The turns corresponding to the intersections of Γ with the conic

$$x^2 + ax\bar{x} + b\bar{x}^2 + cx + f\bar{x} + g=0$$

are the roots of the equation

$$t^4 + ct^3 + (a + g)t^2 + ft + b = 0;$$

hence the equation of the conics circumscribed to the quadrilateral is

$$x^2 + ax\bar{x} + \sigma_4\bar{x}^2 - \sigma_1 x - \sigma_3\bar{x} + \sigma_2 - a = 0.$$

The center is $(2\sigma_1\sigma_4 - a\sigma_3)/(4\sigma_4 - a^2)$.

When $a=0$ the conic is the circumscribed equilateral hyperbola and the center is the mid-point of OP_1. When $\sigma_2=0$, the hyperbola passes through O and we find this theorem:

When a quadrilateral is such that each of the vertices is, in the triangle formed by the three others, the image of the Feuerbach point of the tangential triangle, the circumscribed equilateral hyperbola is the common Jerabek hyperbola of the four triangles formed by the vertices of the quadrilateral.*

When, in the general case, the circumscribed conic passes through O, $a=\sigma_2$ and the tangent at O is

$$\sigma_1 x + \sigma_3\bar{x} = 0.$$

The tangent at the circumcenter to the circumscribed conic passing through that point is parallel to the Simson lines of the extremities of the circumdiameter passing through the point P_1.

Circumscribed parabolas will be obtained when $a=\pm(\sigma_4)^{1/2}$; their equation is therefore

$$(x \pm \bar{x}\sigma_4^{1/2})^2 - \sigma_1 x - \sigma_3\bar{x} + \sigma_2 \mp 2\sigma_4^{1/2} = 0.$$

The axes are

$$x \pm \bar{x}\sigma_4^{1/2} = (\sigma_1 \pm \sigma_3/\sigma_4^{1/2})/4;$$

they meet at right angles at G and are parallel to the diagonals of the square formed by the points L_i.

9. Various theorems. In the triangle $A_1A_2A_3$, s_2/s_1 is the Feuerbach point of the tangential triangle. But

$$s_2/s_1 + t_4 = \sigma_2/s_1.$$

Hence, *the distances from O to the straight lines joining A_1, A_2, A_3, A_4 to the Feuerbach points of the tangential triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$, are inversely proportional to the distances from O to the orthocenters of these triangles.*

Similarly, the point s_3/s_2 is the inverse of the orthocenter of $A_1A_2A_3$ with respect to Γ; but

$$s_3/s_2 + t_4 = \sigma_3/s_2.$$

* Isogonal conjugate to the Euler line.

The distances from O to the mid-points of the joins of A_1, A_2, A_3, A_4 to the inverse, with respect to Γ, of the orthocenters of the triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ are inversely proportional to the distances from O to these orthocenters.

DISCUSSIONS AND NOTES

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The department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

ON GENERALIZED HERMITIAN MATRICES

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1. Let

$$A = (a^{(1)}a^{(2)} \cdots a^{(n)}) = \begin{pmatrix} a'_{(1)} \\ a'_{(2)} \\ \cdot \\ \cdot \\ \cdot \\ a'_{(m)} \end{pmatrix} = (a^{\nu}_{\mu}), \begin{pmatrix} \nu = 1, \cdots, n \\ \mu = 1, \cdots, m \end{pmatrix}$$

be a matrix with n columns $a^{(\nu)}$ and m rows $a'_{(\mu)}$ over a field $\mathfrak{F}$. A matrix A of rank r will be called a P_r-matrix if there is a set of r indices $\mu_1, \mu_2, \cdots, \mu_r$ out of the numbers $1, 2, \cdots$, Min (m, n) such that the r rows

$$(1) \qquad a'_{(\mu_1)}, a'_{(\mu_2)}, \cdots, a'_{(\mu_r)}$$

and the r columns

$$(2) \qquad a^{(\mu_1)}, a^{(\mu_2)}, \cdots, a^{(\mu_r)}$$

are at once linearly independent. Important examples of P_r-matrices are the symmetric, hermitian and skew-symmetric matrices. The following theorem will be proved here:

In every P_r-matrix A there is an r-rowed principal submatrix A_r of rank r.

This is a natural generalization of a well known theorem for all the special P_r-matrices mentioned;* however the notion of P_r-matrix suggests the following simple proof:

* Cf. C. C. MacDuffee, The theory of matrices, Berlin 1933, p. 12, where the proof is based on a rather complicated lemma of G. Kowalewski, or J. H. M. Wedderburn, Lectures on matrices, New York 1934, p. 89. This proof refers to some properties of the characteristic polynomial of A. Our proof makes use only of the notion of linear dependence, thus showing the elementary character of the theorem.

Consider the "section matrix" A_r of the rows (1) and the columns (2), *i.e.* the square array of the r^2 elements of A which belong to one of the rows (1) as well as to one of the columns (2). This obviously is a principal submatrix of A; for the elements $a_{\mu_\rho}^{\mu_\rho}$ ($\rho=1, \cdots, r$) form the main diagonal of A_r. Now let s be the rank of A_r; then $s \leqq r$ and s columns with the indices $\mu_{\rho_1}, \mu_{\rho_2}, \cdots, \mu_{\rho_s}$ are linearly independent whilst the other $r-s$ columns of A_r are linearly dependent on these s ones. Let $\bar{A}$ be the matrix with n columns whose rows are the r rows (2). The ν-th column $\bar{a}^{(\nu)}$ of this matrix $\bar{A}$ depends linearly on $\bar{a}^{(\mu_1)}, \bar{a}^{(\mu_2)}, \cdots, \bar{a}^{(\mu_r)}$ and these being linear combinations of $\bar{a}^{(\mu_{\rho_1})}, \bar{a}^{(\mu_{\rho_2})}, \cdots, \bar{a}^{(\mu_{\rho_s})}$, the column $\bar{a}^{(\nu)}$ will also be linearly dependent on these s columns of $\bar{A}$. Thus the rank of $\bar{A}$ is s. But $\bar{A}$ has r linearly independent rows; hence $s=r$ which proves the theorem.

2. The preceding theorem is useful in the transformation theory of ϕ-symmetric matrices.* Departing slightly from Albert's method this notion may be introduced in the following way. Let α, β be elements of the infinite field $\mathfrak{F}$. Let $\phi(\alpha)$ be a transformation of $\mathfrak{F}$ into itself such that either $+\phi(\alpha)$ or $-\phi(\alpha)$ is an automorphism of $\mathfrak{F}$ which leaves invariant all elements of a certain subfield $\mathfrak{F}^{(\phi)}$ of $\mathfrak{F}$ and only these. Then one has

$$\phi(\alpha+\beta) = \phi(\alpha)+\phi(\beta),\ \phi(\alpha\beta) = \phi(1)\phi(\alpha)\phi(\beta),\ \phi(1) = \pm 1, \tag{3}$$

for all α, β in $\mathfrak{F}$ and

$$\phi(\lambda) = \phi(1)\cdot\lambda \text{ for all } \lambda \text{ in } \mathfrak{F}^{(\phi)}, \tag{4}$$

whilst for all other α in $\mathfrak{F}$ one has $\phi(\alpha) \neq \phi(1)\cdot\alpha$. If $-\phi$ is an automorphism then ϕ may be called an antimorphism. The repeated operation $\phi(\phi(\alpha))$, shortly denoted by ϕ^2, is always an automorphism.

Let $\mathfrak{A}_n$ be the ring of all n-rowed square matrices A over $\mathfrak{F}$, ($n=1, 2, 3, \cdots$). Every auto- or antimorphism ϕ in $\mathfrak{F}$ induces a transformation in $\mathfrak{A}_n$. If $A=(a_\mu^\nu)$ we put

$$A_\phi = (\phi(a_\mu^\nu)).$$

Then by (3)

$$(A+B)_\phi = A_\phi + B_\phi, \qquad (AB)_\phi = \phi(1)A_\phi B_\phi$$

for any two matrices A, B in $\mathfrak{A}_n$. Further $(A')_\phi = (A_\phi)'$ if A' is the transpose of A.

Particularly consider the matrices A in $\mathfrak{A}_n$ for which

$$A_\phi = A'. \tag{5}$$

Then

$$A_\phi^2 = (A_\phi)_\phi = (A')_\phi = (A_\phi)' = A'' = A.$$

Further let T be any non-singular matrix out of $\mathfrak{A}_n$. The matrix

* The content of §§2 and 3 is not new; cf. A. A. Albert, Modern Higher Algebra, Chicago 1937, Chapter V. However in §2 of the present paper a reason is given why in the study of ϕ-congruence one has to restrict oneself to involutions ϕ.

$$B = T'_\phi A T$$

may be called ϕ-congruent with A. When does this ϕ-congruence transformation not destroy the property (5) of a matrix? Evidently

$$B_\phi = \phi(1)(T'_\phi A)_\phi T_\phi = \phi(1)\phi(1)T'_{\phi^2}A_\phi T_\phi = T'_{\phi^2}A'T_\phi$$
$$B' = T'A'T_\phi$$

and hence $B_\phi = B'$ if and only if ϕ^2 is the identity. Thus one is led to restrict oneself to the involutory auto- and antimorphisms ϕ. If ϕ is such an involution a matrix A which satisfies (5) is called ϕ-symmetric.

A ϕ-symmetric matrix A is symmetric if $\phi(\alpha)=\alpha$, skew-symmetric if $\phi(\alpha)=-\alpha$, hermitian if $\phi(\alpha)=\bar{\alpha}$ where $\mathfrak{F}$ is a field of complex numbers and $\bar{\alpha}$ the conjugate complex number of α.

3. Now the first step in the reduction of a ϕ-symmetric matrix A to a "congruence normal form" $T'_\phi AT$, *i.e.* the elimination of the singular case, can be carried out by means of the theorem of §1. Indeed, every ϕ-symmetric matrix A of rank r is a P_r-matrix: If $a^{(\mu_1)}, \cdots, a^{(\mu_r)}$ is a maximum set of linearly independent columns the rows

$$a'_{(\mu_\rho)} = (a^{(\mu_\rho)})'_\phi \qquad (\rho = 1, \cdots, r)$$

form a set of r linearly independent rows of A. Let A_r be the section matrix of these rows and columns. If P further is the permutation matrix which has resp. as first, second, $\cdots$, r-th column the μ_1-th, μ_2-th, $\cdots$, μ_r-th column of the unit matrix, one obtains because of $P_\phi = \phi(1)P$

$$P'_\phi AP = \phi(1)P'AP = \phi(1)\begin{pmatrix} A_r & * \\ * & * \end{pmatrix},$$

whence follows that one can suppose $\mu_\rho = \rho$, *i.e.* the first r columns and rows in A linearly independent.

Thus one has

$$a^{(\mu)} = \sum_{\rho=1}^{r} \sigma_\rho^\mu a^{(\rho)} \qquad (\mu = r+1, \cdots, n)$$

and by (5)

$$a_{(\mu)} = \phi(1)\sum_{\rho=1}^{r} \phi(\sigma_\rho^\mu)a_{(\rho)}. \tag{6}$$

With the coefficients σ_ρ^μ defined by these relations we form in the usual way the matrices of the "elementary transformations." Let $E^{(\rho,\mu)}$ be the matrix which at the place (ρ, μ) has a one, and zeros at all other places; put

$$S^{(\mu)} = E - \sum_{\rho=1}^{r} \sigma_\rho^\mu E^{(\rho,\mu)} \qquad (\mu = r+1, \cdots, n)$$

where E is the unit matrix. Then $AS^{(\mu)}$ arises from A by replacing the μ-th column by the zero column. Further because of $E_\phi^{\rho,\mu}=\phi(1)E^{(\rho,\mu)}$ one has

$$\phi(1)S_\phi^{(\mu)} = E - \phi(1)\sum_{\rho=1}^{r}\phi(\sigma_\rho^\mu)E^{(\rho,\mu)};$$

therefore, as $E^{(\rho,\mu)'}=E^{(\mu,\rho)}$ and by (6), the matrix $\phi(1)S_\phi^{(\mu)'}AS^{(\mu)}$ arises from $AS^{(\mu)}$ by replacing the μ-th row of this latter matrix by the zero-row. Now put

$$S = S^{(r+1)}S^{(r+2)}\cdots S^{(n)},$$

and correspondingly

$$S_\phi = \phi(1)^{n-r-1}S_\phi^{(r+1)}S_\phi^{(r+2)}\cdots S_\phi^{(n)}.$$

Then*

$$S'_\phi AS = \phi(1)\begin{pmatrix} A_r & 0 \\ 0 & 0 \end{pmatrix}.$$

APPROXIMATION TO A CIRCULAR ARC

J. M. Bruce, Chicago, Ill.

We give here an approximation to a circular arc in terms of its chord and the chord of half the arc. Let a be the length of an arc of a circle of radius r, subtending a central angle θ. Let c be the length of the chord of the arc, and b the length of the chord of half the arc. We shall find it convenient to put $\phi=\theta/4$. Then

$$a = 4r\phi, \qquad b = 2r\sin\phi, \qquad c = 2r\sin 2\phi = 4r\sin\phi\cos\phi.$$

Hence

$$b = 2r(\phi - \phi^3/3! + \phi^5/5! + \cdots),$$

and so

$$2b/a = 1 - \phi^2/3! + \phi^4/5! + \cdots.$$

We also have

$$c/2b = \cos\phi = 1 - \phi^2/2! + \phi^4/4! + \cdots.$$

Hence

$$6b/a - c/2b = 2 - \phi^4/60 + \cdots.$$

As an approximation we can therefore take

$$6b/a - c/2b = 2,$$

* This is the theorem 7 of Albert, l.c. p. 105.

which gives

$$a = 12b^2/(c + 4b). \tag{1}$$

The error in this approximation is easily obtained by expanding the right-hand side in powers of ϕ.

$$\begin{aligned} 12b^2/(c + 4b) &= 48r^2 \sin^2 \phi/(4r \sin \phi \cos \phi + 8r \sin \phi) \\ &= 12r \sin \phi/(2 + \cos \phi) \\ &= r(4\phi - \phi^5/45 - \phi^7/378 + \cdots) \\ &= a(1 - \phi^4/180 - \phi^6/1512 + \cdots) \\ &= a(1 - \theta^4/46080 - \theta^6/6193152 + \cdots). \end{aligned}$$

Thus the relative error is about $\theta^4/45000$. This is about .014%, or 45″, when $\theta = 90°$.

From (1) we can get a good approximation to $\cos \phi$. For we have approximately

$$\begin{aligned} 3c/a &= (c^2 + 4bc)/4b^2 = (c/2b)^2 + 2(c/2b) \\ &= \cos^2 \phi + 2 \cos \phi. \end{aligned}$$

This gives

$$\cos \phi = -1 + \sqrt{1 + 3c/a}.$$

There is an error of about 3′24″ when $\theta = 90°$.

A PROOF OF STURM'S THEOREM

M. F. Smiley, Lehigh University

Our purpose is to give a proof of Sturm's theorem* (including the case of multiple roots) which, in the opinion of the writer, exhibits in two lemmas the essential facts on which the theorem is based.

Let us consider the equations

$$f_0 = f_1 q_1 - f_2,\ f_1 = f_2 q_2 - f_3,\ \cdots,\ f_{i-1} = f_i q_i - f_{i+1},\ \cdots,\ f_{k-1} = f_k q_k \tag{1}$$

where $f_0, \cdots, f_k, q_1, \cdots, q_k$ are polynomials with real coefficients. Let $v(x)$ denote the number of variations in sign of the sequence

$$f_0(x),\ f_1(x),\ \cdots,\ f_k(x).$$

LEMMA 1. *If $f_0(c) \neq 0$, then $v(x)$ is constant near*† $x = c$.

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* See, for example, Dickson, First Course in the Theory of Equations, pp. 76–82.

† We shall use the phrase "near $x = c$" in place of the longer "for all x on some interval containing c as an interior point." Similar interpretations should be given the phrases "just before $x = c$" and "just after $x = c$" used in Lemma 2.

Proof. If no $f_i(c)=0$ $(i=1,\cdots,k)$ the conclusion is obvious. If some $f_i(c)=0$ $(i=1,\cdots,k)$, then $f_{i-1}(x)f_{i+1}(x)<0$ near $x=c$. To see this notice that the equations (1) require that

$$f_{i-1}(c) = f_i(c)q_i(c) - f_{i+1}(c) = -f_{i+1}(c). \tag{2}$$

But $f_{i-1}(c)\neq 0$, because otherwise the equations (1) would yield

$$f_{i-2}(c) = f_{i-1}(c)q_{i-1}(c) - f_i(c) = 0,$$

and eventually $f_0(c)=0$, contrary to hypothesis. From the equation (2) we now see that $f_{i-1}(c)f_{i+1}(c)<0$, and hence that $f_{i-1}(x)f_{i+1}(x)<0$ near $x=c$. It is now easy to check that $v(x)$ remains constant near $x=c$.

Remark. Note that we *do not* require that f_1 be the derivative of f_0 in Lemma 1.

LEMMA 2. *If $f_1=f_0'$ and $f_0(c)=0$, then $v(x)$ is constant just before $x=c$ and just after $x=c$, and $v(x)$ is one greater just before $x=c$ than just after $x=c$.*

Proof. We write $f_0(x)=(x-c)^{\mu}\phi(x)$, where $\phi(c)\neq 0$. It follows that

$$f_1(x) = f_0'(x) = (x-c)^{\mu-1}[\mu\phi(x) + (x-c)\phi'(x)].$$

Define $F_i=(x-c)^{1-\mu}f_i$. Then we obtain the equations

$$F_1 = q_2F_2 - F_3, \cdots, F_{i-1} = q_iF_i - F_{i+1}, \cdots, F_{k-1} = q_kF_k$$

from the equations (1). Note that $F_1(c)=\mu\phi(c)\neq 0$. Apply Lemma 1 and we find that the number of variations in sign of the sequence

$$F_1(x), F_2(x), \cdots, F_k(x)$$

is constant near $x=c$. But $F_0(x)=(x-c)\phi(x)$ and $F_1(x)=\mu\phi(x)+(x-c)\phi'(x)$ have opposite signs just before $x=c$ and the same signs just after $x=c$. Thus the number, $V(x)$, of variations in sign of the sequence

$$F_0(x), F_1(x), F_2(x), \cdots, F_k(x)$$

decreases just one at $x=c$. But the number of variations in sign of a sequence is unchanged by multiplication by a nonzero number, and hence $V(x)=v(x)$ except possibly at $x=c$. The conclusions of Lemma 2 are now obvious.

We conclude with a statement of Sturm's theorem, which follows immediately from Lemmas 1 and 2.

THEOREM. *If $a<b$, $f_0(a)\neq 0$, $f_0(b)\neq 0$, and $f_1=f_0'$ then the number of distinct roots of $f_0(x)=0$ on the interval $a\leqq x\leqq b$ is $v(a)-v(b)$.*

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

The Dozen System. By George S. Terry, Longmans, Green and Company, 1941. 53 pages. $.50.

Mr. Terry has set forth a comparison between the decimal and duodecimal systems of numbers in this book. He stresses throughout the advantages of the use of twelve instead of ten as a base for numbers. The book is written clearly and in language that an intelligent school boy can understand. There are exercises, problems, charts, and tables to demonstrate and clarify his points.

The comparative ease of calculation in the fundamental processes with the duodecimal system is shown. The advantages of the system in factoring, finding factorials, and powers and roots of numbers are demonstrated. The convenience of expressing many simple fractions exactly in what he refers to as "point form" is one of the major advantages of the system.

Commerce has long used the dozen-base for buying and selling in dozens and gross but has yet to take advantage of the dozen system for multiplication of feet and inches, or expressing large numbers in fewer digits.

Mr. Terry has made a duodecimal slide rule, adapted Napier's Bones to aid in calculation and, in his earlier work, "Duodecimal Arithmetic," given sets of tables, to make the dozen system practical. He is convinced of the practicality of the use of duodecimal system and is doing everything in his power to convince others.

C. A. LESTER

Tables of Natural Logarithms. Volume III. Contains the Logarithms of the Decimal Numbers from 0.0001 to 5.0000. (Prepared by the Federal Works Agency Works Projects Administration for the City of New York. Conducted under the Sponsorship of the National Bureau of Standards.) New York, Work Projects Administration, 1941. 17+501 pages. $2.00.

This is Volume III of a series of four volumes of natural logarithms. It contains the natural logarithms of decimal numbers from 0 to 5 at intervals of 0.0001.

The corrections of errors in the Wolfram Tables, and the explanation of the procedure for direct and inverse interpolation are repeated from the earlier volumes. The arrangement of the page and the safe-guards for a high degree of accuracy are the same as in the preceding volumes. A fourth volume will contain the natural logarithms of decimal numbers from 5 to 10 at intervals of 0.001.

VIRGIL SNYDER

Statistical Method from the Viewpoint of Quality Control. By Walter A. Shewhart with the editorial assistance of W. Edwards Deming. The Department of Agriculture Graduate School. Washington, D. C., 1939, 9+155 pp.

The industrial world is indebted to Dr. Shewhart for his introduction of statistical methods to the problem of quality control. A realization of the practical significance of his contributions can best be attained by a study of his earlier text "Economic Control of Quality" and of the numerous technical articles published by Dr. Shewhart and his associates at the Bell Telephone Laboratories. This latest book is an important contribution to both industrial and scientific thought. It is based on a series of four lectures delivered by Dr. Shewhart to the Graduate School of Agriculture and, consequently, is not a beginner's textbook. Derivations of standard statistical formulae are omitted and thereby a critical viewpoint of the rôle of statistical methods in quality control is obtained by the reader without the interruptions of mathematical manipulations. A beautiful parallel is drawn between mass production and scientific methods: the author regards the cycle of specification, production and inspection in industry as corresponding to the cycle of hypothesis, experiment and test of hypothesis in science.

Briefly described, Dr. Shewhart's monogram is a criticism of statistical methods but it is definitely constructive. One cannot refute his claim that the language of the statistician is "emotive" rather than scientific, when one considers his list of such common statistical phrases as "statistical facts," "confidence limit," "probable error," "most probable value" and "best estimate." In the opinion of this reviewer, both the scientist and the industrial statistician will profit by a careful study of this sincere and interesting monogram.

WALTER BARTKY

College Geometry. By Paul H. Daus. New York, Prentice-Hall, Inc., 1941. 15 +200 pages. $2.50.

This text is a good introduction to the synthetic goemetry of the triangle and circle and related topics. It is by no means exhaustive, but it does introduce many of the topics which should be made a part of the curriculum of prospective teachers of mathematics.

After a brief historical introduction the author takes up the subject of similar and homothetic figures, with special emphasis on homothetic circles.

The second chapter is devoted to the subjects of concurrency and collinearity. The theorems of Menelaus and Ceva are discussed and applications of these theórems are given. The second part of this chapter deals with harmonic division and orthogonal circles and the harmonic properties of the complete quadrilateral and quadrangle. Although this chapter is brief it affords a good introduction to these subjects.

The author next introduces the subject of inversion, discussing the inverse of the point, line and circle, the angle and distance relations and geometric con-

structions. This is followed by a short chapter on the pole and polar relation with respect to the circle and a brief discussion of the trilinear polar and isogonal conjugates.

This is followed by a rather complete discussion of coaxal circles, emphasizing their harmonic properties. The involution determined by a coaxal system of circles on a line, and the corresponding involution of lines through a point are introduced.

The author devotes two chapters to a discussion of the notable points, lines and circles associated with a triangle. Although this discussion is brief it is sufficiently complete to serve as a good introduction to the interesting geometry connected with the nine-point circle, the orthocentric quadrilateral, the Simson line and associated theorems. The last chapter contains an interesting collection of ruler and compass constructions.

The student with the limited mathematical background provided by the usual high school courses in plane geometry and trigonometry will have little difficulty with this text. The discussions are clear and the diagrams numerous and well done. Ample problems for a short course are included. A novel feature of this text is the key to the solution of the problems which the author has included at the end of the text. Your reviewer was dubious about the value of this feature until he had observed its use in practice. At first glance it appears that the author has offered a crutch to the poor student, as indeed he has, but your reviewer was agreeably surprised to find that his students used the key only as a last resort or as a check on their own work.

Your reviewer believes that a text on college geometry to be of greatest service to prospective teachers of high school mathematics should contain a chapter or two devoted to construction problems involving geometric loci and triangles from indirect elements. The practice in developing the analysis, synthesis and discussion of results of such problems would prove to be a valuable item in his training and also of great value in developing many of the theorems and problems contained in a text of this sort. This text is weak at this point.

The reviewer has found this book teachable and, from the point of view of both teacher and student, readable. It is an excellent addition to the texts on this beautiful and interesting field of synthetic geometry.

H. N. HUBBS

El Sexo desde el Punto de Vista Estadistico. By José González Galé. Buenos Aires, Imprenta de la Universidad, 1941. 54 pp.

Vital statistics appear to show, with remarkable unanimity, that "masculinity" (the ratio of male to female births) is roughly 22/21, with circumstantial aberrations. The present memoir is devoted to an investigation of this and other statistics related to the determination of sex. Though without technical interest to a mathematician, the material is presented in a lucid and entertaining way. The first two chapters (about half the pamphlet) are given over to description and historical background, in which vital statistics are found to have been origi-

nally collected at the direction of Henry VIII during the epidemics of the early sixteenth century. The third chapter presents data on "masculinity." In the last chapter, the determination of sex is explained on the basis of chromosomes and the Mendelian laws, and it is indicated that no satisfactory account of "masculinity" has yet been found.

F. A. FICKEN

Algebraic Solid Geometry. An Introduction. By S. L. Green. Cambridge, University Press; New York, The Macmillan Company, 1941. 9+133 pages. 14 figures. $1.75.

This little book is based on a course of lectures given repeatedly at Queen Mary College for general students. It assumes a knowledge of elementary algebra, including determinants, and an outline of trigonometry. The salient feature is its directness and brevity. Much of the discussion is based on the interpretation of the sides of a plane, thus avoiding most of the ambiguity arising from signs. The treatment is entirely in terms of rectangular cartesian coördinates. The subjects include planes, lines, spheres, central quadric surfaces, paraboloids and cones, with an introduction to generators of central quadrics, poles and polars, harmonic section, and inversion. A generous list of well-chosen exercises is inserted at frequent intervals. Answers are not provided. The printing and press-work are excellent.

VIRGIL SNYDER

NEW BOOKS RECEIVED

Tables of Natural Logarithms. Volume III Contains the Logarithms of the Decimal Numbers from 0.0001 to 5.0000. (Prepared by the Federal Works Agency Works Projects Administration for the City of New York. Conducted under the Sponsorship of the National Bureau of Standards.) New York, Works Projects Administration, 1941. 17+501 pages. $2.00.

Methods of Correlation Analysis. By Mordecai Ezekiel. Second Edition. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Limited, 1941. 19+531 pages. $5.00.

An Introduction to the Theory of Newtonian Attraction. By A. S. Ramsey. Cambridge, at the University Press; New York, The Macmillan Company, 1940. 9+184 pages. $2.50.

Dimension Theory. By Witold Hurewicz and Henry Wallman. Princeton Mathematical Series, Volume 4. Princeton University Press; London, Humphrey Milford and Oxford University Press, 1941. 7+165 pages. $3.00.

Algebraic Solid Geometry. An Introduction. By S. L. Green. Cambridge, England, at the University Press; New York, The Macmillan Company, 1941. 9+133 pages. $1.75.

Science and Sanity. By A. Korzybski. Second edition. Chicago, Illinois, Institute of General Semantics, 1941. 382 pages. $3.50.

To Discover Mathematics. By G. M. Merriman. New York, John Wiley and Sons, 1942. 435 pages. $3.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT AND J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Brown University, Providence, R. I.

A MATHEMATICAL CONTEST

A contest like the one that follows has been tried several times at Wellesley College and has seemed to interest the students. Perhaps it may appeal to Mathematics Clubs in other colleges. It is obviously not suitable for a club meeting, since it involves the use of histories of mathematics, anthologies of poetry and so on. However, the officers of a club might like to sponsor such a contest and offer a prize to its members or to all students taking mathematics in the institution in question.

HELEN A. MERRILL
MARION E. STARK

"A prize is offered for the best set of answers. Any student who has taken a course in mathematics in college may compete. To win the prize a student must have at least fifteen answers correct. After each answer give an exact reference, stating where it was found."

A. Name the following:

1. A mathematician, now dead, who wrote very popular books for children.
2. The man responsible for the coördinates x and y of a point.
3. A woman mathematician who married at the age of eighteen in order to escape from Russia.
4. A Greek mathematician who was also a musician and a philosopher.
5. A mathematician whose name furnishes the title of a well-known poem by Robert Browning.
6. The man who wrote the oldest mathematical textbook still in actual use.
7. A father and daughter who were both mathematicians of note.
8. A professor of romance languages who has written an unusually interesting arithmetic.
9. The first famous woman mathematician.
10. A man deflected by a fire from mathematics to architecture as a profession.
11. An English mathematician who knew only two tunes. One of them was "God save the Queen" and the other wasn't, and he recognized the first by the fact that people stood up to sing it.
12. A mathematician who is a fine violinist.

13. A mathematician who wrote on geometry, astronomy, and algebra, but who is far better known as the author of a single poem.

14. A brilliant mathematician who at twenty-five gave up the subject for a life of religious retirement.

15. A Scotch mathematician who entered the university at the age of eleven.

B. Name the authors, all well-known, of the following:

16. "Let braggarts vow to do and dare
And right abuses.
He'd rather sit at home and square
Hypotenuses."

17. "Mathematics is the queen of the sciences, and arithmetic is the queen of mathematics."

18. "Every one versed in the matter will agree that even the elements of a scientific study of nature can be understood only by those who have a knowledge of at least the elements of differential and integral calculus."

19. "I'm very well acquainted too with matters mathematical,
I understand equations, both the simple and quadratical.
About binomial theorem I'm teeming with a lot o' news;
With many cheerful facts about the square of the hypotenuse."

20. "Yet what are all such gayeties to me
Whose thoughts are full of indices and surds?
$x^2-7x-53=11/3$."

21. "Two and two will make four if you leave 'em alone ever so. But fifteen and twelve don't make twenty-seven—not of themselves. Not till you do them in a sum. Like this."

22. "Passages abound in these speeches which to almost any literary taste are arresting for the simple beauty of their English, a beauty characteristic of one who had learned to reason with Euclid and learned to feel and to speak with the same authors of the Bible."

23. "If, with the same materials, I can make both God and dragon, of what use is higher mathematics?"

24. "Howe'er it be, it seems to me
'Tis only fair ourselves to please;
Dry eyes are more than indices."

25. "Euclid alone has looked on beauty bare."

CLUB REPORTS, 1940–41

Mathematics Club, Woodrow Wilson Junior College, Chicago

At the twelve bi-monthly meetings held during the year the following topics were presented by faculty members: The abacus, Problems of map making, Zero—the trouble maker in algebra, Moebius surfaces, What is the fourth dimension, Paradoxes of infinity, Proofs new and old for the theorem of Pythagoras. Topics discussed by student members included: Measurement of distances and sizes of heavenly bodies, Problems of permutation and combination, The decimal vs. the duo-

decimal number system, The meaning of the relativity of motion, Codes and ciphers. Officers were: President, Earl Clendenon; Vice-President, Morton Zeman; Secretary, Mary MacNamara; Faculty Sponsor, Dr. Luise Lange.

Rho Theta, St. Louis University

The society held regular monthly meetings throughout the year. At the first meetings of each semester new members were formally installed. At one of these, Dr. Regan proposed several problems in plane geometry which were worked out by the members and discussed at the next meeting. Mr. Ed Walters, inactive member of Rho Theta and graduate student gave a talk at another meeting on Seismic prospecting and in this discussion emphasized the applications of mathematics in this branch of geophysics. The year closed with the annual banquet. Officers were: President, Alex Yokubaitis; Vice-President, Harry Brueggemann; Secretary-Treasurer, John McCann; Faculty Adviser, Dr. Francis Regan.

Mathematics Club, Cooper Union Institute of Technology

The report for the year shows the following topics presented: The slide rule, by W. A. Sherwood; Dimensional analysis, by S. Manson; Higher geometry, by H. Grad; Vector analysis, by M. Rubinowitz. The prize for excellence in first year mathematics was a slide rule and was awarded to Gerald Weiss. Officers were: President, Samuel Manson; Vice-President, Theodore Gold; Secretary-Treasurer, Murray Klamkin; Faculty Adviser, Professor F. H. Miller.

Pi Mu Epsilon, Michigan State College

Programs for the year were in all but one case in direct charge of faculty members. This chapter reports that talks presented by faculty members are more popular and help to maintain interest more than do student papers. Subjects discussed at meetings were: Simple topological problems, by Dr. G. B. Van Schaack; Demonstration of the slide rule, by Professor J. E. Powell and Mr. J. Sheedy; Determination of orbits of comets, by Professor E. T. Welmers; Music and continued fractions, by Professor J. M. Barbour of the department of music; Problems in probability, by Mr. C. Nordstrom; Descriptive geometry, by Mr. J. Zimmer; Solitaire on a checkerboard, by Dr. B. M. Stewart. The only student paper presented was Determination of volumes and surfaces of eggs, by Mr. M. Rottenstein. One meeting was devoted to the following motion pictures: Plane and solid geometry, Frequency curves, and Einstein's theory of relativity. At the annual banquet in January, Professor Ayres of the University of Michigan spoke on Elementary problems of topology. Officers were: President, S. P. Schlesinger; Vice-President G. Elaine Van Aken; Secretary, Ruth L. Winegar; Treasurer, Charles F. Michalski; Faculty Director, Professor Everett T. Welmers.

Mathematics Club, Eastern Illinois State Teachers College

This club held fourteen meetings during the year. Speakers and their subjects were: Dr. Taylor on Observations made in German schools, Dr. Heller on The purpose of the mathematics club, D. Trulock on Tests and measurements in mathematics, Orval Rice on Mathematical wrinkles, Wilma Bond on Plans for a mathematics field day, Maxine Rennels on Mathematical fallacies, W. Bails on Personal experiences in aviation, Jean Fullen on Mathematics in Lewis Carroll's *Alice in Wonderland*, Dr. Taylor on Life of Mark Twain. Officers were: President, Maxine Rennels; Vice-President, Orval Rice; Secretary, Wilma Bond; Treasurer, Edwin McKittrick; Faculty Adviser, Dr. Heller.

Pi Mu Epsilon, Ohio State University

Two meetings of the Ohio Alpha chapter were held during the year. A lecture by Dr. Samuel Eilenberg of the University of Michigan was held jointly with the meeting of the graduate mathematics club. At the annual initiation and banquet, Professor Lester R. Ford of Illinois Institute of

Technology spoke on Fractions. Officers were: President, Lawrence Ringenberg; Vice-President, Paul Weaver; Secretary, William Scott; Treasurer, John Ault.

Mathematical Society, Trinity College

Each year this society tries to have at least one speaker from the alumnae who was a former member or a major in mathematics and who is making use of her mathematical training in the teaching profession, in business, or in the field of statistical research. This meeting always proves to be the most popular on the year's program. The speaker this year was Miss Rita Buechert of the class of 1935 who is head of the statistical research department of the District of Columbia Council of Social Agencies. She spoke on Statistical research in the field of sociology and social work. Two prizes were awarded during the year, the Sister Marie Cecilia Memorial Prize to the senior maintaining an average of over 90 per cent in her major mathematics courses was won by Mary Charlotte Crook and the prize essay contest was won by Elizabeth Mahan for her paper on The problem of the apportionment of representatives. Topics discussed included: A geometric interpretation of the transformation HAH^{-1} and some of its algebraic applications, by Sister Thomas; Old age assistance and survivors insurance, by Mr. J. Jacques of the Social Security Board; Inversion, by Ellen Schofield; The four color problem, by Teresa Karnes; The abacus in China and Japan, by Sister Marie Raymond. A gift of $10.00 was given to the department for the purchase of books, by the society. Officers were: President, Marie Straukamp; Vice-President, Marie Lee; Secretary, Marie Kehoe; Treasurer, Barbara Archibald; Faculty Adviser, Sister Thomas Marie.

Mathematics Club, Oshkosh State Teachers College

The program for the year consisted of the following: The fourth dimension, by Paul Haworth and a review of the book *Flatland*, by Sarah J. Richards; Cryptographs and Ciphers, by Ruth Savinsky and Why study mathematics, by Roland Hahn; Adapting the mathematics curriculum to our era, by Irvin Schudlick; Mathematical pastimes, by Leslie Kornowski; Hypsometer and Clinometer, by Henry Grabowski; Alexandria—shrine of mathematics, by Lucille Diedrick. New members were welcomed at a meeting at which mathematical games were played and Marian Pohl spoke on The mathematics of the suspension bridge. Officers were: President, Leslie Kornowski; Vice-President, Irvin Schudlick; Secretary, Sarah Jane Richards; Treasurer, Joan Miller; Historian, Ednie Kiddie; Advisers, Dr. M. M. Beenken and Dr. Irene Price.

Pi Mu Epsilon, Brooklyn College

The topic studied at the meetings of the second semester was Plucker coördinates. Professor Johnson gave an introductory talk followed by further papers by A. Francis Bausch and Harvey Casson. Results of the annual contest sponsored by this chapter for colleges in metropolitan New York are as follows: Brooklyn College, first team, 41 pts.; Brooklyn College, second team, 36 pts.; Cooper Union, 33 pts.; Queens College, 16 pts.; New York University, 13 pts.; Columbia University, 13 pts.; Yeshiva College, 7 pts. There were five men on every team and the contest had 15 time problems. Richard Bellman of Brooklyn College won the individual prize. For the first time in the history of the contest, it was won by one school—Brooklyn College, entitling the school to retain permanent possession of the plaque which had been awarded annually. Officers were: Director, Professor R. A. Johnson; President, Leonard Greenstone; Secretary, Paul Rosenbluth; Treasurer, Francis Bausch; Librarian, Harvey Casson.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 511. *Proposed by W. E. Bleick, U. S. Naval Academy, Annapolis*

A whistle, which emits a sound of constant pitch, is attached to a bomb. An observer at a fixed point on the ground sees the bomb dropped from rest at an initial angular elevation B. Assume that the bomb falls with a constant acceleration g and hits the ground at a distance L from the observer. The observer hears a variable whistle pitch because of the Doppler effect. The apparent pitch heard by the observer is a maximum when the component of the bomb's velocity in his direction is a maximum. Find the angular elevation of the bomb at the moment when the apparent pitch is a maximum. Show that, when the initial angular elevation B is small, the maximum apparent pitch is heard when the bomb has lost one-third of its initial altitude.

E 512. *Proposed by V. Thébault, San Sebastián, Spain*

Find the first n odd numbers whose sum divides the sum of their fourth powers.

E 513. *Proposed by N. A. Court, University of Oklahoma*

A line revolves about a fixed point in such a manner that the segment intercepted on it by two intersecting planes has its mid-point in a third given plane. Show that the locus of the variable line is a cone of the second degree.

E 514. *Proposed by J. A. Bullard, University of Vermont*

Find the sum $\sum_{k=0}^{n-1} e^{kx} \sin (y+kz)$.

E 515. *Proposed by H. T. R. Aude, Colgate University*

Find all the triangles with integral sides which have one side equal to 16 units and the cosine of an adjacent angle equal to $-\frac{1}{4}$.

SOLUTIONS

Pythagorean Triads

E 472 [1941, 337]. *Proposed by V. Thébault, San Sebastián, Spain*

Find positive integers x, y, z (less than 100), such that $x^2+y^2=z^2$ and $X^2+Y^2=Z^2$, where X, Y, Z are derived from x, y, z by inserting an extra digit (the same for all) on the left.

Solution by W. E. Buker, Pittsburgh Public Schools

If x, y, $z<10$, we have $(10p+x)^2+(10p+y)^2=(10p+z)^2$ and $x^2+y^2=z^2$. Hence $x+y+5p=z$. But since $x+y>z$, p would be negative.

Suppose x, $y<10$, $z\geqq 10$. Then $(10p+x)^2+(10p+y)^2=(100p+z)^2$. But this with $x^2+y^2=z^2$ requires that $x+y=10z+490p$. Here again p would be negative.

There remains the possibility that $x<10$; y, $z\geqq 10$. Then

$$(10p + x)^2 + (100p + y)^2 = (100p + z)^2$$

and

$$p = 2(z - y) - x/5. \tag{1}$$

Clearly x, being divisible by 5, must be equal to 5. The only set of Pythagorean integers satisfying these requirements is

$$x = 5, y = 12, z = 13.$$

From (1), $p=1$.

Also solved by R. K. Allen, D. H. Browne, M. L. Constable, William Douglas, E. P. Starke, and the proposer.

Reciprocal Planes with respect to a Tetrahedron

E 473 [1941, 337]. *Proposed by N. A. Court, University of Oklahoma*

Two variable transversal planes PQR, $P'Q'R'$, reciprocal with respect to a given tetrahedron $DABC$, meet the edges DA, DB, DC in the pairs of points P and P', Q and Q', R and R'. Show that the line of centers of the two spheres $DPQR$, $DP'Q'R'$ passes through a fixed point. (Two transversal planes are said to be *reciprocal* with respect to a tetrahedron if their traces on each edge are equidistant from the midpoint of the edge. See the proposer's *Modern Pure Solid Geometry*, p. 122, art. 354.)

Solution by Howard Eves, Pittsburgh, Pa.

Let U, V, W be the midpoints of DA, DB, DC and let $\overline{U}$, $\overline{V}$, $\overline{W}$ be the midpoints of DU, DV, DW. Let O, O', S be the centers of the spheres $DPQR$, $DP'Q'R'$, $DUVM$. Finally, Let $\overline{P}$ and $\overline{P}'$ be the midpoints of DP and DP'. Then, since U is the midpoint of PP', $\overline{U}$ is the midpoint of $\overline{P}\overline{P}'$. But the planes through $\overline{P}$, $\overline{P}'$, $\overline{U}$ perpendicular to DA pass respectively through O, O', S, whence it is clear that the plane through $\overline{U}$ passes through the midpoint of OO'. A similar argument holds for the planes through $\overline{V}$ and $\overline{W}$ perpendicular

to DB and DC respectively. But the three planes through $\overline{U}$, $\overline{V}$, $\overline{W}$ intersect in S. Hence S is the midpoint of OO', and the theorem is proved.

The corresponding theorem for the plane can be established similarly. (See V. Thébault, *Mathesis*, 1928, p. 231, Question 2442.)

Also solved by the proposer.

Iterated Quadratic Surds

E 474 [1941, 337]. *Proposed by Roy MacKay, Eastern New Mexico College*

For $k>1$, define $a_1=\{k(k-1)\}^{1/2}$, $a_n=\{k(k-1)+a_{n-1}\}^{1/2}$, $b_1=k^{1/2}$, $b_n=(kb_{n-1})^{1/2}$. Prove that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = k.$$

Solution by Eduardo Gaspar, Rosario, Argentina

Since the sequences $\{a_n\}$ and $\{b_n\}$ are monotonic increasing, they have a limit, finite or infinite. We shall show that both sequences are bounded and consequently have a finite limit. We observe that

$$a_1 = \{k(k-1)\}^{1/2} < k.$$

If the same inequality holds for a_{n-1}, we have also

$$a_n = \{k(k-1) + a_{n-1}\}^{1/2} < \{k(k-1) + k\}^{1/2} = k$$

for every n. Similarly, since $k>1$, we have

$$b_1 = k^{1/2} < k.$$

If the same inequality holds for b_{n-1}, we have also

$$b_n = (kb_{n-1})^{1/2} < k.$$

Thus both sequences are bounded.

Let us put

$$\lim a_n = \lim a_{n-1} = a, \quad \lim b_n = \lim b_{n-1} = b.$$

Then from $a_n=\{k(k-1)+a_{n-1}\}^{1/2}$ and $b_n=(kb_{n-1})^{1/2}$ we obtain

$$a = \{k(k-1) + a\}^{1/2}, \quad b = (kb)^{1/2}.$$

Both equations have the single positive root $a=k$, $b=k$. This shows that

$$\lim a_n = \lim b_n = k.$$

Also solved by R. K. Allen, D. H. Browne, E. P. Starke, and the proposer.

Duplicating the Cube

E 475 [1941, 337]. *Proposed by J. Goodfellow, West Rumney, N. H.*

Let the diameter AB of a circle S meet a perpendicular chord HH' at O. Take points C and D on AB, such that $CO=OB$ and $OD=OH$. Let G be one of the

points of intersection of S with the circle on CD as diameter. Show that we have approximately

$$OG^3 = AO \cdot OB^2.$$

How close an approximation does this construction provide for the classical problem of "duplicating the cube"?

Solution by D. H. Browne, Buffalo, N. Y.

Let $OA = a^2$, $OB = OC = b^2$, so that $OD = OH = ab$. Draw the perpendicular GE from G to AB. Then we have

$$\begin{aligned} EG^2 &= AE \cdot EB = (a^2 + OE)(b^2 - OE) \\ &= CE \cdot ED = (b^2 + OE)(ab - OE), \end{aligned}$$

whence

$$OE = \frac{ab^2}{a + 2b}, \qquad EG^2 = \frac{2ab^3(a+b)^2}{(a+2b)^2}, \qquad OG^2 = \frac{ab^3(2a+b)}{a+2b}.$$

To test the given relation, we have

$$\left(\frac{OG^3}{AO \cdot OB^2}\right)^2 = \frac{b}{a}\left(\frac{2a+b}{a+2b}\right)^3.$$

For the duplication of the cube we put $a = \sqrt{2}$, $b = 1$, and find

$$OG^3 = 1.997 \cdots.$$

As a/b increases, the error becomes greater.

Also solved by the proposer.

Editorial Note. The identity,

$$\frac{b}{a}\left(\frac{2a+b}{a+2b}\right)^3 = 1 - \frac{a+b}{a}\left(\frac{a-b}{a+2b}\right)^3,$$

shows that the approximation is very close when a and b are nearly equal. For the duplication of the cube, the above expression becomes

$$1 - (\sqrt{2} - 1)^5/4.$$

When $a/b = 9/8$, it is $1 - 0.00012 \cdots$, so that $OG^3/(AO \cdot OB^2) = 0.99994 \cdots$.

Isosceles Right Triangles

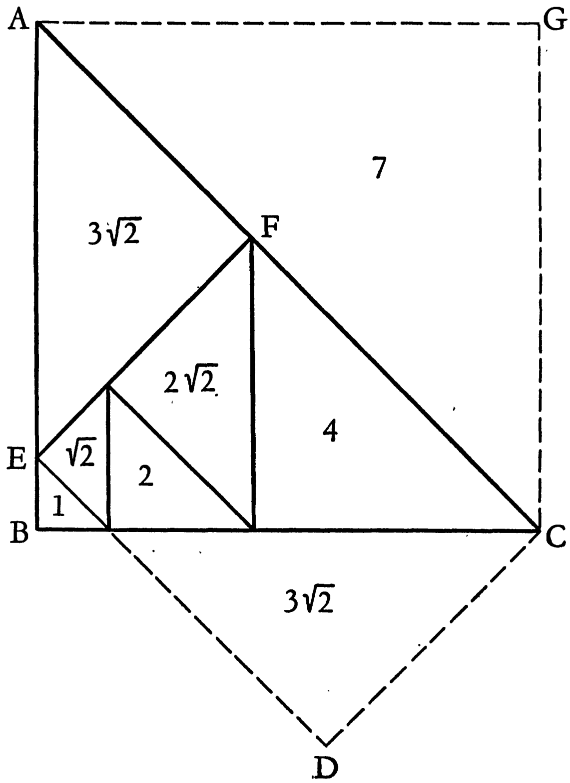
E 476 [1941, 405]. *Proposed by A. H. Stone, Graduate College, Princeton*

Show that it is possible to fit together six isosceles right triangles, all of different sizes, so as to make a single isosceles right triangle.

Solution by Michael Goldberg, Washington, D. C.

The right isosceles triangle ABC is made up of six right isosceles triangles of different sizes. (The number in a triangle is the length of a leg.) Incidentally,

five of these triangles can be arranged to form a rectangle $CDEF$; the addition of a seventh triangle gives the square $ABCG$.



Also solved thus by W. E. Buker, Howard Eves, E. P. Starke, and the proposer. William Douglas and E. P. Starke give a second solution, in which there is a triangle with leg 3 instead of $2\sqrt{2}$, while the other five triangles are as before (only differently arranged). Eves remarks that it is consequently possible to fit together any greater number of different right isosceles triangles to make a single right isosceles triangle, and wonders whether 6 is the smallest such number. The proposer raises the following more difficult question: Can a right isosceles triangle be dissected into a finite number of right isosceles triangles, *no two having a common side*?

Spheres and Tetrahedron

E 477 [1941, 405]. *Proposed by V. Thébault, San Sebastián, Spain*

Consider four spheres (S_1), (S_2), (S_3), (S_4), whose centers are the vertices of a tetrahedron $S_1S_2S_3S_4$. Let (G_1) be the sphere whose center is the centroid of the face $S_2S_3S_4$ and which passes through the points of intersection of spheres (S_2), (S_3), (S_4). Defining (G_2) and (G_3) similarly, prove that the three spheres

(G_1), (G_2), (G_3) intersect on the radical axis of (S_1), (S_2), (S_3). (A similar problem for three circles was discussed in *Mathesis*, 1891, p. 238.)

Solution by Howard Eves, Allen Academy, Bryan, Texas

Let (G_1), (G_2), (G_3) intersect in P and Q, and let (S_1), (S_2), (S_3) intersect in M and N. Invert the entire figure with respect to a sphere with center P. Then (S_1), (S_2), (S_3), (S_4) invert into four spheres (S_1'), (S_2'), (S_3'), (S_4'); and (G_1), (G_2), (G_3) invert into planes (G_1'), (G_2'), (G_3') through the respective radical axes of the triads of spheres

$$(S_2'),\ (S_3'),\ (S_4'); \qquad (S_3'),\ (S_4'),\ (S_1'); \qquad (S_4'),\ (S_1'),\ (S_2').$$

Consequently Q', the intersection of the planes (G_1'), (G_2'), (G_3'), is the radical center of the four spheres (S_1'), (S_2'), (S_3'), (S_4'). Hence Q', M', N' lie on a line (L'). This means that Q, M, N must lie on a circle (L) through the center of inversion P. Since the center of (L) lies on both the parallel planes $S_1S_2S_3$ and $G_1G_2G_3$, it follows that (L) is actually a straight line, and the theorem is proved.

The corresponding theorem for three circles can be proved similarly.

Also solved by Peter Chiarulli and the proposer.

Successive Differences

E 478 [1941, 405]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that the successive differences of ath powers (in the notation $\Delta r^a = (r+1)^a - r^a$) satisfy the relation

$$\sum_{n=0}^{a} (-1)^n \Delta^n 1^a = 0.$$

Solution by N. E. Sheppard, University of Toronto

For any sequence u_k, we have

$$\begin{aligned}\sum_{n=0}^{a} (-1)^n \Delta^n u_1 &= \frac{1+\Delta^{a+1}}{1+\Delta} u_1 \\ &= (1+\Delta^{a+1})E^{-1}u_1 \\ &= (1+\Delta^{a+1})u_0 \\ &= u_0 + \Delta^{a+1}u_0.\end{aligned}$$

If u_k is a polynomial of degree a, $\Delta^{a+1}u_k = 0$. In the present case $u_k = k^a$; therefore $u_0 = 0$, and the desired relation is established.

Also solved by Howard Eves, Solomon Kullback, and E. P. Starke.

Locus Problem

E 479 [1941, 405]. *Proposed by Daniel Arany, Budapest, Hungary*

In the plane of a given triangle ABC, find the locus of a point from which the sides BC and CA subtend equal angles.

Bibliographical Note by N. A. Court, University of Oklahoma

This problem is neither as innocent nor as new as it looks. It was first formu-

lated by J. Steiner for two segments having no common end, and for two equal or supplementary angles. Steiner states (without proof) that the locus consists of two circular cubics, in the *Journal für die reine und angewandte Mathematik*, vol. 45, 1853, p. 375. It was further discussed by P. H. Schoute in the same journal, vol. 99, 1886, p. 98. G. de Longchamps solved the problem in the *Journal de Mathématiques Spéciales*, series 2, vol. 5, 1886, p. 39, and considered the special case when the two segments are collinear in *Journal de Mathématiques Élémentaires*, series 2, vol. 5, 1886, p. 16.

The question was proposed in *Nouvelles Annales de Mathématiques*, series 3, vol. 3, 1884, p. 351, and was solved some thirty years later by H. Brocard in the same journal, series 4, vol. 15, 1915, p. 138. In the same volume F. G. Teixeira contributed a special article on this locus (p. 362). Many other illustrious names, such as Chasles and Salmon, may be mentioned in this connection. Finally, the problem was solved both analytically and synthetically in this MONTHLY, vol. 22, 1915, pp. 20–22.

For the usual meaning of "subtend," the locus consists of part of a strophoid through A and B, with its double point at C, together with the external segment AB.

Also solved by Paul Brock, W. B. Clarke, L. M. Kelly, and the proposer.

Construction of a Pentagon

E 480 [1941, 405]. *Proposed by D. E. Lynch, Jr., Brooklyn, N. Y.*

Construct a pentagon whose sides and diagonals are all commensurable. (For definiteness, suppose there are four equal sides, and three equal diagonals.)

Solution by E. P. Starke, Rutgers University

Put together side by side three congruent isosceles triangles, whose sides are a, a, d, with $a\sqrt{2}<d<2a$, to form a pentagon $PQRST$ such that $PQ=QR=RS=ST=a$, $PR=QS=RT=d$. Let b be the length of the fifth side PT, and let c be that of each remaining diagonal, $TQ=SP$. Note that PS and RS are respectively parallel to RQ and TQ. Let $\theta=\sphericalangle RPQ=\sphericalangle RTQ=\sphericalangle PRQ=\sphericalangle RQS\cdots$. We have at once $\cos\theta=d/2a$. Since

$$\sphericalangle TRP = \sphericalangle SRQ - 2\theta = (\pi - 2\theta) - 2\theta,$$

we find, in the isosceles triangle RTP,

$$b = 2d\cos 2\theta = 2d(2\cos^2\theta - 1) = d(d^2 - 2a^2)/a^2.$$

In the triangle TRQ, with sides a, d, c and respective angles θ, 2θ, $\pi-3\theta$, we have

$$c = a\sin 3\theta/\sin\theta = a(4\cos^2\theta - 1) = (d^2 - a^2)/a.$$

Thus it is only necessary that a and d be commensurable in order that all the lines be commensurable as required. The condition $a\sqrt{2}<d<2a$ corresponds to $\frac{1}{4}\pi>\theta>0$, and provides that the pentagon be convex.

Also solved by the proposer, who finds a nearly-regular pentagon with integral sides and diagonals by taking $a=125$, $d=200$.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4027. *Proposed by Morgan Ward, California Institute of Technology*

In the projective plane P, a triangle with sides A, B, C and vertices D, E, F is the only linear configuration forming a Boolean algebra of order eight with respect to the operations of union and cross-cut. Here the union of A and B is P, the cross-cut of E and D is the null space Z, and so on. In a three dimensional space P, two types of Boolean algebra of order eight are possible; (i) the configuration of three planes A, B, C through a point Z meeting in three lines D, E, F; and (ii) the configuration of two planes A and B through a line D and a line C skew to D meeting A and B in points E and F. Here Z again is the null space.

Show that in a projective space of n dimensions, the total number of distinct types of linear configuration forming a Boolean algebra of order eight is asymptotically equal to $n^3/3!3!$. Also show that the corresponding number for a Boolean algebra of order 2^r is asymptotically equal to $n^r/r!r!$.

4028. *Proposed by P. D. Thomas, Norman, Okla.*

Find the equation of a family of surfaces, each surface satisfying $(OP)^2 = (OQ)^2$, where O is the origin, P is any point on the surface, and Q is the point in which the normal to the surface at P meets the xy-plane.

Find the envelope of the family.

4029. *Proposed by H. L. Dorwart, Washington and Jefferson College*

Let d_1, d_2, d_3, d_4 be the distances in order from the sides of a square of length k units to any interior point P. Then

$$(\sqrt{d_1d_2} \pm \sqrt{d_3d_4})/k \quad \text{and} \quad (\sqrt{d_1d_4} \mp \sqrt{d_2d_3})/k$$

represent the sines and cosines of two angles θ_1 and θ_2, since the sum of the squares of these expressions equals

$$(d_1d_2 + d_3d_4 + d_1d_4 + d_2d_3)/k^2 = (d_1 + d_3)(d_2 + d_4)/k^2 = 1.$$

Give a geometric interpretation for the angles θ_1 and θ_2.

4030. *Proposed by V. Thébault, San Sebastián, Spain*

The resultant of the nine forces represented by the distances to the sides of a triangle ABC from the centers of the equilateral triangles constructed exteriorly (or interiorly) on its sides is represented in magnitude and direction by the vector $\overline{GG'}$ (or $-\overline{GG'}$) equipollent to the vector $\overline{OH'}$ (or $-\overline{OH'}$), where G, O, H' are the centroid of ABC, the circumcenter of ABC, and the orthocenter of the orthic triangle (that is, the triangle with vertices at the feet of the altitudes of ABC).

SOLUTIONS

The Δ_2 Lines

3968 [1940, 574]. *Proposed by Frank Ayres, Jr., Dickinson College.*

Let the line through the vertex A_i, $(i=1, 2, 3)$, and parallel to the opposite side of the triangle $A_1A_2A_3$ meet the circumcircle in the point D_i. Show that: (1) The Δ_2 lines [3929, 1939, 601] of the pairs of points A_i, D_i intersect on the nine-point circle of $A_1A_2A_3$ midway between A_i and the orthocenter of the given triangle. (2) The Δ_2 lines of D_i intersect in the symmetrics of A_1, A_2, A_3 as to the nine-point center.

Solution by J. W. Clawson, Ursinus College

The equation of the Δ_2 line of any point T on the circumcircle is

$$\frac{z}{t+t_1+t_2+t_3}+\frac{tt_1t_2t_3\,\bar{z}}{t_1t_2t_3+tt_1t_2+tt_2t_3+tt_1t_3}=1.$$

Hence, the Δ_2 lines of A_i or t_i and of D_i or t_jt_k/t_i are, respectively,

$$\frac{z}{2t_i+t_j+t_k}+\frac{t_it_jt_k\,\bar{z}}{2t_jt_k+t_it_j+t_it_k}=1, \quad t_iz+t_it_jt_k\bar{z}=(t_i+t_j)(t_i+t_k).$$

(1) The midpoint of A_i or t_i and H or $t_1+t_2+t_3$ is $(2t_i+t_j+t_k)/2$. This satisfies both of these equations.

(2) The symmetric of A_i or t_i with respect to the nine-point center or $(t_1+t_2+t_3)/2$ is t_j+t_k. This satisfies the equations of the Δ_2 lines of D_j and D_k.

Solved also in a similar manner by E. F. Allen and the proposer.

Editorial Note. The theorem of the problem can be extended to space of n dimensions after defining the $(n-1)$-dimensional plane Δ_2. Let S be an orthocentric simplex with the vertices A_i, $1\leqq i\leqq n+1$, with the orthocenter at H, the centroid at G, and the circumsphere (C) with its center at C and radius of length R. Let also the sphere (C') have its center at C' on HC so that $HC'=HC/n$ and its radius R/n; and we now define the $(n-1)$-dimensional planes Δ_2 and Δ_1 corresponding to a given point T on (C) in the following:

THEOREM. *A straight line $H\beta_i$ through H parallel to the straight line TA_i intersects the plane of the face π_i opposite to A_i in the point β_i. The $n+1$ points β_i lie*

in a plane Δ_2. The plane through H perpendicular to TA_i intersects the plane of face π_i in the $(n-2)$-dimensional space α_i. The $n+1$ intersections α_i lie in a plane Δ_1. The planes Δ_1 and Δ_2 are tangent to a quadric surface of revolution $\{C'\}$ with foci H, H' and center C', and $\{C'\}$ is tangent also to the $n+1$ faces π_i. The sphere (C') is the auxiliary sphere for the focal axis.

We take H as the origin of vectors $\mathbf{a}_i$ to the vertices A_i. The vectors of C, G, C' are denoted by $\mathbf{c}$, $\mathbf{g}$, $\mathbf{c}/n$; see the *Editorial Note* on 3963 [1942, 132]. The points H and G are centers of similitude for (C) and (C'), so that if TG is produced to P, where $GP=TG/n$, the point P lies on (C'). Let $\overline{A}_i$ be the point on HA_i so that $H\overline{A}_i=HA_i/n$; then $\overline{A}_i$ lies on (C'). If A_iG is produced to G_i so that $GG_i=A_iG/n$, then G_i lies on (C_i) and it is the centroid of the face π_i; and, if we denote by $\mathbf{g}_i$ the vector of G_i, then $\mathbf{a}_i+n\mathbf{g}_i=2\mathbf{c}$. Thus PG_i is parallel to TA_i and its length is TA_i/n. Denote the vectors of T, P by $\mathbf{y}$, $\mathbf{z}$; then $\mathbf{y}+n\mathbf{z}=2\mathbf{c}$. Since $\overline{A}_iG_i$ is a diameter of (C'), $P\overline{A}_i$ is perpendicular to PG_i and also to TA_i and $H\beta_i$. We shall suppose at first that H does not fall upon a vertex so that m is not zero, $m=\mathbf{a}_i\cdot\mathbf{a}_j$, $i\neq j$. Thus the vector for β_i satisfies the equation

$$(1)\qquad (\mathbf{z}-\mathbf{a}_i/n)\cdot\mathbf{x}=0;$$

and since β_i lies in π_i, its vectors satisfies also the equation

$$(2)\quad \pi_i:\qquad \mathbf{a}_i\cdot\mathbf{x}=m.$$

Hence, the vector for β_i satisfies the equation

$$(3)\quad \Delta_2:\qquad n\mathbf{z}\cdot\mathbf{x}-m=0.$$

This proves that the $n+1$ points β_i lie in a plane perpendicular to HP. The vector $n\mathbf{z}$ is the vector of a point P_c on (C); and (3) shows that Δ_2 is the polar plane of P_c with respect to the polar sphere (H).

The equation of the plane through H perpendicular to TA_i or to PG_i is

$$(4)\qquad \left(\mathbf{z}-\frac{2\mathbf{c}-\mathbf{a}_i}{n}\right)\cdot\mathbf{x}=0.$$

Hence, the points of intersection of this plane and π_i satisfy both (2) and (4); and, therefore, the equation of the locus of these points of intersection is

$$(5)\quad \Delta_1:\qquad (2\mathbf{c}-n\mathbf{z})\cdot\mathbf{x}-m=0.$$

If P_c' is the point with the vector $2\mathbf{c}-n\mathbf{z}$, the midpoint of P_cP_c' has the vector $\mathbf{c}$, and thus this midpoint is C. Hence P_c' lies also on (C). From (5) we see that Δ_1 is the polar plane of P_c'. The polar of A_i is the plane of π_i with the equation (2). The polar of (C) is a quadric surface of revolution $\{C'\}$ with one focus at H and tangent to the $n+1$ planes π_i and to the planes Δ_1 and Δ_2. The inverse of (C) is (C'), and hence (C') and $\{C'\}$ are tangent at two points on HC. This shows that (C') and $\{C'\}$ have the same center C', and that the other focus H'

must have the vector $2\mathbf{c}/n$. Also, the points H, H' must be isogonal conjugates with respect to S.

If $m=0$, the orthocenter H must fall upon a vertex, say A_{n+1}; and then for $i=n+1$ the equation (1) becomes $\mathbf{z}\cdot\mathbf{x}=0$, and it will be seen that we do not need the equation of π_{n+1}. For other values of i the equations (1), (2), (3) are the above with $m=0$. Also, for $i=n+1$ the equation (4) becomes the one in (5) after setting $m=0$; hence, the equations for Δ_2 and Δ_1 are given by (3) and (5) with $m=0$. In this case, the foci H and H' lie on (C') and $\{C'\}$ and the latter surface degenerates to these two points; also, $G_{n+1}=H'$. This suffices for $m=0$.

We now consider the extended theorem of the problem. The sphere (C) cuts the plane through A_i parallel to the opposite face π_i in an $(n-2)$-dimensional sphere. Let T_i, D_i be points at the ends of a diameter of this latter sphere with vectors $\mathbf{y}_i$, $\mathbf{y}_{di}$. Then to T_i, D_i there correspond P_i, P_{di} at ends of a diameter of the intersection of (C') and π_i; in particular, two such points are G_i, H_i, where H_i is the foot of the altitude from A_i. Let the vectors of P_i, P_{di} be $\mathbf{z}_i$, $\mathbf{z}_{di}$; then we have

$$\mathbf{z}_i + \mathbf{z}_{di} = \frac{2\mathbf{c} - \mathbf{a}_i}{n} + \frac{m\mathbf{a}_i}{\mathbf{a}_i^2},$$

where $\mathbf{a}_i\cdot\mathbf{z}_i=m$, $\mathbf{a}_i\cdot\mathbf{z}_{di}=m$. The Δ_2 plane for T_i, or P_i, has the equation

$$\Delta_2(P_i):\qquad n\mathbf{z}_i\cdot\mathbf{x} - m = 0,\quad \text{or}\quad n\mathbf{z}_i\cdot\mathbf{x} - \mathbf{z}_i\cdot\mathbf{a}_i = 0.$$

Hence $\Delta_2(P_i)$ passes through $\overline{A}_i$ with the vector $\mathbf{a}_i/n$ for all such points P_i, or T_i. This is part (1) of the problem. For part (2) we take the special case where $T_i=A_i$, $P_i=G_i$, $P_{di}=H_i$, which is the case of the problem for $n=2$; and we now consider $m\neq 0$. We now have

$$\Delta_2(H_i):\qquad n\left(\frac{m\mathbf{a}_i}{\mathbf{a}_i^2}\right)\cdot\mathbf{x} - m = 0,\quad \text{or}\quad n\mathbf{a}_i\cdot\mathbf{x} - \mathbf{a}_i^2 = 0.$$

The symmetric of A_j with respect to C' has the vector $2\mathbf{c}/n-\mathbf{a}_j$, and we shall show that $\Delta_2(H_i)$ passes through this symmetric. We have

$$n\mathbf{a}_i\cdot\left(\frac{2\mathbf{c}}{n} - \mathbf{a}_j\right) - \mathbf{a}_i^2 = 2\mathbf{c}\cdot\mathbf{a}_i - nm - \mathbf{a}_i^2 = nm - nm = 0,\quad i\neq j.$$

Hence, if j is fixed, $\Delta_2(H_i)$, $i\neq j$, passes through the symmetric of A_j. This completes part (2).

Associated Triangles

3969 [1940, 574]. *Proposed by Frank Ayres, Jr., Dickinson College*

Let the line through the vertex A_i $(i=1, 2, 3)$, and parallel to the opposite side of the triangle $A_1A_2A_3$ meet the circumcircle in the point D_i. Show that: (1) The orthocenter of $D_1D_2D_3$ lies on the join of the circumcenter and isogonal conjugate point of the nine-point center of $A_1A_2A_3$. (2) The join of the orthocenters of $D_1D_2D_3$ and $A_1A_2A_3$ is the image line of the Steiner point of the latter triangle.

Solution by A. M. Peiser, Ithaca, N. Y.

Let the vertices of the triangle be t_i, $(i=1, 2, 3)$. The line through t_i parallel to side t_2t_3 is given by

$$\frac{x}{t_2t_3} + y = \frac{t_1}{t_2t_3} + \frac{1}{t_1};$$

and setting $y=1/x$, we find the second intersection with the unit circle to be t_2t_3/t_1. This is the point D_1. Similarly, D_2 and D_3 are t_1t_3/t_2 and t_1t_2/t_3, respectively.

The orthocenter of a triangle whose vertices are turns is the sum of the turns. Hence, if we let the orthocenter of $D_1D_2D_3$ be H, and its conjugate be $\overline{H}$, we have

$$\overline{H} = \frac{t_1}{t_2t_3} + \frac{t_2}{t_1t_3} + \frac{t_3}{t_1t_2} = \frac{t_1^2 + t_2^2 + t_3^2}{t_1t_2t_3} = \frac{S_1^2 - 2S_2}{S_3},$$

where the S's are the symmetric functions of the t's. Hence $H=(S_2^2-2S_1S_3)/S_3$. The line joining H to the circumcenter of $t_1t_2t_3$ is $(S_1^2-2S_2)x=(S_2^2-2S_1S_3)y$. The isogonal conjugate of a point a is $(a-S_1+\bar{a}S_2-\bar{a}^2S_3)/(a\bar{a}-1)$. Hence the isogonal conjugate of $S_1/2$, the nine-point center, is $(2S_1S_3-S_2^2)/(4S_3-S_1S_2)$. And it can readily be shown that this value, together with its conjugate, satisfies the equation of the line OH. Hence, as was to be shown, the three required points are collinear.

The image line of any point on the unit circle is given by $Tx-S_3y=TS_1-S_2$ (Musselman, this MONTHLY, vol. 45, 1938, p. 426). For the Steiner point, the line of images is

$$S_3(S_1^2 - 3S_2)x - S_3(S_2^2 - 3S_1S_3)y = S_3(S_1^3 - 3S_1S_2) - S_2(S_2^2 - 3S_1S_3).$$

And by direct substitution, it can be shown that both S_1 and H lie on this line of images, S_1 being the orthocenter of triangle $t_1t_2t_3$, and H being the orthocenter of triangle $D_1D_2D_3$, as was to be shown.

Solved also in a similar manner by J. W. Clawson and the proposer.

Desmic Systems

3972 [1940, 662]. *Proposed by N. A. Court, University of Oklahoma*

With the traces of a plane on the edges of a tetrahedron as centers, spheres are drawn orthogonal to the circumsphere of the tetrahedron. Show that the twelve points of intersection of the six spheres with the respective edges form a desmic system.

Solution by the Proposer

Let X_0, Y_0, Z_0, U_0, V_0, W_0 be the traces of a plane Σ on the edges BC, CA, AB, DA, DB, DC of the tetrahedron $DABC$. The six spheres $(X_0), \cdots, (W_0)$ having these points for centers and orthogonal to the circumsphere (O) of $DABC$ form a coaxal net (N) (see Court, *Modern Pure Solid Geometry*, p. 191, art. 602 f.)

whose conjugate pencil (P) is determined by the sphere (O) and Σ as radial plane.

Let (R) be any sphere of (P), and (A), (B), (C), (D) the spheres with A, B, C, D as centers and orthogonal to (R). The three spheres (B), (C), (X_0) are orthogonal to (R), and their centers are collinear; hence these spheres are coaxal. Now (X_0) being orthogonal to (O), its points of intersection X, X' with the edge BC are harmonically separated by the vertices B, C; hence (X_0) is the sphere of similitude of the spheres (B), (C) (*ibid.*, p. 186, art. 591).

Thus pairs of points X, X'; Y, Y'; $\cdots$; W, W' are the centers of similitude of the four spheres (A), (B), (C), (D) taken in pairs; hence the proposition (*ibid.*, p. 240, art. 736).

Rectangle Inscribed in Circle

3897 [1938, 696]. *Corrected. Proposed by V. Thébault, San Sebastián, Spain*

Let $ABCD$ be a rectangle inscribed in a circle with center O, and P a point on the equilateral hyperbola circumscribing $ABCD$. The straight lines PA, PB, PC, PD cut the circle again in A', B', C', D'. The perpendiculars from P to the sides of the quadrilateral $A'B'C'D'$ cut $A'B'$ in A'', $B'C'$ in B'', etc.

(1'') The diagonals $A''C''$ and $B''D''$ are perpendicular and intersect in a point Q on the straight line OP. (2'') The ratio of the lengths of these diagonals is the same as the ratio of the sides of the rectangle. (3'') The quadrilateral $A''B''C''D''$ is inscribed in a circle and circumscribes a conic with foci P and Q. (4'') The Newton line for $A'B'C'D'$ passes through P and is perpendicular to the Newton line for $A''B''C''D''$.

Solution by R. Bouvaist, France

Rectangular axes of coördinates Px and Py are chosen with P as origin parallel to the sides CD and DA of the rectangle. The projections of P on the sides DA, BC, AB, CD are denoted respectively by P_1, P_2, P_3, P_4; and we set $PP_1=a$, $PP_2=-a'$, $PP_3=b$, $PP_4=-b'$, so that we have as the coördinates of the vertices of $ABCD$

$$A(a, b),\ B(-a', b),\ C(-a', -b'),\ D(a, -b').$$

Since P is on the equilateral hyperbola $\{ABCD\}$ we must have

$$aa' = bb'. \tag{1}$$

Let A_1, B_1, C_1, D_1 be the vertices of the antipedal quadrilateral of $ABCD$ with respect to P, where the projection of P on D_1A_1 is A, of P on A_1B_1 is B, *etc.* Then by the use of (1) we find the coördinates

$$\begin{aligned} &A_1[a-a', b+b'],\quad B_1[-(a+a'), b-b'],\quad C_1[a-a', -(b+b')],\\ &\qquad\qquad D_1[a+a', b-b']. \end{aligned} \tag{2}$$

Let P' be the symmetric of P with respect to O; then the diagonals A_1C_1 and

B_1D_1 are perpendicular, intersecting in P', and are parallel respectively to BC and CD. The quadrilateral $A_1B_1C_1D_1$ is inscribed in the circle with P as center and radius $\rho=(a^2+a'^2+b^2+b'^2)^{1/2}$.

(1″) The quadrilateral $A''B''C''D''$ is the inverse with respect to P of the quadrilateral $A_1B_1C_1D_1$, the power of inversion being the power of P with respect to the circle $(O)=(ABCD)$, or $K^2=-2aa'=-2bb'$. These two quadrilaterals are homothetic with respect to P. For, we have

$$PA_1\cdot PA'' = PB_1\cdot PB'' = PC_1\cdot PC'' = PD_1\cdot PD'' = K^2;$$

and from this follows that

$$\frac{PA''}{PA_1}=\frac{PB''}{PB_1}=\frac{PC''}{PC_1}=\frac{PD''}{PD_1}=\frac{K^2}{(PA_1)^2}=-\frac{2aa'}{\rho^2}. \tag{3}$$

The diagonals $A''C''$, $B''D''$ are parallel to Px, Py and intersect in a point Q, the homothetic of P' in the above homothetic relation.

(2″) We have

$$A''C''=\frac{2K^2(b+b')}{\rho^2}, \qquad B''D''=\frac{2K^2(a+a')}{\rho^2};$$

and from this it follows that

$$\frac{A''C''}{B''D''}=\frac{b+b'}{a+a'}=\frac{BC}{AB}.$$

(3″) The quadrilateral $A_1B_1C_1D_1$ circumscribes the conic with the foci P and P', and with the circle (O) as the auxiliary circle for the focal axis. The quadrilateral $A''B''C''D''$, which is homothetic to $A_1B_1C_1D_1$ with respect to P, therefore circumscribes the homothetic conic with foci P and Q and is inscribed in the circle homothetic to (O).

(4″) We find for the coördinates of the vertices

$$A'\left(\frac{K^2a}{a^2+b^2},\ \frac{K^2b}{a^2+b^2}\right), \qquad B'\left(-\frac{K^2a'}{a'^2+b^2},\ \frac{K^2b}{a'^2+b^2}\right),$$

$$C'\left(-\frac{K^2a'}{a'^2+b'^2},\ -\frac{K^2b'}{a'^2+b'^2}\right), \qquad D'\left(\frac{K^2a}{a^2+b'^2},\ -\frac{K^2b'}{a^2+b'^2}\right).$$

The Newton line for this quadrilateral has the equation $(a-a')x=(b-b')y$. It is then easily seen that this line is perpendicular to the Newton line for $A_1B_1C_1D_1$, and to its homothetic, the Newton line for $A''B''C''D''$.

Remarks. If P is assumed to be any point in the plane of $ABCD$, the circles (PA_1C_1), (PB_1D_1) are always orthogonal, and consequently the diagonals $A''C''$ and $B''D''$ are perpendicular. For, take for axes the parallels to the sides of the rectangle drawn through P'; and set $P'A_1=\beta$, $P'C_1=\beta'$, $P'D_1=\alpha$, $P'B_1=\alpha'$.

Let x_1, y_1 be the coördinates of P. The conic with foci P, P' and having circle (O) for the focal auxiliary circle is inscribed in $A_1B_1C_1D_1$, and its tangential equation is

$$K^2(u^2+v^2)-W(ux_1+vy_1+W)=0.$$

From this results the relations

$$\frac{1}{\alpha\alpha'}+\frac{1}{\beta\beta'}+\frac{1}{K^2}=0,\qquad \frac{1}{\alpha}+\frac{1}{\alpha'}+\frac{x_1}{K^2}=0,\qquad \frac{1}{\beta}+\frac{1}{\beta'}+\frac{y_1}{K^2}=0$$

We then find the following equations

$$(PB_1D_1):\quad x^2+y^2+\frac{xx_1\alpha\alpha'}{K^2}-\mu y+\alpha\alpha'=0,$$

$$(PA_1C_1):\quad x^2+y^2-\mu' x+\frac{yy_1\beta\beta'}{K^2}+\beta\beta'=0.$$

The two circles are orthogonal if

$$\frac{\mu'\alpha\alpha' x_1}{K^2}+\frac{\mu\beta\beta' y_1}{K^2}+2(\alpha\alpha'+\beta\beta')=0,$$

or

$$\frac{\mu' x_1}{\beta\beta'}+\frac{\mu y_1}{\alpha\alpha'}=2.$$

Since these circles pass through $P(x_1, y_1)$, we have

$$\mu y_1=x_1^2+y_1^2+\frac{x_1^2\alpha\alpha'}{K^2}+\alpha\alpha',$$

$$\mu' x_1=x_1^2+y_1^2+\frac{y_1^2\beta\beta'}{K^2}+\beta\beta';$$

and we then have

$$\frac{\mu y_1}{\alpha\alpha'}+\frac{\mu' x_1}{\beta\beta'}=(x_1^2+y_1^2)\left(\frac{1}{\alpha\alpha'}+\frac{1}{\beta\beta'}\right)+\frac{x_1^2+y_1^2}{K^2}+2=2$$

by use of an equation above.

We can also state the following property: Given a circular cubic passing through its singular focus (locus of the foci of the conics of a tangential pencil) and two conjugate points P and P' of that curve (foci of a conic of the pencil), if through P' we draw two perpendicular chords, cutting the cubic again in A_1 and C_1, B_1 and D_1, respectively, the circles (PA_1C_1), (PB_1D_1) are orthogonal.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to C. O. Oakley, Haverford College, Haverford, Pennsylvania.

This department seeks information as to institutions and persons who are now giving or who plan to give Engineering, Science and Management Defense Training courses during the coming summer. Please send any such information to the editor of *News and Notices.*

Brown University is continuing during the summer (June 15–August 29) and the next academic year the program of advanced instruction and research in mechanics already in operation for a year. In the summer 80 participants will be accepted (of which 20 will be engaged entirely in research) and 40 in the academic year. Since this program is supported by the United States Office of Education, the Carnegie Corporation of New York, and the Rockefeller Foundation, no fees will be charged.

For the Summer the following courses will be given: By Professor Brillouin: Introduction to partial differential equations, Advanced dynamics. By Professor Prager: Geometrical foundations of mechanics, Plasticity, Special topics in research. By Dr. Bergman: Theory of flight, Review of special topics in pure mathematics, Fluid dynamics, Special topics in research. By Professor Sokolnikoff: Elasticity, Advanced elasticity, Special topics in research. By Professors Tamarkin and Feller: Differential and integral equations of physics. By Dr. Schelkunoff: Electromagnetic waves. By Professor von Mises: Advanced fluid dynamics.

For the next academic year eight courses are scheduled. Several substantial fellowships are available for highly qualified participants.

Dr. Walter Bartky, associate dean, Division of Physical Sciences at the University of Chicago, has been advanced from associate professor of astronomy to professor of applied mathematics.

Dr. E. G. Begle and Dr. E. R. Kolchin have been awarded National Research Fellowships for 1941–1942. They are studying at the University of Michigan and the Institute for Advanced Study, respectively.

Assistant Professor P. O. Bell, on leave from the University of Kansas, has an instructorship at Princeton University.

Assistant Professor A. C. Berry of Columbia University has been appointed to an associate professorship at Lawrence College.

Assistant Professor E. A. Cameron of the University of North Carolina has been promoted to an associate professorship.

Dr. C. E. Clark of Purdue University has been promoted to an assistant professorship.

Dr. Max Coral of Wayne University has been promoted to an assistant professorship.

Paul Eberhart of Washburn Municipal University of Topeka, Kansas, has been promoted to an assistant professorship.

Assistant Professor E. A. Hazlewood of Texas Technological College has been promoted to an associate professorship.

Professor Einar Hille of Yale University is on leave of absence and is at Stanford University.

Professor W. A. Hurwitz has been granted a sabbatical leave of absence for the second semester of the year 1941–42.

Professor W. R. Hutcherson of Berea College, on leave of absence, is spending the second semester at Brown University.

Assistant Professor S. C. Kleene of the University of Wisconsin has been appointed to an associate professorship at Amherst College.

G. R. Kraus of Cathedral College has been made head of the department at Gannon School of Arts and Science, Erie, Pennsylvania.

Dr. Roy MacKay of Eastern New Mexico College has been appointed an associate professor at New Mexico State College of A. and M. A.

Dr. R. J. Michel of the University of Missouri has been appointed to a professorship at Southeast Missouri State Teachers College.

Dr. F. W. Owens, professor of mathematics and head of the department at Pennsylvania State College is on leave of absence during the second semester of 1941–42.

Dr. E. S. Quade of the University of Florida has been promoted to an assistant professorship.

Dr. A. C. Schaeffer of Stanford University has been promoted to an assistant professorship.

Professor Mary Emily Sinclair of Oberlin College, while on leave for the second semester, is studying at Columbia University.

C. E. Stevens of Hofstra College has been promoted to an assistant professorship.

Dr. John Williamson of Johns Hopkins University is on leave of absence for a year and has been appointed to an associate professorship at Queens College.

The following appointments to instructorships have been announced:
University of California, Berkeley: R. W. Shepard
University of California, Davis: Charles Bubb
Case School of Applied Science: Dr. E. L. Crow, Dr. P. E. Guenther
University of Colorado: George Ulrich

Compton Junior College: A. E. Marston
Cornell University: Dr. C. E. Rhodes; Part-time: Charles Hatfield, Jr.
University of Florida: R. D. Specht
Harvard University: Dr. Irving Kaplansky, Dr. C. E. Rickart
University of Iowa: Dr. W. D. Berg
Johns Hopkins University: Dr. L. I. Wade
University of Kansas: Dr. R. S. Pate
University of Maryland: E. N. Nilson
Massachusetts Institute of Technology: Dr. O. G. Owens
Michigan State Normal College: Dr. Edith R. Schneckenburger
University of Michigan: Dr. Charles Thorne
North Carolina State College: Dr. R. L. Anderson
Northwestern University: Dr. J. W. Givens
Pennsylvania State College: R. P. Bentz, Dr. W. J. Harrington
Princeton University: Dr. Warren Ambrose
Santa Barbara State College: Dr. S. E. Rauch
Smith College: Dr. Jeanne S. LeCaine
Stanford University: Dr. G. E. Forsythe
U. S. Naval Academy: Dr. Byron Cosby, Jr., C. B. Lindquist, Dr. A. W. McGaughey, Dr. H. T. Muhly, Dr. Seymour Sherman
Vassar College: Alexandra I. Forsythe
Wayne University: Morris Friedman
Wellesley College: Katharine E. Hazard
University of Wisconsin: Dr. R. E. Johnson

Professor C. S. Atchison, head of the department of mathematics at Washington and Jefferson College since 1912, died November 21, 1941. He was a charter member of the Mathematical Association.

Dr. W. V. N. Garretson, professor of mathematics at the Oklahoma A. and M. College since 1929, died January 17, 1942, at the age of sixty-five. He was a charter member of the Mathematical Association.

THE LETTER OF ADMIRAL NIMITZ

"When secondary schools eliminate not only trigonometry but also algebra and geometry from their programs, and then most of the reasoning problems of arithmetic, since pupils say they are too difficult, and offer as substitutes general mathematics in the ninth grade, social mathematics in the tenth grade, and review of arithmetic in the eleventh or twelfth grade as the total mathematical program of the school, where along the educational ladder are pupils to obtain experience in reasoning and in practice in solving progressively more difficult mathematical problems? Where in the course of the four years are youth to find mathematical problems which will extend their intellectual horizons and stretch their mental muscles?"

The preceding queries from a leaflet issued by the University of Michigan were called forth by a letter written last November by Admiral C. W. Nimitz, at that time Chief of the Bureau of Navigation of the United States Navy. In late October he had visited the University of Michigan. The letter was in amplification of remarks made during the visit and was written in response to a letter from Professor Bredvold.

In December Admiral Nimitz was made Commander in Chief of the Pacific Fleet.

It has been the wish of many that this exchange of letters be printed in the MONTHLY. It is hoped that the members of the Association will bring the facts and figures cited by Admiral Nimitz to the attention of school boards and other school authorities to the end that there may be some improvement in the deplorable conditions of which he writes.

The letters follow:

Captain F. U. Lake October 30, 1941
Head of the Training Division
Bureau of Navigation, Washington, D. C.
My dear Capital Lake:

When Admiral Nimitz visited the campus of the University of Michigan the other day, he mentioned that there had been some difficulty in finding students in American colleges other than engineering who were sufficiently prepared in mathematics to make them available for training for commissions in the Navy. This situation ought to be called to the attention of educators in colleges and secondary schools throughout the country. I should deeply appreciate receiving a statement from you on this matter, especially if you could give me such facts and figures as would constitute a self-evident argument. I hope also that it will not be necessary to set any restrictions on the use of such information. It seems to me that educators should promptly recognize the danger, if there is any, from our past softening of our educational programs.

Very truly yours,
LOUIS I. BREDVOLD
Member of the University Advisory
Committee on Military Affairs

November 12, 1941

My dear Professor Bredvold:

Thank you for your letter of October 30. While we have not felt that it was our business to compile exhaustive data on our observations of the products of the educational systems of this country, we are in a position to give you some information on this subject.

A carefully prepared selective examination was given to 4,200 entering freshmen at 27 of the leading universities and colleges of the United States. Sixty-eight per cent of the men taking this examination were unable to pass the arithmetical reasoning test. Sixty-two per cent failed the whole test, which included also arithmetical combinations, vocabulary, and spatial relations. The majority of failures were not merely borderline, but were far below passing grade. Of the 4,200 entering freshmen who wished to enter the Naval Reserve Officers' Training Corps, only 10% had already taken elementary trigonometry in the high schools from which they had graduated. Only 23% of the 4,200 had taken more than one and a half years of mathematics in high school.

This same lack of fundamental education presented and continues to present a major obstacle in the selection and training of midshipmen for commissioning as ensigns, V-7. Of 8,000 applicants —all college graduates—some 3,000 had to be rejected because they had had no mathematics or insufficient mathematics at college nor had they ever taken plane trigonometry. Almost 40% of

the college graduates applying for commissioning had not in the course of their education taken this essential mathematics course.

The experience which the Navy has had in attempting to teach navigation in the Naval Reserve Officers' Training Corps Units and in the Naval Reserve Midshipmen Training Program (V-7) indicates that 75% of the failures in the study of navigation must be attributed to the lack of adequate knowledge of mathematics. Since mathematics is also necessary in fire control and in many other vital branches of the naval officer's profession, it can readily be understood that a candidate for training for a commission in the Naval Reserve cannot be regarded as good material unless he has taken sufficient mathematics.

The Navy depends for its efficiency upon trained men. The men are trained at schools conducted for this purpose and the admission of men to these schools is based upon the meeting of certain carefully established requirements. However, in order to enroll the necessary number of men in the training schools, it was found necessary at one of the training stations to lower the standards in 50% of the admissions. This necessity is attributed to a deficiency in the early educations of the men involved. The requirements had to be lowered in the field of arithmetical attainment. Relative to the results obtained in the General Classification Test, the lowest category of achievement was in arithmetic.

A study has been made of the grades received in the examinations of candidates for enlistment in the Navy, classified geographically according to the location of the recruiting station through which the candidates applied for enlistment. It is to be noted that the proficiency in arithmetic in the eastern part of the country was strikingly greater than that of the middle west and west. The lowest average mark east of the Mississippi was equal to the highest average mark west of the Mississippi. The three highest average attainments in arithmetic were achieved by the recruting stations in Troy, Brooklyn, and Buffalo—all in New York State.

May I express the hope that this information will be of assistance to you.

Sincerely yours,
C. W. Nimitz,
Chief of Bureau,

(Signed) F. U. Lake,
By direction.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-seventh Annual Meeting, New York, N. Y., December 30–31, 1942.

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

Allegheny Mountain
Illinois, Decatur, May 8–9, 1942
Indiana, Crawfordsville, May 1–2, 1942
Iowa, Mt. Pleasant, April 17–18, 1942
Kansas, Hays, March 27–28, 1942
Kentucky, Lexington, April 11, 1942
Louisiana-Mississippi, Jackson, Miss., March 6–7, 1942
Maryland-District of Columbia-Virginia, Ashland, Va., May 2, 1942
Metropolitan New York, New York, April 18, 1942
Michigan
Minnesota, Northfield, May 9, 1942
Missouri, Kansas City, April 17, 1942
Nebraska, Omaha, May 9, 1942
Northern California, San Francisco, Jan. 30, 1943
Ohio, Columbus, April 2, 1942
Oklahoma, Oklahoma City
Philadelphia, Philadelphia, Nov. 28, 1942
Rocky Mountain, Golden, Colo., April 17–18, 1942
Southeastern, Emory University, Ga., March 26–27, 1942
Southern California, Los Angeles, March 14, 1942
Southwestern, State College, N. M., April 27–28, 1942
Texas, Lubbock, April 3–4, 1942
Upper New York State, Rochester, May 2, 1942
Wisconsin, Oshkosh, May 2, 1942

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association. Through two subsequent gifts the prize is now awarded every three years. The last award was made in December 1941 to Professor Saunders Mac Lane for his two papers in the *American Mathematical Monthly:* "Modular fields," volume 47, 1940, pp. 259-274 and "Some recent advances in algebra," volume 46, 1939, pp. 3-19.

As determined more recently by the Trustees, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included; they carry their own reward.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

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THE FALL MEETING OF THE KENTUCKY SECTION

A joint meeting of the Kentucky Section of the Mathematical Association of America and the Kentucky Section of the National Council of Teachers of Mathematics was held at the University of Kentucky on Saturday, October 25, 1941. Professor Tryphena Howard, president of the Kentucky Section of the Council, presided at the morning meeting.

There were sixty-two in attendance, including the following nineteen members of the Association: N. B. Allison, M. C. Brown, L. W. Cohen, H. H. Downing, L. A. Fair, Charles Hatfield, Tryphena Howard, E. D. Jenkins, Fritz John, C. G. Latimer, F. Elizabeth LeStourgeon, W. L. Moore, Sister Charles Mary Morrison, R. S. Park, Sallie E. Pence, D. W. Pugsley, D. E. South, Guy Stevenson, H. A. Wright.

The chairman of the Kentucky Section of the Mathematical Association of America, Professor L. A. Fair, presided at the luncheon meeting which followed the morning session.

The following papers were presented:

1. "Some psychological aspects of the teaching of arithmetic" by Professor M. E. Schell, Western Kentucky State Teachers College, introduced by Professor Howard.

2. "Errors on freshman entrance examinations" by Professor H. A. Wright, Transylvania College.

3. "Mathematics enters naturally into practical problems" by Professor O. T. Koppius, University of Kentucky, introduced by Professor Howard.

4. "Mathematics and science in the national defense program" by Professor D. W. Pugsley, Berea College.

5. "Mathematics and the environment" by Sister M. Raymond, Ursuline College, introduced by Professor Howard.

6. "On the invariants of a conic section" by Professor C. G. Latimer, University of Kentucky.

7. "Mathematical instruments" by Edith Wood, Okolona High School, Louisville, introduced by Professor Howard.

8. "Treatment of a certain improper integral" by Professor W. C. Wineland, Morehead State Teachers College, introduced by Professor Fair.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor Schell stressed the following points (a) the language of mathematics is an attempt by mankind to express accurately his mental reactions as employed in quantitative thinking; (b) the intelligent use of this mathematics is based upon a correct comprehension of the mental processes the language represents; (c) the solving of a problem is a mental process or a series of correctly related mental processes, commonly known as analysis; (d) every problem intelligently solved must necessarily be correctly analyzed; (e) the teacher's leading approach in developing the ability to do correct quantitative

thinking is through the language of mathematics; (f) the correct solution of a problem is based upon correct quantitative thinking, hence more emphasis should be placed on developing this ability rather than upon the mechanical processes of expression as is so often done.

2. A brief study of two sets of freshman entrance examinations was made by Professor Wright, using Kentucky Mathematics Test, Form KM 38 and Form KM 40. The problems in the latter differ slightly in wording from the former but not in principle. A comparison of the two sets of papers indicated a lack of ability to analyze statements accurately. The errors on each set indicate a lack of knowledge of the fundamental principles of arithmetic and algebra.

3. Professor Koppius applied elementary mathematics to a problem in geophysics. In deep bore holes, such as deep oil wells, the course of the pipe frequently veers from the vertical to a great extent, and may be in any azimuth in a horizontal plane. Applying simple principles of geometry and trigonometry together with the fact that the definite integral between x_1 and x_2 is the area under the curve, it is shown that the position of the bottom of deep wells with respect to the origin taken at the well mouth on the surface can be determined in terms of δ, α, and L, where δ is the angle which a pipe section dL makes with the vertical at any point, α is the angle which the projection of dL makes with the N–S direction in the horizontal plane, and L is the total length of pipe in the hole. For finding the true vertical depth, only the δ's at the corresponding L's are necessary. A device was described for observing the various δ, α, and L.

4. Professor Pugsley described the Engineering defense training program in introductory Engineering subjects as carried out in Pennsylvania during the summer of 1941. The program was outlined as ten week tuition free courses in mathematics, physics, chemistry, drawing, and mechanics for "superior" high school graduates who had had two years of mathematics and one year of science, and who did not plan to enter college. Supervised study and special lectures supplemented the class work. About 3200 students enrolled (1948 completed the work) in 150 classes held in 100 or more cities and towns. Three hundred teachers were employed. As far as possible the teachers were selected from those who had had college teaching experience and at least a master's degree. The actual results of the program are difficult to state at this time. Data seemed to indicate that about one-fourth of the group had already been employed in industry, that many were going on with further study in evening classes elsewhere and would probably be employed soon. The circumstances of employment were governed largely by the local situations.

5. Sister Raymond, stated that mathematics is acknowledged to be a tool subject but the question, "A tool for what?" is often left unanswered. In order to offset this weakness in the teaching of mathematics, Ursuline College has arranged extension courses for the soldiers who are serving their country at Bowman Field. These special courses center entirely around problems of flying. For example, when the soldiers have completed the study of mechanics wherein they are to be taught the principles of analyzing a law and then deriving their

own equations to express the law, they will be given a problem of airplane design and performance in which they must, using all the basic physical laws of mechanics, calculate all necessary data used in the design of airplanes. By this means the college hopes to bring in closer contact mathematics and the environment.

6. Professor Latimer obtained the well-known invariants of a conic by a method involving very little computation and no algebraic material beyond high school algebra. In most analytic geometry texts, considerable computation is used on this problem and of course the elegant methods of higher algebra cannot be used.

7. The use of instruments in the classroom, material things which students can see, handle, and measure, makes the teaching of mathematics more real and vital. Instruments made by the students themselves have more meaning than those factor made, though the latter have their place in accuracy of measurement. A sundial can easily be made by any skillful mathematics student with a little instruction. Geometry and trigonometry underlie its construction. Miss Wood stated that other simple, inexpensive instruments easily made by students and illustrating many theorems in geometry and algebra are the pantograph, proportional dividers, center-square, angle trisector, T-square, astrolabe, clinometer, and a telescope with a magnification of 72.

8. Professor Wineland discussed the method used by H. Bethe, Ann. der Phys. 5, 325 (1930), in treating a certain improper integral which arises in the application of the Born approximate method to scattering by a Coulomb field. In this treatment, the divergent integral $\int_0^\infty \sin Kr\,dr$ is replaced by its mean value over a range of K from $K-\Delta K$ to $K+\Delta K$, the order of integration is reversed and the resulting integral is evaluated. It was shown that this procedure is indicated by the physical consideration that the parameter K, which is a function of the scattering angle, is not constant but varies over some small range of values.

D. E. SOUTH, *Secretary*

THE NOVEMBER MEETING OF THE MICHIGAN SECTION

The fall meeting of the Michigan Section of the Mathematical Association of America was held at the University of Detroit, Detroit, Michigan, on Saturday, November 15, 1941. Professor T. R. Running, chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was sixty-three including the following thirty-nine members of the Association: N. H. Anning, J. W. Baldwin, W. D. Baten, E. F. Beckenbach, F. A. Beeler, E. G. Begle, E. E. Blanche, J. M. Bookstein, W. M. Borgman, J. B. Brandeberry, C. J. Coe, A. H. Copeland, Max Coral, C. C. Craig, P. S. Dwyer, Peter Field, K. W. Folley, V. G. Grove, J. F. Heyda, T. H. Hildebrandt, E. E. Ingalls, L. S. Johnston, P. S. Jones, Wilfred Kaplan, L. C. Karpin-

ski, E. D. McCarthy, D. C. Morrow, A. L. Nelson, J. E. Powell, G. Y. Rainich, E. D. Rainville, Arthur Rosenthal, L. J. Rouse, T. R. Running, E. R. Stabler, B. M. Stewart, G. B. Van Shaack, Fern Welker, E. T. Welmers.

The following program of twelve papers was presented:

1. "Reduction of systems of linear differential equations" by Professor Max Coral, Wayne University.

2. "Stereographic projections with applications to navigation" by Professor L. S. Johnston, University of Detroit.

3. "Independence and the rank of a probability matrix" by Professor A. H. Copeland, University of Michigan.

4. "Solution of the general cubic by hyperbolic and circular functions" by Dr. E. E. Blanche, Michigan State College.

5. "On mean curvature" by Professor Samuel Eilenberg, University of Michigan, introduced by the Secretary.

6. "Establishing the price of a work of art by Birkhoff's formulas" by Professor L. C. Karpinski, University of Michigan.

7. "Solitaire on a checkerboard" by Dr. B. M. Stewart, Michigan State College.

8. "The Hamilton differential" by Professor V. C. Poor, University of Michigan, introduced by the Secretary.

9. "Slide rule for a road spacing problem" by Dr. E. E. Ingalls, Albion College.

10. "Isogonal trajectories of a pencil of planes" by D. K. Kazarinoff, University of Michigan, introduced by the Secretary.

11. "Rotations in space with rational direction cosines" by Professor N. H. Anning, University of Michigan.

12. "Testing advertising effectiveness through eye movement photography" by Dr. J. J. McNamara, University of Detroit, introduced by Professor Johnston.

Abstracts of the papers follow, numbered as above:

1. Professor Coral reviewed the known theory concerning the reduction of the order of a system of linear differential equations,

$$y_i' = a_{ik}(x)y_k, \qquad (i, k = 1, 2, \cdots, n; x_1 \leqq x \leqq x_2).$$

If p linearly independent solutions $y_{is}(x)$, $(s=1, 2, \cdots, p)$, of the system are known, for which the determinant $|y_{rs}|$, $(r, s=1, 2, \cdots, p)$, is never zero on $x_1 \leqq x \leqq x_2$, then the system can be reduced by a suitable transformation to one of order $n-p$. Using a theorem due to Bliss, Professor Coral showed the theorem can be proved without the hypothesis concerning the determinant $|y_{rs}|$.

2. Professor Johnston pointed out the simplicity of the stereographic projection of the sphere on the plane from the standpoints of construction and measurement and showed the graphical solution of the spherical triangle as applied to great circle sailing, to mapping and to practical astronomy. He urged the study of this projection in the courses in solid geometry and trigonometry, par-

ticularly in view of its value in navigation and "avigation" and other phases of civil and military activity connected with the war effort. His paper was illustrated with a set of fifteen large charts.

3. Professor Copeland represented the probability that two random variables x and y will take on the values a_i and b_j respectively by an m by n matrix (W_{ij}). He showed that various types of dependence are associated with the rank of this matrix. Thus strict independence is equivalent to the condition that (W_{ij}) be of rank 1. The constancy of the regression curve implies that (W_{ij}) is of rank less than m and n. The vanishing of the correlation coefficient places no restriction on the rank in the general case, but for $m=n=2$ all three types of dependence coincide.

4. Dr. Blanche showed that the reduced cubic $x^3+3Hx+G=0$ may be solved by the use of the formulas for sin $3x$, cos $3x$, sinh x, cosh x, sin $(u+iv)$. He discussed first the case in which H and G are real, showing how solutions may be obtained under the various possible combinations of signs of H and G. He also discussed the general case in which H and G are complex, calling attention to similarities to the real case.

5. Professor Eilenberg considered first a function f defined and continuous on the surface S^n of a unit sphere in euclidean $(n+1)$ space, and he defined $F(p)$ to be the average of f on the $(n-1)$ sphere equatorial relative to p. He then showed that f and F have the same integral over S^n. It thus appears that if $f(p_1)+f(p_2)+\cdots+f(p_{n+1})=k$ whenever $p_1, p_2, \cdots, p_{n+1}$ are mutually orthogonal unit vectors, then the average of f on S^n is $k/(n+1)$. As an application of this result he pointed out that the mean curvature of a Riemannian manifold usually obtained by contracting the curvature tensor can be likewise obtained using the averaging method of harmonic analysis.

6. Professor Karpinski first discussed the criteria which might be employed to determine the validity of an application of mathematical formulas to new domains and applied his conclusions to Birkhoff's formulas for aesthetic measure and ethical value. Assuming the validity of the methods and formulas given by Birkhoff in a popular lecture, "A Mathematical Approach to Ethics" in the *Rice Institute Pamphlet* for January, 1941, Professor Karpinski added a third formula and with its aid calculated with perfect accuracy the price at which a great work of art was recently sold.

7. Dr. Stewart established necessary and sufficient conditions for clearing checker boards of various shapes of all the checkers but one, employing the usual king's jump. His paper appeared in this MONTHLY for April, 1941.

8. Professor Poor pointed out the value of the Hamilton differential in the absolute geometric development of vector analysis. He established a necessary and sufficient condition for the existence of the Hamilton differential of a polygenic function.

9. Dr. Ingalls explained the problem of the spacing of logging roads in timber land and its solution by Professor D. M. Matthews of the Department of Forestry of the University of Michigan. He exhibited a practical circular slide

rule by which the optimum distance between roads could be calculated from the cost of skidding logs, cost of road construction and density of timber.

10. Mr. Kazarinoff treated the problem of the determination of a family of curves cutting a given pencil of planes at a given angle α and lying in a plane making a given angle β with the axis of the pencil. He showed that for $\alpha=\beta$ the curves are circles for which the axis of the pencil is a focal line. For $\alpha<\beta$ the trajectories are spirals and for $\alpha>\beta$ no continuous solution exists.

11. Professor Anning exhibited several neat two parameter formulas by which rotations in space with rational direction cosines may be obtained. He discussed in particular rotation about the line $x=y=z$.

12. Dr. McNamara presented a statistical study of the effectiveness of advertisement, the data being collected by an ingenious photographic method. The variates employed were the advertisement itself, its position in the publication and the age and education of the reader.

C. J. COE, *Secretary*

THE SIXTEENTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The sixteenth annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Swarthmore College, Swarthmore, Pennsylvania, on Saturday, November 29, 1941, Professor Arnold Dresden presiding.

The attendance was about fifty, including the following thirty-three members of the Association: E. F. Allen, C. B. Allendoerfer, J. A. Benner, H. W. Brinkmann, P. A. Caris, J. W. Clawson, Richard Courant, E. H. Cutler, J. E. Davis, L. J. Deck, F. L. Dennis, Arnold Dresden, Tomlinson Fort, R. M. Foster, H. S. Grant, J. E. Ikenberry, R. B. Kleinschmidt, J. R. Kline, V. V. Latshaw, W. F. Long, F. L. Manning, Clifford Marburger, R. W. Mariott, A. E. Meder, Jr., W. R. Murray, F. W. Owens, Helen B. Owens, G. E. Raynor, C. J. Rees, I. J. Schoenberg, C. A. Shook, C. V. L. Smith, W. M. Smith.

At the business meeting the following officers were elected for next year: Chairman, C. O. Oakley, Haverford College; Secretary, P. M. Whitman, University of Pennsylvania; Program Committee, H. W. Brinkmann, chairman; J. E. Davis, and J. A. Shohat. It was voted to hold the 1942 meeting at the University of Pennsylvania, Philadelphia, Pennsylvania, on Saturday, November 28, 1942. The Section approved in principle a progress report on the activities of the Committee for the Promotion of Science in Secondary Education for the State of Pennsylvania.

The following papers were presented:

1. "The problem of the square pyramid" by Professor R. P. Bailey, Lafayette College, introduced by Professor Oakley.

2. "Cubic congruences" by Professor H. W. Brinkmann, Swarthmore College.

3. "Recent developments in the Cauchy theory of analytic functions" by Dr. P. T. Maker, Rutgers University, introduced by Professor Oakley.

4. "Problems of stability and instability demonstrated by soap film experiments" by Professor Richard Courant, New York University.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Bailey presented an account of the problem of the square pyramid. After the solutions of Lucas, Watson, and others had been reviewed, it was shown that the truth of Lucas's Theorem may be made to depend on the proof that a certain quartic has no integral solutions, trivial or otherwise. The equivalence of Lucas's system of equations to other interesting systems was shown.

2. Professor Brinkmann discussed the relations between the discriminant of a cubic polynomial and its roots, if any, for a prime modulus. An elementary proof was given for a theorem due to Voronoi.

3. Dr. Maker discussed the contribution of modern measure theory to the problem of reducing the conditions on a function of a complex variable in order that it be analytic. The theory of functions monogenic on general sets, and a generalization of the Cauchy theorem to functions on closed sets, were examined.

4. Professor Courant stated that if a dynamical system permits different states of stable equilibrium, corresponding to relative minima of the potential energy, then there must exist transitions between these two states leading over an intermediary state of unstable equilibrium (maximinimum of energy). In mathematical terms: If a problem of stationary values leads to two different solutions of minimum character, then there must be expected at least one third solution of non-minimum character. This principle can be demonstrated by soap film experiments partly of a novel type, relating to minimal surfaces and systems of such surfaces. Likewise surfaces of constant mean curvature are used. These experiments and their mathematical basis are related to classical minimum problems of elementary geometry. They point to a new field of the calculus of variations where the problem is not to connect two elements in the plane or curves in space by extremal arcs or surfaces, but to connect any number of points or any closed graph in space by systems of extremals that intersect in not prescribed points or lines. In addition, a link between these problems and classical isoperimetric problems will be established.

P. A. CARIS, *Secretary*

WHAT IS THE ERGODIC THEOREM?

G. D. BIRKHOFF, Harvard University

The integral of Lebesgue (1901), founded upon Borel measure, has been a dominating weapon in the striking advance of Analysis during the present century. Perhaps the Ergodic Theorem (1931) is destined to hold a central position in this development. Indeed, Wiener and Wintner in a recent article* refer to it as "the only result of real generality established for the solutions of dynamical systems."

To understand the theorem and the nature of its applications it is necessary first of all to say something about (Borel-Lebesgue) measure, *i.e.*, "probability" in the sense sketched by Poincaré in the third volume of his *Méthodes Nouvelles de la Mécanique Céleste*. We restrict ourselves to the case of a line segment of unit length with coördinate x, $0 \leqq x \leqq 1$. Suppose that we have a set of non-overlapping intervals, finite in number and of total length $l<1$ in this segment. The probability in a certain intuitive sense that a point, *taken at random*, lies in one of these intervals, is l; and the probability that it lies in the complementary set is of course $1-l$.

Now suppose that we are given a point set M containing an infinite number of points, which can be enclosed within an infinite set of non-overlapping intervals of lengths l_1, l_2, $\cdots$ of total length.

$$l_1 + l_2 + l_3 + \cdots = l < 1.$$

Then clearly the probability that a point, taken at random, lies in M, cannot exceed l; and the probability that it lies in the complementary set is at least $1-l$. If now M is of such a nature that it can be enclosed in an infinite set of intervals of total length not exceeding an arbitrarily small quantity ϵ, it is apparent that the probability of a random point falling in M does not exceed ϵ, *i.e.* the probability is 0. Such a set M is said to be of measure 0.

For instance, the set of rational points $x=m/n$ which is everywhere dense on the line segment, is of measure 0. In fact these points may be arranged in order

$$0,\ 1;\ \tfrac{1}{2};\ \tfrac{1}{3},\ \tfrac{2}{3};\ \tfrac{1}{4},\ \tfrac{3}{4};\ \tfrac{1}{5},\ \tfrac{2}{5},\ \tfrac{3}{5},\ \tfrac{4}{5};\ \cdots$$

and the nth one of these points may obviously be enclosed within an interval of length $\epsilon/2^n$. Since we have

$$\frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \cdots = \epsilon,$$

* On the ergodic dynamics of almost periodic systems, American Journal of Mathematics, vol. 63, 1941. For an introduction to the literature see Eberhard Hopf's "Ergodentheorie," Ergebnisse der Mathematik und ihrer Grenzgebiete, Berlin, Springer, 1937. Our discussion here deals only with the "Ergodic Theorem," and not at all with the "Mean Ergodic Theorem" of von Neumann, which stimulated me to reconsider some old ideas, and so led me to the discovery and proof of the Ergodic Theorem, embodying a strong, precise result which, so far as I know, had never been hoped for.

it is evident that this set of rational points is of measure 0.

More generally, if we have a set M such that it can be enclosed within a set of intervals of length $l_1, l_2, \cdots$ with

$$l_1 + l_2 + \cdots \leqq l + \epsilon$$

while the complementary set $\overline{M}$ can be enclosed similarly within intervals $\bar{l}_1, \bar{l}_2, \cdots$ with

$$\bar{l}_1 + \bar{l}_2 + \cdots \leqq (1 - l) + \epsilon$$

for $\epsilon > 0$ arbitrarily small, then $\overline{M}$ is said to be measureable of measure l; and its complementary set M will then clearly be measurable of measure $1-l$. In this case the probability that a random point falls in M is obviously to be regarded as l.

All ordinary infinite sets specifically defined by analytic methods are found to be measureable in this sense.

The gist of the Ergodic Theorem can now be illustrated by means of our line segment.

Suppose that there is given any one-to-one *measure preserving* transformation T of the line segment $0 \leqq x \leqq 1$ into itself; T may have a finite or infinite number of discontinuities. A first simple example is the following: Imagine the line segment $0 \leqq x < 1$ bent into a circle of circumference 1, without any stretching; the first transformation T is merely a rotation of this circle through a certain angle α. A second example is the following: The line segment is divided into the infinite set of intervals,

$$0 \leqq x < \tfrac{1}{2};\ \tfrac{1}{2} \leqq x < \tfrac{3}{4};\ \tfrac{3}{4} \leqq x < \tfrac{7}{8}, \cdots$$

and then the second interval is interchanged with the first, the fourth with the third, etc., thus defining the transformation T. In both cases T is evidently of the stated type, and measure is preserved.

The Ergodic Theorem then says: *For any such measure-preserving transformation T, and for each individual point P (except possibly an exceptional set of measure 0), there is a definite probability that its iterates under T, from P on, namely*

$$P,\ T(P),\ T^2(P), \cdots \quad \text{and} \quad P,\ T^{-1}(P),\ T^{-2}(P), \cdots$$

fall in a given measurable set M.

In other words the proportion of n of these points (beginning with P) which lie in the set M tends toward a definite limit μ_p, as n approaches infinity in either direction.

More generally, a line segment may be replaced by a finite volume M of n-dimensions, $n > 1$, and the points of M may be assigned a variable (integrable) positive weight, $w(P)$. The generalized theorem would then assert that the corresponding weighted means tend toward a limit μ_p. In the simple special case first stated, this weight is 1 for the points of M and 0 for the points not in M.

Or, again, for $n>1$ the discrete transformation T may be replaced by a steady measure-preserving flow T_t in time t, and the analogous theorem holds.

To illustrate this last possibility, suppose that in the square $0 \leqq x < 1$, $0 \leqq y < 1$, the points move with a uniform velocity in a fixed direction, making an angle α with that of the x axis, and leaving the square to return at the homologous point (see the adjoining figure). Evidently such a transformation T_t is area-preserving. Let now M be any selected measurable part of the square, and let P be any point of the square—aside always from a possible exceptional

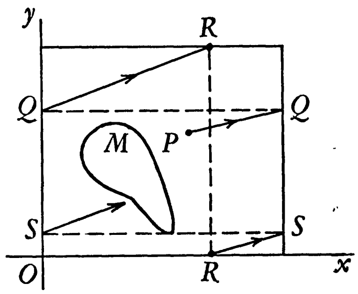

FIG 1.

set of measure 0. On the basis of the same theorem, there is a definite probability in infinite time, $t \geqq 0$ or $t \leqq 0$ that $P_t = T_t(P)$ falls within M, and this probability is the same in both directions. More generally a weight $w(P)$ may be introduced in the case of a "flow" as well as in the discrete case.

In more analytic garb, the theorem states in the two cases respectively that for $n \to \pm\infty, T \to \pm\infty$:

$$\frac{w(P) + w(T(P)) + \cdots w(T^{n-1}(P))}{n} \to \mu_P; \qquad \frac{1}{T}\int_0^T w(P)dP \to \mu_P.$$

The kind of applications to dynamical systems which the Ergodic Theorem affords are exceedingly varied and interesting. Take the simple example of an idealized convex billiard table on which an idealized billiard ball P moves with velocity 1. In the figure let $\phi = \text{arc } OA$, $\phi_1, = \text{arc } OA_1$, $l = AP$, $l^* = AA_1$. We have a transformation $(\theta_1, \phi_2) = T(\theta, \phi)$ defined over a rectangle

$$0 < \theta < \pi; \qquad 0 \leqq \phi \leqq p, \qquad\qquad (p = \text{perimeter of table})$$

in the $\theta\phi$-plane, associated with the motion. It is not hard to prove that T is measure-preserving in the sense that the double integral

$$\int\int \frac{\sin\theta}{\sin\theta_1} d\theta d\phi$$

has the same value when extended over any measurable part of this rectangle

as over its image under T; indeed it would be possible to deform the rectangle so that, over the new region, ordinary areas are preserved.

Furthermore it is clear that, if we associate with any "state of motion" of

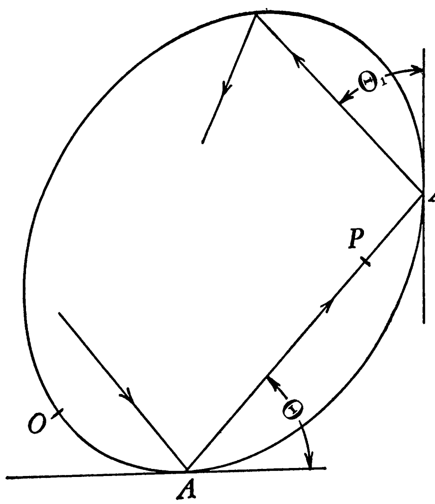

FIG. 2

the billiard ball, as of P, the three coordinates θ, ϕ, l then a steady flow T_t is defined in the corresponding region of three-dimensional $\theta\phi l$-space:

$$0 < \theta < \pi; \qquad 0 \leqq \phi < p, \qquad 0 \leqq l \leqq l^*$$

in which the following volume integral is preserved:

$$\int \left(\int\int \frac{\sin \theta}{\sin \theta_1} d\theta d\phi \right) dl.$$

Thus the theorem applies to this flow.

Here are three obvious applications to this simple but typical dyanamical problem:

(1) the average length of n successive chords of the path tends to a definite limit, the same whether the time t increases or decreases;

(2) the average angle θ at n successive collisions tends to a definite limiting value;

(3) the billiard ball tends in the limit to lie in any assigned area of the table a definite proportion of the time.

There is one especially interesting case, which may in fact be the "general case" as far as we know: It may happen that all of the points of our volume behave in essentially the same way in the mean (aside always from the excepted set of measure 0, of course). If they do not so behave, the underlying space can

be subdivided into *invariant* measurable sets; thus for an elliptical table, the motions lying wholly in the ring outside a smaller confocal ellipse form such a closed invariant set; and this is an integrable problem—a limiting case of geodesics on a flattening ellipsoid.

What the Ergodic Theorem means, roughly speaking, is that for a discrete measure-preserving transformation or a measure-preserving flow of a finite volume, probabilities and weighted means tend toward limits when we start from a definite state P (not belonging to a possible exceptional set of measure 0), and, furthermore, the limiting value is the same in both directions.

The Ergodic Theorem applies to manifold deep problems of analysis and of applied mathematics—as well to the solar system as to our simple billiard ball problem! Thus in G. W. Hill's celebrated idealization of the earth-sun-moon problem (the restricted problem of three bodies) we can at once assert (with probability 1) that the moon possesses a true mean angular state of rotation about the earth (measured from the epoch), the same in both directions of the time.

FOCAL CUBICS ASSOCIATED WITH FOUR POINTS IN A PLANE*

M. G. BOYCE, Western Reserve University

1. Introduction. Let A, B, C, D be distinct fixed points and Z a variable point, all in the same plane. The locus of Z such that the directed angles AZB and CZD are equal, or equal when reduced modulo π, will be referred to as the *equal-angle locus* with respect to the four points. Similarly, if the distance ratios AZ/BZ and CZ/DZ are equal, the locus of Z will be called the *equal-ratio locus*. Any ordered set of four points will be said to form an $\mathcal{A}$*-basis* for a given curve if the curve is an equal-angle locus for the four points in that order, and an $\mathcal{R}$*-basis* is defined in the corresponding manner.

Each of the two loci just described has long been known to be a circular cubic which passes through its singular focus. Such a cubic is called a *focal cubic*, since it is the locus of the foci of the conics cut from a cone of second degree by a pencil of planes whose axis is tangent to the cone and perpendicular to a principal section. Quetelet initiated the study of these focal loci in his inaugural dissertation [Gand, 1819] on the case in which the cone is right circular. The focal curve is then identical with the oblique strophoid. Van Rees† soon afterward discussed the general case, Chasles and others followed with additional contributions, and Teixeira‡ in 1908 included a rather comprehensive treatment of focal cubics in his treatise on special curves.

* A preliminary report on this paper was presented to the Ohio Section of the Mathematical Association of America, April 8, 1939.

† K. Van Rees, Memoire sur les focales, Correspondance mathématique et physique de A. Quetelet, vol. 5, Brussels, 1829, pp. 361–378.

‡ F. Gomes Teixeira, Traité des Courbes Speciales Remarquables, vol. 1, Coïmbre, 1908, pp. 45–58.

The object here will be to study these curves by means of complex number representation and to investigate especially their properties relative to the base points. A complex number will be associated with a point as in the Gauss-Argand diagram. When a point is denoted by a capital letter, its complex number coördinate will be denoted by the corresponding small letter, although when convenient the small letter will be referred to as a point also. Greek letters will designate real numbers, and a bar over a number symbol will denote the conjugate complex number. The viewpoint will be that of real metric Euclidean geometry.

2. Certain points and lines related to an ordered four-point. First, we define the *focal center* of the ordered four-point A, B, C, D to be the point S such that the angles ASB and CSD are equal (not reduced modulo π) and the ratios AS/BS and CS/DS also are equal. Its coördinate s will satisfy the equation

$$(a - s)/(b - s) = (c - s)/(d - s), \tag{1}$$

and hence

$$s = (ad - bc)/(a + d - b - c), \tag{2}$$

provided $a+d \neq b+c$. Obviously, S is on both the equal-angle and the equal-ratio loci.

Next, let M and M' denote the mean proportionals of A and D with respect to S. By this we mean that the numbers $m-s$ and $m'-s$ are the mean proportionals of $a-s$ and $d-s$. From (1) it is readily apparent that B and C have the same mean proportionals with respect to S as do A and D. Hence

$$(m - s)^2 = (a - s)(d - s) = (b - s)(c - s), \qquad m' - s = -(m - s). \tag{3}$$

The points M and M' will be called the *mean points* of the ordered four-point.

The centroid of the four-point will be denoted by G. Its coordinate is

$$g = (a + b + c + d)/4.$$

Finally, the line segment joining the mid-points of AD and BC will be called the *middle line* of the ordered four-point.

The 24 ways of ordering four points give only three different values to the formula (2) for s. Thus an unordered four-point has three focal centers, three pairs of mean points, three middle lines, and one centroid.

3. Equations of the loci. The directed angle AZB is the amplitude of the quotient $(b-z)/(a-z)$, and the distance ratio AZ/BZ is the modulus of $(a-z)/(b-z)$. It follows from the definitions of §1 that the equations of the equal-angle and equal-ratio curves can be written, respectively,

$$\frac{a - z}{b - z} \div \frac{\bar{a} - \bar{z}}{\bar{b} - \bar{z}} = \frac{c - z}{d - z} \div \frac{\bar{c} - \bar{z}}{\bar{d} - \bar{z}}, \tag{4-A}$$

$$\text{(4-R)} \qquad \frac{a-z}{b-z}\cdot\frac{\bar{a}-\bar{z}}{\bar{b}-\bar{z}}=\frac{c-z}{d-z}\cdot\frac{\bar{c}-\bar{z}}{\bar{d}-\bar{z}}.$$

These equations reduce to

$$\text{(5-A)} \qquad \bar{a}_0z^2\bar{z}-a_0z\bar{z}^2+\bar{a}_1z^2-a_1\bar{z}^2-a_2z\bar{z}-\bar{a}_3z+a_3\bar{z}+a_4=0,$$

$$\text{(5-R)} \qquad \bar{a}_0z^2\bar{z}+a_0z\bar{z}^2+\bar{a}_1z^2+a_1\bar{z}^2-b_2z\bar{z}+\bar{b}_3z+b_3\bar{z}+b_4=0,$$

where

$$\text{(6)} \qquad \begin{aligned} a_0 &= a+d-b-c, \qquad a_1 = bc-ad,\\ a_2 &= (\bar{a}+\bar{d})(b+c)-(a+d)(\bar{b}+\bar{c}),\ b_2 = (a+d)(\bar{a}+\bar{d})-(b+c)(\bar{b}+\bar{c}),\\ a_3 &= bc(\bar{a}+\bar{d})-ad(\bar{b}+\bar{c}), \qquad b_3 = ad(\bar{a}+\bar{d})-bc(\bar{b}+\bar{c}),\\ a_4 &= ad\bar{b}\bar{c}-bc\bar{a}\bar{d}, \qquad b_4 = bc\bar{b}\bar{c}-ad\bar{a}\bar{d}. \end{aligned}$$

Equation (5-A) is satisfied by a, b, c, and d; whereas one member of (4-A) is undefined for each of these values of z. For continuity we shall include these points as a part of the equal-angle locus.

The equations (5) are equivalent to their conjugate equations and hence represent curve loci. They are of third degree unless $a+d=b+c$. In the latter case the loci are equilateral hyperbolas, and the base points form a parallelogram in the order $A\ B\ D\ C$. Such an ordered four-point will be designated as a *special* basis* and will be excluded from further consideration in this paper. That is, we assume hereafter that $a+d\neq b+c$.

To simplify the equations (5) take the origin at the focal center S of the base points and the real axis parallel to their middle line. Then, from (3)

$$\text{(7)} \qquad ad = bc = m^2,$$

and the following relations hold among the coefficients (6),

$$\text{(8)} \qquad \begin{gathered} a_0=\bar{a}_0, \qquad a_1=0, \qquad a_2=a_0(g-\bar{g}), \qquad b_2=a_0(g+\bar{g}),\\ a_3=b_3=a_0m^2, \qquad a_4=b_4=0. \end{gathered}$$

By virtue of these relations equations (5) now become

$$\text{(9-A)} \qquad z^2\bar{z}-z\bar{z}^2-2(g-\bar{g})z\bar{z}-\bar{m}^2z+m^2\bar{z}=0,$$

$$\text{(9-R)} \qquad z^2\bar{z}+z\bar{z}^2-2(g+\bar{g})z\bar{z}+\bar{m}^2z+m^2\bar{z}=0.$$

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* Other special cases of interest arise if the four base points are not all distinct. Especially deserving of comment is the equal-angle locus when A and D coincide, that is, the locus of points Z such that angle AZB = angle CZA (mod. π). This locus is a circular cubic passing through A, B, C, and the two Fermat points F_1, F_2 of the triangle. It has a crunode at A, where it cuts itself at right angles. (In particular, if $AB=AC$ the locus is evidently the circumcircle and the perpendicular from A to BC.) Treated inversively in the manner of Morley, such a curve is a biquadratic through the six points A, B, C, F_1, F_2, ∞. In this connection see M. W. Dean, A system of six rectangular biquadratics, American Journal of Mathematics, vol. 52, 1930, pp. 585–600. Further references are there given to this case of a focal cubic with node.

Since $z\bar{z}$ is a factor of the highest degree terms, the loci of equations (9) are circular cubics, and since it is also a factor of the terms of next highest degree, the singular focus is at the origin. Each curve thus passes through its singular focus, and we can state as a conclusion:

THEOREM 1. *The equal-angle and the equal-ratio loci for any non-special set of base points are focal cubics having a common singular focus at the focal center of the base points.*

4. Orthogonality of the curves. Addition of equations (9-A) and (9-R) gives

$$\bar{z}(z^2 - 2gz + m^2) = 0,$$

and subtraction gives the conjugate equation. Hence the finite points of intersection of the two curves are the origin and two points whose coördinates are

$$g + \sqrt{g^2 - m^2}, \qquad g - \sqrt{g^2 - m^2}.$$

It may be observed that the mean proportionals of these two points with respect to the focal center are M and M' and that their centroid is G.

To find the angles at which the two curves meet, first find $dz/d\bar{z}$ for each curve, since this gives the clinant* of the tangent to the curve. We obtain from the equations (9)

$$\text{(10-A)} \qquad \frac{dz}{d\bar{z}} = \frac{z^2 - 2z\bar{z} - 2(g - \bar{g})z + m^2}{\bar{z}^2 - 2z\bar{z} + 2(g - \bar{g})\bar{z} + \bar{m}^2} = \frac{z^2(\bar{m}^2 - \bar{z}^2)}{\bar{z}^2(m^2 - z^2)},$$

$$\text{(10-R)} \qquad \frac{dz}{d\bar{z}} = \frac{z^2 + 2z\bar{z} - 2(g + \bar{g})z + m^2}{\bar{z}^2 + 2z\bar{z} - 2(g + \bar{g})\bar{z} + \bar{m}^2} = -\frac{z^2(\bar{m}^2 - \bar{z}^2)}{\bar{z}^2(m^2 - z^2)}.$$

At any point common to the two curves, one of these clinants is the negative of the other. This is the condition for perpendicularity, and hence the two curves are orthogonal at every point of intersection.

By writing equation (9-A) in the form

$$z\bar{z}[z - \bar{z} - 2(g - \bar{g})] - \bar{m}^2 z + m^2\bar{z} = 0$$

and (9-R) in the analogous form, the asymptotes are found to be

$$\text{(11-A)} \qquad z - \bar{z} = 2(g - \bar{g}),$$

$$\text{(11-R)} \qquad z + \bar{z} = 2(g + \bar{g}).$$

These lines have clinants 1 and -1 respectively, and they intersect at the point $2g$. The foregoing results can be collected to give

THEOREM 2. *The equal-angle and the equal-ratio loci for a non-special basis intersect orthogonally in three finite points and have orthogonal asymptotes. The*

* The clinant of a line is a number of modulus unity and of amplitude equal to twice the inclination of the line to the real axis.

three points of intersection of the curves and that of the asymptotes are vertices of a parallelogram whose center is the centroid of the base points.

5. $\mathcal{A}$ and $\mathcal{R}$ bases for any focal cubic. It is known that a focal cubic has a double infinity of $\mathcal{A}$-bases* and a double infinity of $\mathcal{R}$-bases.† We shall characterize these bases analytically in complex numbers and thereby obtain a new geometric description of them.

Since a focal cubic is a circular cubic which passes through its singular focus, its equation can be written in either of the forms

$$z^2\bar{z} - z\bar{z}^2 - 2\alpha i z\bar{z} - \bar{p}z + p\bar{z} = 0, \tag{12}$$

$$z^2\bar{z} + z\bar{z}^2 - 2\beta z\bar{z} + \bar{q}z + q\bar{z} = 0, \tag{13}$$

where α and β are real numbers. Both of these forms have the origin taken at the singular focus, while the real axis is parallel to the asymptote in (12) and perpendicular to it in (13).

Now, to investigate $\mathcal{A}$-bases, we suppose the given focal cubic to have its equation in the form (12). By identifying (12) with (9-A) we see that for four ordered points a, b, c, d to form an $\mathcal{A}$-basis for (12) it is necessary that

$$ad = bc = p, \tag{14}$$

$$a + d - b - c = \bar{a} + \bar{d} - \bar{b} - \bar{c}, \tag{15}$$

$$a + d + b + c - (\bar{a} + \bar{d} + \bar{b} + \bar{c}) = 4\alpha i. \tag{16}$$

Conversely, those relations in (6), (7), and (8) which concern the equal-angle locus are implied by (14), (15), and (16), provided $g-\bar{g}$ be replaced by αi and m^2 by p, and they in turn imply that (12) is the equal-angle locus for the basis a, b, c, d. By adding and subtracting (15) and (16) an equivalent pair of equations is obtained, and we can state that *the conditions,*

$$ad = bc = p, \tag{17}$$

$$a + d - \bar{a} - \bar{d} = b + c - \bar{b} - \bar{c} = 2\alpha i, \tag{18}$$

are necessary and sufficient that the ordered four-point a, b, c, d form an $\mathcal{A}$-basis for the focal cubic (12).

A non-self-conjugate equation in complex unknowns yields another equation when its conjugate is taken, while of course a self-conjugate equation does not. The two non-self-conjugate equations in (17) and the two self-conjugate ones in (18) are thus seen to give six conditions on the eight unknowns a, b, c, d and their conjugates. This verifies the existence of a double infinity of $\mathcal{A}$-bases for a given focal cubic.

By eliminating d from the two equations in (17) and (18) which involve d, it is seen that a must be on the focal cubic. Likewise b, c, and d must be on the

* Van Rees, op. cit., p. 375.

† Teixeira, op. cit., p. 51.

cubic. In fact, if any two points a and b are chosen on the curve (12), then (17) determines c and d so as to form an $\mathcal{A}$-basis.

Next, to determine $\mathcal{R}$-bases, let the equation of the given focal be (13). By identifying (13) with (9-R) and reasoning as before, we see that the conditions on a, b, c, d are (14) with q replacing p, (15), and

$$(19) \qquad a + d + b + c + \bar{a} + \bar{d} + \bar{b} + \bar{c} = 4\beta.$$

Equations (15) and (19) can be combined into a single non-self-conjugate equation, and we have that *the conditions*

$$(20) \qquad ad = bc = q,$$

$$(21) \qquad a + d + \bar{b} + \bar{c} = 2\beta$$

are necessary and sufficient that a, b, c, d form an $\mathcal{R}$-basis for the focal cubic (13).

The three non-self-conjugate equations (20), (21) yield six conditions in all, indicating that the $\mathcal{R}$-bases of a focal cubic also constitute a doubly infinite system. One point, say a, can be chosen as any point in the plane not on the curve (13). Then equations (20) and (21) determine three other points b, c, d which with a form an $\mathcal{R}$-basis for (13).

6. Mean points and middle lines. By comparing (17) with (7) we see that every $\mathcal{A}$-basis for the focal cubic (12) will have the same mean points, namely, $\pm\sqrt{p}$. To find the mean points for $\mathcal{R}$-bases for the cubic (12), apply the transformation $z = iw$ and obtain

$$w^2\bar{w} + w\bar{w}^2 - 2\alpha w\bar{w} + (-\bar{p})w + (-p)\bar{w} = 0.$$

This equation is of the same form as (13), and hence by (20) all of its $\mathcal{R}$-bases have the mean points $\pm i\sqrt{p}$. Before transformation these points would be $\pm\sqrt{p}$, and we have

THEOREM 3. *All of the $\mathcal{A}$-bases and $\mathcal{R}$-bases of a focal cubic have the same mean points.*

Thus the mean points of the bases are characteristic points of the curve, and we may speak of them as the *mean points of the focal cubic.* If the equation of a focal cubic is in either of the forms (12) or (13), then its mean points are the square roots of the coefficient of $\bar{z}$.

The line parallel to the asymptote and midway between it and the singular focus is called the *middle line* of a focal cubic. For the cubics (12) and (13) the middle lines are, respectively,

$$(22) \qquad z - \bar{z} = \alpha i, \qquad z + \bar{z} = \beta.$$

A focal cubic is uniquely determined when its mean points and middle line are given.

From (18) it is seen that $(a+d)/2$ and $(b+c)/2$ satisfy the first of equations (22) and hence the middle line of the focal cubic contains the middle lines of all

the $\mathcal{A}$-bases of the cubic as segments. Again, by using (21) and its conjugate, the second of the lines (22) can be shown to be perpendicular to the segment joining $(a+d)/2$ and $(b+c)/2$ and through its midpoint. Hence the middle line of the focal cubic is the perpendicular bisector of the middle lines of all of its $\mathcal{R}$-bases. We can now translate the necessary and sufficient conditions of §5 into the geometrical terminology of mean points and middle lines, as in the following theorems.

THEOREM 4. *An ordered four-point is an $\mathcal{A}$-basis for a focal cubic if and only if its mean points are the mean points of the cubic and its middle line is along the middle line of the cubic.*

THEOREM 5. *An ordered four-point is an $\mathcal{R}$-basis for a focal cubic if and only if its mean points are the mean points of the cubic and its middle line is bisected orthogonally by the middle line of the cubic.*

7. Common bases for a pair of orthogonal focal cubics. Consider any two focal cubics which have the same singular focus and orthogonal asymptotes and which are orthogonal at all points of intersection. We inquire now as to what conditions an ordered four-point must satisfy in order to be an $\mathcal{A}$-basis for one of the pair of curves and an $\mathcal{R}$-basis for the other. First, we note that the two focal cubics can have their equations taken in the forms (12) and (13). Next, by adapting the clinant formulas (10) to these equations, it can be shown that p and q in (12) and (13) must be equal if the two curves are to be orthogonal at all intersections. Since the mean points for focals (12) and (13) are the square roots of the coefficients of $\bar{z}$, we have

THEOREM 6. *Any two orthogonal focal cubics with a common singular focus have the same mean points.*

By virtue of this theorem we see that the same four-point can satisfy the hypotheses of both Theorem 4 and Theorem 5 for the focal cubics of an orthogonal pair. Hence

THEOREM 7. *An ordered four-point is an $\mathcal{A}$-basis for one of two orthogonal focal cubics which have a common singular focus and is an $\mathcal{R}$-basis for the other, if and only if the mean points of the four-point coincide with those of the two curves and the middle line of the four-point is along that of the first curve and is bisected by that of the second.*

Analytic conditions for a common basis for a pair of orthogonal focal cubics can be obtained by combining the conditions of §5. Thus, if $p=q$ in (12) and (13), then (17), (18), *and* (21) *are necessary and sufficient conditions that a, b, c, d form an $\mathcal{A}$-basis for* (12) *and an $\mathcal{R}$-basis for* (13).

The equations (17), (18), and (21) comprise two self-conjugate and three non-self-conjugate equations. This would seem to give eight conditions, but it is found that (18) and (21) imply the conjugate of (21), and hence there are only seven independent conditions on eight unknowns. Hence

THEOREM 8. *A pair of orthogonal focal cubics with the same singular focus has a single infinity of common $\mathcal{A}$- and $\mathcal{R}$-bases.*

8. Centers of inversion. The inverse of z with respect to a circle with center c and radius ρ is the point y given by

$$(z - c)(\bar{y} - \bar{c}) = \rho^2.$$

This transformation applied to a focal cubic gives in general a bi-circular quartic. However, if the center is on the curve, the inverse curve is again a focal cubic. In particular, it is well known that a focal cubic inverts into itself if the center is taken at a point where the tangent is parallel to the asymptote and if the radius is the length of a tangent from such a point to the curve.

To find the centers of self inversion of a given focal cubic, we take the equation in the form (9-A) and set its clinant (10) equal to -1. This gives

$$\bar{m}z - m\bar{z} = 0, \qquad \bar{m}z + m\bar{z} = 0.$$

Since m and $-m$ are the mean points of the curve, we have

THEOREM 9. *The centers of inversion of a focal cubic are the points of intersection, other than the singular focus, of the curve with two lines, one being through the mean points and the other the perpendicular bisector of the segment joining them.*

The elimination of $\bar{z}$ between $\bar{m}z - m\bar{z} = 0$ and (9-A) gives

$$z(z^2 - 2m\gamma z + m^2) = 0,$$

where γ is the real number $(g - \bar{g})/(m - \bar{m})$. Thus the roots are $z_0 = 0$ and

$$z_1 = m(\gamma + \sqrt{\gamma^2 - 1}), \qquad z_2 = m(\gamma - \sqrt{\gamma^2 - 1}).$$

If $\gamma^2 > 1$, then $\sqrt{\gamma^2 - 1}$ is a non-zero real number and the three roots are all found to satisfy both $\bar{m}z - m\bar{z} = 0$ and (9-A). If $\gamma^2 = 1$, then $z_1 = z_2$ and the focal cubic has a node at this point. However, the node is not a center of inversion. Finally, if $\gamma^2 < 1$, then z_1 and z_2 do not satisfy $\bar{m}z - m\bar{z} = 0$.

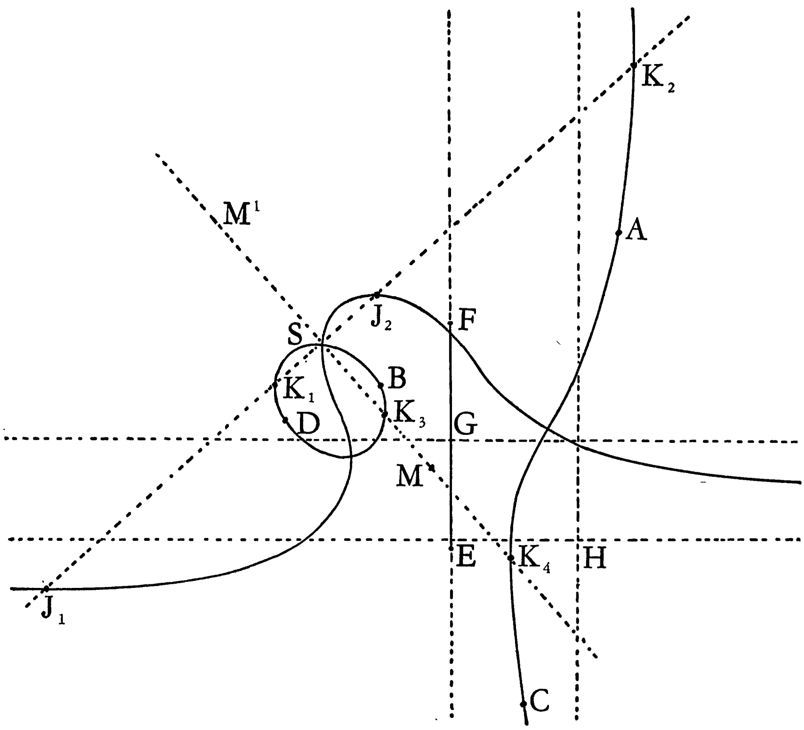
A similar argument applied to the line $\bar{m}z + m\bar{z} = 0$ shows that it always intersects the curve in two actual points besides the singular focus.

Since the points $g - \bar{g}$ and $m - \bar{m}$ are twice as far from the real axis as are the middle line and the mean points, respectively, the foregoing results can be expressed as in the following theorem. The relation between the number of real centers of inversion and the number of circuits of a circular cubic gives at the same time a criterion for the number of circuits.

THEOREM 10. *A focal cubic has two or four real centers of inversion, and thus has one or two circuits, according as its mean points are or are not on opposite sides of its middle line.*

In the figure one set of base points $ABCD$ is shown. The two-circuited curve is an equal-angle locus for this basis and the one-circuited curve is an equal-

ratio locus. The middle lines are the lines through G, and the asymptotes are those through H. The line segment EF is the middle line of the four-point



$ABCD$. The centers of inversion of the one-circuited curve are designated by J's and those of the two-circuited curve by K's.

MATHEMATICS PLACEMENT AT THE UNIVERSITY OF OREGON

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Our problem is that of placing the entering college freshman in a mathematics course that best fits his preparation and his ability. A few years ago this problem was almost non-existent, since at that time students were assigned to one standard beginning course, usually College Algebra. In recent years the amount of high school mathematics given, as well as the caliber of courses taught has become so variable that colleges have been forced to introduce new freshman mathematics courses. With the advent of these new courses came the problem of placement. I would like to present the work done by the staff of the Department of Mathematics of the University of Oregon in its attempt to solve this problem.

Experience has shown that the student's mathematics record does not give enough information to place him properly in one of the possible freshman

courses. What is taught as eleventh grade algebra in one school does not always correspond to the same titled course in another school. Further, what is considered as B work in one school may rate no better than a low C in another. To help remedy this lack of placement information, a placement test was constructed and given to all students enrolling in freshman mathematics. The student's score on this test, his high school record, and his score on an intelligence test were used in an attempt to place him in the proper freshman course. The statistical approach of multiple linear regression was used. In using this method it is necessary that the variables used be numerical. To make the high school mathematics record numerical we studied the average college mathematics grade made by students having the same high school mathematics record. This divided our data into ten groups. The first group contained the students with no high school algebra, the second contained students who received an F in 9th grade algebra with no 11th grade algebra, the third contained those with a D in 9th grade algebra, and so on to the tenth group which was made up of students with A's in both 9th and 11th grade algebra. The increase in the average freshman mathematics grade from one group to the next was approximately constant throughout the ten groups. This fact enabled us to use the following table in translating the high school mathematics record to a numerical score.

Mathematics score	No math	Grade in 9th grade algebra with no 11th	Average grade in 9th and 11th grade algebra
1	x		
2		F	
3		D	
4		C	
5		B	
6		A	
7			D
8			C
9			B
10			A

Let x_1 = placement test score,
x_2 = high school mathematics score,
x_3 = psychological decile,*
x_4 = scholastic decile,†
x_5 = number of years since high school graduation.

* The students were given a psychological decile from their scores on the Ohio State University Psychological Test, form 20.

† The scholastic decile was obtained by ranking the students according to the number of A's and B's in their complete high school record. From this ranking a decile score was obtained for each student. If the student's rank in his graduating class is available, it could be used in place of the scholastic decile.

The problem was to predict the grade of work a student would do in the various first term mathematics courses from the above five variables. This was done by predicting an achievement score from the record of the student, where the achievement score is related to the possible grades in the mathematics courses by the following table:

Achievement score, y	Elementary algebra	Intermediate algebra	College algebra	Elementary analysis*
1	F			
3	D			
5	C	F		
7	B	D		
9	A	C		
11		B	F	
13		A	D	F
15			C	D
17			B	C
19			A	B
21				A

That is, a student with an achievement score of 9 would be expected to make an A in Elementary algebra, a C in Intermediate algebra, and a failure in any more advanced course.

From studying the records of 174 students who had finished their first term mathematics course and who had been given the placement test at the beginning of the term, the predicting or regression equation was found to be

$$y = .41173 + .12629x_1 + .66590x_2 + .11512x_3 + .11041x_4 + .23031x_5.$$

The standard error for the predicted y's was equal to 2.3548. If one considered only x_1 and x_2 in his prediction he would have the regression equation,

$$y = 1.51279 + .1330046x_1 + .7089780x_2,$$

with the standard error equal to 2.4228. This indicates that in placement work the factors x_3, x_4, and x_5 are almost superficial when considered with x_1 and x_2, and hence in the rest of our work these factors were disregarded.

From the standard error given above one can see that approximately two-thirds of the students actually received their predicted grades, while approximately 95% received grades which were within one level of the predicted grade.

It was decided that a student should not be placed in a course unless his predicted achievement score indicated that the student should be able to do C work or better in that course. In case this rule allowed two placements for a student, he should be assigned to the more advanced course. Using this rule the following chart was made and used in placing students.

* The text for this course is Milne and Davis, "Introductory College Mathematics."

PLACEMENT CHART

Assign to	High school mathematics record								
	Grade in 9th grade algebra					Average grade in 9th and 11th			
	F	D	C	B	A	D	C	B	A
Elementary algebra	0 to 40	0 to 33	0 to 28	0 to 23	0 to 17	0 to 12	0 to 7	0 to 1	
Intermediate algebra	41 to 84	34 to 78	29 to 73	24 to 68	18 to 62	13 to 57	8 to 52	2 to 46	0 to 41
College algebra	85 a o	79 a o	74 a o	69 to 83	63 to 76	58 to 72	53 to 67	47 to 61	42 to 56
Elementary analysis				84 a o	77 a o	73 a o	68 a o	62 a o	57 a o

In this chart the intervals in the body of the chart refer to placement test scores. Thus a student with a *C* in 9th grade algebra and no 11th grade algebra who receives between 29 and 73 points on the test would be assigned to Intermediate algebra.

One of the main criteria used in the evaluation of any placement scheme is whether the placements keep course dropping at a minimum. In the actual use of this chart we have reduced the percentage of drops or changes of courses after registration from 29 per cent to 9 per cent of the enrollment. In considering whether or not 9 per cent is near the minimum one must bear in mind the fact that at the University of Oregon the regulations governing the dropping of courses are so liberal that most students drop courses instead of failing them. We hope that in future years our placements with the aid of the placement chart will be as accurate as they appear to be this year. However, we are continuing our work in this field in the effort to improve our technique.

AN APPLICATION OF VECTOR ANALYSIS TO THERMODYNAMICS

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1. Introduction. If E denote the internal energy of a one-component body of unit mass, S the entropy, V the volume, T the absolute temperature, and P the imposed pressure, then $E=E\ (S,\ V)$, and equilibrium subsists if

$$(1) \qquad T = E_S, \qquad P = -E_V.$$

The subscripts mean partial derivatives taken with respect to the quantity so indicated. A necessary and sufficient condition that the equilibrium be not unstable (*i.e.*, that it be stable or neutral) is that

$$(2) \qquad \delta E - T\delta S + P\delta V \geqq 0,$$

where $\delta E=E'-E$, $\delta S=S'-S$, and $\delta V=V'-V$, and T and P are to be computed for the state $(S,\ V)$ by (1). The unprimed letters refer to the state whose stability is in question; the primed letters denote any other possible state.

It is our purpose to show how the geometrical properties of four fundamental thermodynamic surfaces may be deduced from a vectorial analogue of (2), and to indicate other applications of vector analysis to these surfaces and to plane diagrams derived from them.

2. Development of the criterion. If the rectangular coördinates x, y, z be identified respectively with S, V, and E, (2) becomes

$$(3) \qquad \delta z - z_x\delta x - z_y\delta y \geqq 0.$$

Letting $\bar{\mathbf{r}}$ denote the position vector of a point $R\ (x,\ y,\ z)$,

$$(4) \qquad \delta\bar{\mathbf{r}} = [\delta x,\ \delta y,\ \delta z].$$

The normal to the surface $z=z(x,\ y)$ at R is defined as

$$(5) \qquad \bar{\mathbf{N}} = \bar{\mathbf{r}}_x \times \bar{\mathbf{r}}_y = [-z_x,\ -z_y,\ 1].$$

From (4) and (5),

$$(6) \qquad \bar{\mathbf{N}}\cdot\delta\bar{\mathbf{r}} = \delta z - z_x\delta x - z_y\delta y.$$

The scalar product $\bar{\mathbf{N}}\cdot\delta\bar{\mathbf{r}}$ affords a convenient geometrical criterion. The vectors $\bar{\mathbf{N}}$ and $\delta\bar{\mathbf{r}}$ have a common origin at R, and $\bar{\mathbf{N}}$ is perpendicular to the tangent plane at R, making an acute angle with the positive direction of the z-axis. If the criterion exceed zero, therefore, any point R' on the surface, and, hence, the whole surface, falls above the tangent plane at R; if the criterion equal zero, R' lies in the tangent plane; if it be negative, the surface falls below the tangent plane.

For the surface $E=E(S,\ V)$, the sign of the criterion is given by (2). To ascertain its value for other surfaces, however, the variables must be expressed in terms of S and V. Thus

$$x = x(S, V), \qquad y = y(S, V), \qquad z = z(x, y),$$

whence

$$z_x = J\left(\frac{z, y}{S, V}\right) \div J\left(\frac{x, y}{S, V}\right) \tag{7}$$

and

$$z_y = J\left(\frac{z, x}{S, V}\right) \div J\left(\frac{y, x}{S, V}\right). \tag{8}$$

To study the surface near a point, the variations may be taken sufficiently small so that orders of infinitesimals exceeding the second are negligible. Then

$$\overline{\mathbf{N}} \cdot \delta\bar{\mathbf{r}} = \tfrac{1}{2}\{z_{xx}(\delta x)^2 + 2z_{xy}\delta x\delta y + z_{yy}(\delta y)^2\} = \tfrac{1}{2}q. \tag{9}$$

3. Applications. The four fundamental thermodynamic surfaces correspond to the equations

$$E = E(S, V), \quad A = A(T, V), \qquad H = H(P, S), \quad F = F(P, T),$$

where the functions A (the work content), H (the enthalpy), and F (the free energy) are defined as follows:

$$A = E - TS, \tag{10}$$

$$H = E + PV, \tag{11}$$

$$F = E + PV - TS. \tag{12}$$

Hence, by (7) and (8), and the equilibrium relations (1),

$$A_T = -S, \quad H_P = V, \quad F_P = V, \qquad A_V = -P, \quad H_S = T, \quad F_T = -S.$$

The criteria for the four surfaces, then, are expressible, by (6), as follows:

$$E\text{-surface, } \overline{\mathbf{N}} \cdot \delta\bar{\mathbf{r}} = \delta E - T\delta S + P\delta V, \tag{13}$$

$$\begin{aligned} A\text{-surface, } \overline{\mathbf{N}} \cdot \delta\bar{\mathbf{r}} &= \delta E - T\delta S + P\delta V - \delta T\delta S \\ &= -\{(-\delta E) - T'(-\delta S) + P'(-\delta V)\} - \delta P\delta V, \end{aligned} \tag{14}$$

$$\begin{aligned} H\text{-surface, } \overline{\mathbf{N}} \cdot \delta\bar{\mathbf{r}} &= \delta E - T\delta S + P\delta V + \delta P\delta V \\ &= -\{(-\delta E) - T'(-\delta S) + P'(-\delta V)\} + \delta T\delta S, \end{aligned} \tag{15}$$

$$F\text{-surface, } \overline{\mathbf{N}} \cdot \delta\bar{\mathbf{r}} = -\{(-\delta E) - T'(-\delta S) + P'(-\delta V)\}. \tag{16}$$

In light of (2), the signs of the criteria for the various surfaces may be read immediately from (13)–(16). The signs of the quadratic form q, its discriminant $z_{xx}z_{yy} - (z_{xy})^2$, and the partial differential coefficients z_{xx} and z_{yy} then follow. The signs of the principal curvatures and the forms of Dupin's Indicatrix, as well as the positions of the surfaces relative to their tangent planes, thus are manifest.

In (2), the inequality sign holds for bivariant states, the equality for invariant states, and for univariant, or two-phase, states, the relation applies as written. For points in the portion of the E-surface corresponding to univariant states, therefore, $q \geqq 0$, and, hence, the discriminant, D, vanishes. This field, consequently, is developable. Let $\bar{\mathbf{u}}$ denote the difference between the position vectors of the two points which lie on a generator and which correspond respectively to the two phases capable of coexistence. Then, if $\overline{\mathbf{N}}$ be the normal at any point on the generator,

$$\bar{\mathbf{u}} \cdot \overline{\mathbf{N}} = 0.$$

Let the tangent plane which contains the generator rotate about it through an infinitesimal angle and let the consequent new value of $\overline{\mathbf{N}}$ be $\overline{\mathbf{N}}'$. Then

$$\bar{\mathbf{u}} \cdot \overline{\mathbf{N}}' = 0.$$

Hence,

$$\bar{\mathbf{u}} \cdot (\overline{\mathbf{N}}' - \overline{\mathbf{N}}) = \bar{\mathbf{u}} \cdot \mathbf{d}\overline{\mathbf{N}} = 0.$$

This is the vectorial equivalent of the Clapeyron equation. In terms of components,

$$\bar{\mathbf{u}} \cdot \mathbf{d}\overline{\mathbf{N}} = [\Delta S, \Delta V, \Delta E] \cdot [-\,dT, dP, 0] = -\,\Delta S dT + \Delta V dP = 0.$$

To convert the equation to its usual form, we replace ΔS by its equivalent L/T, where L denotes the specific latent heat of the change of state at the temperature T. Then

$$dP/dT = L/(T\Delta V).$$

4. Some aspects of plane sections. Points on the E-surface in regions of stability are characterized by a positive definite quadratic form, and, consequently, $D > 0$. Fields of instability, on the other hand, contain points for which the criterion is negative for certain continuous variations. Hence, q is indefinite, and $D < 0$. From continuity considerations, therefore, boundary points between these fields must satisfy the relation

$$E_{SS}E_{VV} - (E_{SV})^2 = 0. \tag{17}$$

The critical state corresponds to such a boundary point.

The properties of the isentropic, isometric, isothermal, and isobaric curves at the critical point are of interest. Consider the SV diagram obtained by projecting the E-surface on its base plane. The gradients of entropy, volume, temperature, and pressure are as follows:

$$\nabla S = \bar{\mathbf{i}},$$

$$\nabla V = \bar{\mathbf{j}},$$

$$\nabla T = \left(\frac{\partial T}{\partial S}\right)_V \bar{\mathbf{i}} + \left(\frac{\partial T}{\partial V}\right)_S \bar{\mathbf{j}} = E_{SS}\bar{\mathbf{i}} + E_{SV}\bar{\mathbf{j}},$$

$$\nabla P = \left(\frac{\partial P}{\partial S}\right)_V \bar{\mathbf{i}} + \left(\frac{\partial P}{\partial V}\right)_S \bar{\mathbf{j}} = - E_{VS}\bar{\mathbf{i}} - E_{VV}\bar{\mathbf{j}}.$$

From (17), it appears that the gradients of T and P are linearly dependent at the critical point, and, therefore, the isothermal and the isobaric have a common tangent there in the SV diagram. But the operator ∇ is formally invariant with respect to a transformation of coordinates. The property of tangency, consequently, must hold in any diagram whatever which represents states of the body uniquely. In particular, it follows that in any intensity-extensity diagram, constant intensity curves have horizontal tangents at the critical point.

A consideration of the gradients, in connection with the fact that the critical state is stable and that, therefore, the temperature must increase with the entropy at constant pressure, reveals further that in circumscribing the critical point in a clockwise direction, the iso-curves are encountered in the order S, T, P, V.

If W denote the work done by the body in undergoing a cyclical process, it is apparent that

$$W = \int_0 PdV = \int_0 P\nabla V \cdot \mathbf{d}\bar{\mathbf{r}}$$

where the integral is to be taken about the circuit in question. By Green's Theorem, therefore,

$$W = \iint \bar{\mathbf{k}} \cdot \text{rot } (P\nabla V)da. \tag{18}$$

In the xy coördinate system, (18) may be converted into its Cartesian equivalent, becoming

$$W = \iint J\left(\frac{P, V}{x, y}\right)dxdy. \tag{19}$$

The work associated with a closed path thus is given in terms of the area enclosed. Dimensional homogeneity requires that the jacobian of (19) have the dimensions of energy x area^{-1}. J may be regarded, therefore, as the reciprocal of the scale of work in the xy-diagram. It is evident that the scale, in general, will vary with the position of a point in the diagram. If x and y be identified respectively with V and P, however, $J=-1$, and, hence, the VP diagram exhibits work to constant scale. Similarly, the ST diagram is of constant scale. The VS diagram, on the other hand, is of variable scale.

In the mapping of points in a variable scale diagram upon the VP plane, a sufficient assurance of biunique correspondence is that $J\neq 0$. The sense of rotation, however, will be preserved or reversed according as J is positive or negative; *i.e.*, the region in which $J<0$ will be inverted in mapping on the VP plane. In view of the continuity between the two regions, such an inversion is attain-

able only by "folding" along the line upon which J vanishes. In the VP diagram for certain substances, consequently, there will occur regions of points which are double-valued with respect to the corresponding thermodynamic states. Moreover, this property may be expected in any diagram of constant scale, as is evident from the consideration that circuits described in the same direction in a diagram of variable scale may be associated with quantities of work of opposite sign; whereas, this is not possible in a constant scale diagram.*

The deformations necessary to transform the VS plot into various other plane diagrams may be illustrated through the action of variable, planar dyadics. Let

$$\bar{\mathbf{r}} = V\bar{\mathbf{i}} + S\bar{\mathbf{j}} \quad \text{and} \quad \Phi = \bar{\mathbf{i}}\bar{\mathbf{i}} + \frac{P}{S}\bar{\mathbf{j}}\bar{\mathbf{j}}.$$

Then

$$\bar{\mathbf{r}}' = \Phi\cdot\bar{\mathbf{r}} = V\bar{\mathbf{i}} + P\bar{\mathbf{j}}.$$

The transforms are evidently position vectors of points in the VP diagram, the effect of Φ being to slide all points vertically so that the isobars become horizontal lines spaced linearly with respect to P. Again, let

$$\Psi = \frac{T}{V}\bar{\mathbf{i}}\bar{\mathbf{i}} + \bar{\mathbf{j}}\bar{\mathbf{j}}.$$

Then

$$\Psi\cdot\bar{\mathbf{r}}' = T\bar{\mathbf{i}} + P\bar{\mathbf{j}}.$$

The VP diagram thus is transformed into the TP plot. A single dyadic Ω which duplicates in action the two dyadics given above is defined by the equation

$$\Omega = \Phi\cdot\Psi.$$

5. Interrelationships. Mutual and reciprocal properties of the four surfaces will appear from an inspection of the following table in which the quantities E, H, A, and F are displayed, for the sake of uniformity, as scalar triple products. The expressions may be verified easily by expansion and reference to the defining equations (10)–(12).

Quantity Surface	SVE	PTF	TVA	PSH
E	z	$\bar{\mathbf{r}}_x\times\bar{\mathbf{r}}_y\cdot\bar{\mathbf{r}}$	$\bar{\mathbf{r}}\times\bar{\mathbf{r}}_x\cdot\bar{\mathbf{j}}$	$\bar{\mathbf{r}}\times\bar{\mathbf{r}}_x\cdot\bar{\mathbf{j}}$
F	$\bar{\mathbf{r}}_x\times\bar{\mathbf{r}}_y\cdot\bar{\mathbf{r}}$	z	$\bar{\mathbf{i}}\cdot\bar{\mathbf{r}}_y\times\bar{\mathbf{r}}$	$\bar{\mathbf{i}}\cdot\bar{\mathbf{r}}_y\times\bar{\mathbf{r}}$
A	$\bar{\mathbf{r}}\times\bar{\mathbf{r}}_x\cdot\bar{\mathbf{j}}$	$\bar{\mathbf{r}}\times\bar{\mathbf{r}}_x\cdot\bar{\mathbf{j}}$	z	$\bar{\mathbf{r}}_x\times\bar{\mathbf{r}}_y\cdot\bar{\mathbf{r}}$
H	$\bar{\mathbf{i}}\cdot\bar{\mathbf{r}}_y\times\bar{\mathbf{r}}$	$\bar{\mathbf{i}}\cdot\bar{\mathbf{r}}_y\times\bar{\mathbf{r}}$	$\bar{\mathbf{r}}_x\times\bar{\mathbf{r}}_y\cdot\bar{\mathbf{r}}$	z

* A case in point is furnished by water. In the VS diagram, $J = -(\partial P/\partial S)_V$. The line of fold here is the curve which represents liquid states of maximum density; for along this curve, J vanishes, and on opposite sides of it, J assumes opposite signs.

A NOTATION FOR INFINITE MANIFOLDS

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I wish to introduce a notation for the content of a manifold of geometric objects depending on arbitrary functions. Our symbolism has been found useful in problems of geometry and physics, especially those depending on systems of partial differential equations. The notation was first mentioned in my review of Riquier's treatise on differential systems. (See *Bulletin of the American Mathematical Society*, 1912.)

The content of a set of geometric objects, each uniquely determined by n arbitrary constants, is denoted by the familiar notation ∞^n. Thus the number of points in ordinary space is ∞^3, and the totality of events in Einstein space is ∞^4. There are ∞^4 straight lines in space. The conics of the plane are ∞^5 in number, whereas the number in space is ∞^8. There are ∞^9 quadric surfaces in space. The number of orbits of conceivable planets in the solar system obeying Kepler's laws is ∞^5. A rigid body moving freely in space has 6 degrees of freedom. The number of possible positions is ∞^6.

Although the preceding sets are quite large in content, we soon meet manifolds which are very much more extensive. For example, consider the totality of curves in space; this depends on two arbitrary functions of a single variable, that is, y and z are arbitrary functions of x. This set would be of the type ∞^∞ since the number of arbitrary constants is endless. But this vague symbol would represent also the number of surfaces in space, which depends on one arbitrary function of two variables, that is, z is an arbitrary function of x and y. Therefore to distinguish between these two obviously distinct manifolds, we suggest that the number of curves in space be denoted by the symbol $\infty^{2f(1)}$ and the number of surfaces by the symbol $\infty^{1f(2)}$.

As another example, the number of cylinders in space is $\infty^{2+1f(1)}$, because a cylinder is determined by selecting an arbitrary curve in a fixed plane as base and an arbitrary direction in space for the generators. The 2 denotes two arbitrary constants and may be denoted by $2f(0)$. Hence the totality of cylinders in space is $\infty^{2f(0)+1f(1)}$. Similarly, we find that the total set of cones in space is $\infty^{3f(0)+1f(1)}$. Of course usually we may omit $f(0)$ without ambiguity.

We make the following definition: *Let a manifold M depend on r_0 constants, r_1 functions of 1 variable, r_2 functions of 2 variables, $\cdots$, r_k functions of k variables. Then the content of M is expressed by the symbol*

$$(S) \qquad \infty^{r_0f(0)+r_1f(1)+r_2f(2)+\cdots+r_kf(k)}.$$

The complete exponent represents the number of degrees of freedom.

In a given plane field of force, there are ∞^3 dynamical trajectories. However, the number of possible plane fields of force, since this depends on two functions of two variables, will be $\infty^{2f(2)}$. A given field of force in space possesses ∞^5 dynamical trajectories. The complete set of fields of force in space is $\infty^{3f(3)}$. (See my *Princeton Colloquium Lectures* 1912, 1934.)

The conformal and equilong groups of the plane each consist of $\infty^{2f(1)}$ transformations. The group of all lineal element transformations preserving the set of all isothermal families is much larger, and is expressed by the symbol $\infty^{1f(0)+4f(1)}$. On the other hand, the group preserving the dual-isothermal type is expressed by the symbol $\infty^{9f(1)}$. If we ask how many isothermal families of curves are possible, we find the symbol $\infty^{2f(1)}$. The same is true for the dual-isothermal type.

It is known that ∞^m and ∞^n, where $m \neq n$, are topologically distinct (Theorem of Brouwer). Now I wish to propose two fundamental questions. Firstly, are $\infty^{mf(k)}$ and $\infty^{nf(k)}$, where $k>0$ and $m \neq n$, topologically distinct? Secondly, how can we topologically compare $\infty^{f(k)}$ and $\infty^{f(l)}$ where $k \neq l$? For example, does there exist a one-to-one continuous correspondence between all the curves and all the surfaces of space? In general, what is the effect of changing the coefficients or the arguments or both in the exponential symbol for infinite manifolds? Evidently in any rigorous discussion we must refine the type of "arbitrary function" admitted (as regards continuity, differentiability). In most applications it is perhaps wise to confine ourselves to *analytic* functions. This is the case in the publications referred to in the final paragraphs. (Ambiguities may arise in counting the functions.)

Before closing, I wish to give the following additional examples of ∞^n. The content of the opulence of lineal elements of the plane is ∞^3. In the opulence are contained the totalities of ∞^4 geometric turbines, ∞^6 limacon series, ∞^3 flat fields, ∞^4 spherical fields, and ∞^9 quadric fields.

The turbine group, defined by the preservation of the manifold of ∞^4 geometric turbines, is isomorphic to the projective group of space and consists of ∞^{15} lineal element transformations. An important subgroup of this is the Lie group consisting of the ∞^{10} contact transformations of the plane preserving the set of ∞^3 circles. Also we may mention the Moebius and Laguerre groups, each possessing ∞^6 correspondences, which are themselves subgroups of the turbine and Lie groups. As a final illustration, we may give the whirl-motion group, studied extensively by myself and De Cicco, which consists of the ∞^6 lineal element transformations obtained by compounding turns, slides, and motions. This is a subgroup of the turbine group but not of the Lie, Moebius, and Laguerre groups. (See *Bull. Amer. Math. Soc.*, 1938.)

The symbol (*S*) suggested above has been used for many years in my seminar, in papers by Kasner and De Cicco, Comenetz, Fialkow, and recently in an extensive memoir on the inverse problem of the calculus of variations by Douglas. (See *Trans. Amer. Math. Soc.*, 1941.)

DISCUSSIONS AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

MANTISSA AND CHARACTERISTIC

P. C. HAMMER, Oregon State College

The state of confusion existing in textbooks concerning the definitions of the terms *mantissa* and *characteristic* of the common logarithm has been pointed out in a recent note by C. B. Read.* This confusion does not serve a real purpose in simplifying the teaching of logarithms. That a certain lack of conciseness may contribute to the ease of teaching, according to some at least, appears evident from the looseness with which such terms as *variable, constant, function,* and *limit* are customarily introduced to the student. Overemphasis on the niceties of rigor tends to kill enthusiasm even for more advanced students. This point has been well brought out by Dr. Scheffé† in a note concerned with teaching advanced calculus students.

Logarithms present a didactic problem of some difficulty for several reasons. In the first place they are based on exponents, a subject which has difficulties of its own on the elementary level. Second, the connection between a logarithm of a number and the number is not immediately accessible to simple algebraic treatment. For example, by what means could the student check $10^{.30103} \doteq 2$? Third, the large number of new words of imposing appearance having no relevant connotation to the student affords a semantic obstacle. These words include *base, logarithm, antilogarithm, cologarithm, interpolation, characteristic,* and *mantissa.* In view of the hit and run treatment given logarithms in many calculus courses it is important to present as clear a picture of them as possible at the elementary stage. It is here proposed that rigorous definitions of *characteristic* and *mantissa* do not contribute to confusion but aid in clarity.

The following presentation of *characteristic* and *mantissa* agrees with usage: If x is a positive number, then x can be written in only one way in the form

$$x = d \cdot 10^p \tag{1}$$

where $1 \leq d < 10$ and p is an integer, negative, zero, or positive. Then,

$$\log_{10} x = p + \log_{10} d. \tag{2}$$

We call p the *characteristic* of $\log_{10} x$ and $\log_{10} d$ the *mantissa* of $\log_{10} x$.

* Is a mantissa necessarily positive?, this MONTHLY, vol. 48, 1941, pp. 203–204.

† At what level of rigor should advanced calculus be taught?, this MONTHLY, vol. 47, 1940, pp. 635–640.

From our definition it is relatively a simple matter to prove that a *mantissa* is non-negative and less than one. In looking up $\log_{10} x$ it is found that only $\log_{10} d$ is tabulated and p is determined by the decimal point in the decimal representation of x. This definition may differ from usage only in that we have included the possibility $d=1$. All the questions discussed in the note by C. B. Read mentioned above are immediately resolved on the basis of this simple presentation. The logarithm is represented *uniquely* as a sum of two numbers. Thus if $\log_{10} x = -2.34251$ we have only one way of writing it in the form (2), i.e.,

$$\log_{10} x = -3 + .65749.$$

To lay the illusion of some students that writing -2.34251 as $-3+.65749$ or as $7.65749-10$ involves some high degree of understanding, we find it effective to emphasize that the value of a number is not changed if we add zero to it. We may add zero in the form $3-3$ or $10-10$ if we choose. Ample illustrations would remove any doubt as to the meaning of the definitions.

Another advantage of so defining the words *characteristic* and *mantissa* appears in the fact that if numbers were written in an expansion which is other than the decimal system, the form of (1) immediately suggests generalization to the new base. However, if we are to use numbers in decimal representation the generalization might take another form. We illustrate in case of natural logarithms. Let equation (1) hold as before with the same restrictions on d and p. Then

$$\log_e x = p\cdot\log_e 10 + \log_e d \tag{3}$$

and we call $p\cdot\log_e 10$ the *characteristic* and $\log_e d$ the *mantissa* of $\log_e x$. Then, $\log_e d$ would be tabulated and a separate table of $p\cdot\log_e 10$ given. If x is in decimal form the use of these tables is a simple matter. Now it is no longer true that a mantissa is less than one nor that the characteristic is an integer but the concept of a *convenient unique decomposition* of the logarithm into a sum of two parts persists.

This latter extension of the meaning of *characteristic* and *mantissa* appears to be in line at least with the etymological forebear of the word *mantissa*. This word comes from a Latin word meaning *makeweight*. It would appear that the characteristic was thought of as a rough approximation of the logarithm of a number *characteristic* of the location of the decimal point of the number and that the *mantissa* was considered as a necessary correction or makeweight.

NOTE ON THE INVERSION OF A CENTROSYMMETRIC MATRIX

EDWARD SAIBEL, Carnegie Institute of Technology

In many applications of finite differences or the operational calculus it is necessary to find the inverse of a square non-singular matrix. It is usually the inversion of this matrix which consumes the major portion of the time spent in

solving the problem. This matrix frequently turns out to be centrosymmetric; that is, if $A = [a_{rs}]$ is the matrix, and n the order, $a_{rs} = a_{n+1-r,n+1-s}$. In this case the inverse may be found in considerably less than half the usual time by the following procedure:

Let us designate by H_{ij} the unit matrix with the element 1 inserted in the (ij) position, $i \neq j$. Then $H_{ij}AH_{ij}^{-1}$ has the effect on A of adding the jth row to the ith row and of subtracting the ith column of A from the jth.*

Whether A is of even order, $2m \times 2m$ or of odd order $(2m+1)\times(2m+1)$, m of these operations, *viz.*, adding the first row of A to the last row and subtracting the last column from the first, adding the second row of A to the second from the last row and subtracting the second to the last column of A from the second column, etc. will transform A to the form

$$HAH^{-1} = \begin{bmatrix} P_1 & Q_1 \\ O_1 & R_1 \end{bmatrix} \tag{1}$$

where

$$H = H_{m+1,m} \cdots H_{2m-1,2}H_{2m,1} \text{ for the even case}$$

or

$$H = H_{m+2,m} \cdots H_{2m,2}H_{2m+1,1} \text{ for the odd case.}$$

P, Q, R, and O are submatrices. In the even case the orders of these submatrices are $m\times m$ and in the odd case $m\times m$, $m\times(m+1)$, $(m+1)\times(m+1)$ and $(m+1)\times m$ respectively. O consists entirely of zeros.

The above operations performed on A^{-1}, which is also centrosymmetric, put it in a form similar to (1), viz.,

$$HA^{-1}H^{-1} = \begin{bmatrix} P_1 & Q_1 \\ O_1 & R_1 \end{bmatrix}. \tag{2}$$

Since HAH^{-1} and $HA^{-1}H^{-1}$ are reciprocal, $PP_1 = I_m$ and $RR_1 = I_{m \text{ or } m+1}$. If the reciprocals of P and R respectively are found, the elements of A^{-1} are readily obtained. As an example, consider the case of a 5×5 centrosymmetric matrix.

If the unknown A^{-1} is

$$\begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & l & k \\ j & i & h & g & f \\ e & d & c & b & a \end{bmatrix}$$

* Turnbull and Aitken, Theory of Canonical Matrices, Blackie and Son.

then

$$HA^{-1}H^{-1} = \begin{bmatrix} a-e & b-d & c & d & e \\ f-j & g-i & h & i & j \\ 0 & 0 & m & l & k \\ 0 & 0 & 2h & g+i & f+j \\ 0 & 0 & 2c & b+d & a+e \end{bmatrix} = \begin{bmatrix} P_1 & Q_1 \\ O_1 & R_1 \end{bmatrix}.$$

Having found P_1 and R_1 as inverses of P and R, we can compute the elements of Q_1 very simply, since

$$c = (2c)/2, \quad d = [(b+d) - (b-d)]/2, \quad e = [(a+e) - (a-e)]/2, \quad \text{etc.}$$

In a similar manner the elements of A^{-1} can be easily obtained. In the even case all the elements of A^{-1} are obtained from the sums and differences of pairs. In this manner the inversion is replaced by two inversions but considerable saving in time is effected.

It is evident from the above that the characteristic equation of A may be replaced by two characteristic equations of orders m and m or m and $(m+1)$ respectively. This procedure can also effect considerably saving in time in vibration and stability problems where the characteristic equation must be solved for one or more of the roots.

GEOMETRICAL ASPECTS OF THE POWER FUNCTION

LUISE LANGE, Woodrow Wilson Junior College

In the following are shown some geometrical interpretations of the formulas for the differentiation and integration of the power function $y=ax^n$ which do not seem to be pointed out in the textbooks.

Construction of the tangent. The expression for the slope of the tangent, $dy/dx=anx^{n-1}$ can be written in the form ny/x. It thus appears that the tangent at any point (x_1, y_1) of the curve can be constructed by laying off from the origin on the y-axis the directed distance $-(n-1)y_1$ and connecting this point with the point (x_1, y_1), thus obtaining a line of required slope ny_1/x_1. For the parabola $(n=2)$ this construction, which makes the subtangent equal to $2y_1$, is well known. That it holds equally for all values of n seems less well known. ($n=0$ and $n=1$ are special, trivial cases for a horizontal and oblique straight line respectively.) An instructive figure illustrating this relation is obtained by plotting in one diagram several members of the family of curves $y=ax^n$, say for $n=3, 2, 1/2, -1$, and constructing their tangents at the common point of intersection $(1, a)$ which cut the y-axis respectively at $-2a$, $-a$, $+a/2$, $+2a$.

Interpretation of areas. The expression for the area between the x-axis and the curve $y=ax^n$ between $x=0$ and $x=x_1$, $A=\int_0^{x_1} ydx=ax_1^{n+1}/(n+1)$ can be written in the form $A=x_1y_1/n+1$. It then appears that this area is equal to $1/(n+1)$ of

the rectangle $A_r = x_1y_1$ whose one vertex is at the point (x_1, y_1), and whose two sides are on the coördinate axes.

This relation holds immediately only for $n \geqq 0$ ($n=0$ and $n=1$ are again special, trivial cases giving the full rectangle and the triangle respectively), since for negative values of n the curve goes towards infinity as x approaches zero. However for $-1<n<0$ the above relation still holds in the sense that $A_r/(n+1)$ represents the limit which the area from x to x_1 approaches as x approaches zero. (For instance for $n=-1/2$ this limiting area equals twice the rectangle A_r.) For $n<-1$ the area from 0 to x_1 becomes infinite. But now the area from x_1 to ∞ becomes finite. The limiting value of this area is at once seen to be $-1/(n+1)$ times the rectangle A_r. (For example, the area under the curve $y=ax^{-2}$ from x_1 to x, as x increases indefinitely, approaches the full rectangle A_r; that for $y=ax^{-3}$ half the rectangle, etc.)

Interpretation of volumes of revolution. If the curve $y=ax^n$ from $x=0$ to $x=x_1$ revolves around the x-axis the volume so generated is $V=\pi\int_0^{x_1}y^2dx$ $=\pi a^2x_1^{2n+1}/(2n+1)$. If this is written in the form $V=\pi y_1^2x_1/2n+1$ it appears that this volume equals $1/(2n+1)$ times the volume V_c of a cylinder of base πy_1^2 and altitude x_1, that is, of a cylinder with the same base and altitude as the solid of revolution. (The formula for the volume of a cone thus appears merely as the special case of this formula for $n=1$.)

As regards negative values of n the above relation remains valid for $-1/2<n<0$ in the sense that the expression $V_c/(2n+1)$ represents the limit which the volume between x and x_1 approaches as x approaches zero. For $n<-1/2$ the expression $-V_c/(2n+1)$ is the limit of the volume from x_1 to ∞. (For instance, for the hyperbola $y=ax^{-1}$ this limiting volume is equal to that of the whole cylinder V_c.)

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

A First Year of College Mathematics. By Henry J. Miles. New York, John Wiley and Sons, 1941. 17+607 pages. $3.00.

This text, written by a member of the faculty of the University of Illinois, is intended for the use of any group of college students who need the subject matter of college algebra, plane trigonometry, and plane and solid analytic geometry. It aims to give a solid foundation for the mathematics of the sophomore year.

The chapter titles are (1) Rectangular coördinates, (2) Graphs, (3) Functions, (4) Systems of linear equations, (5) Quadratic equations, (6) Angles, polar

coördinates and trigonometric functions, (7) The straight line, (8) Loci, (9) The conics, (10) Transformation of rectangular coördinates, (11) Oblique triangles, (12) Exponents—logarithms, (13) Complex numbers, (14) The derivative and its applications, (15) Progressions, (16) Permutations, combinations and probability, (17) Theory of equations, (18) Empirical equations, (19) Solid analytic geometry, (20) Foundations of algebra. Comments will be made on the treatment of some of them.

The book is carefully prepared and attractively presented. Explanations are well interspersed with helpful drawings. After a substantial background is given, problems are placed at the end of each chapter to challenge the prospective scientist or engineer.

There is a novel blending of algebra, trigonometry and analytic geometry throughout the work. The trigonometric functions are defined by means of polar coördinates and the usefulness of their periodicity noted, e.g., in describing the strength of an alternating electric current or in describing the position of a point on a rotating wheel. The laws of sines and of cosines are obtained by polar coördinates and after the section on rotation of axes, the formulas for the sine and the cosine of the sum of two angles are effected by a single rotation through the angle $(\theta+\phi)$.

In the section on loci which precedes the study of conics, suggestions are made for examining an equation for intercepts, symmetry, extent and asymptotes. Several problems are solved completely, representing a fair level of difficulty.

The chapter on conics cites interesting applications to astronomy and engineering. It includes the equation of the general conic and construction for the conics.

The derivative is approached geometrically. Consideration is given to increasing and decreasing functions, maximum and minimum values of a function and later in the text to finding the real roots of an equation. Differentiation is carried through the exponential function with problems paralleling the work sufficient to be of real use in other courses during the latter part of the freshman year. By means of the calculus, further, the coefficients in the binomial theorem are determined.

Chapter 18 deals with empirical equations or curve fitting. Four types are discussed: (1) The linear, (2) the parabolic, (3) the power, (4) the exponential. The linear type is presented both by the method of averages and of least squares; the parabolic by the method of averages. In (3) one may plot the points on logarithmic coördinate paper and in (4) on semi-logarithmic paper.

SARA C. WALSH

Mathematics—Its Magic and Mastery. By Aaron Bakst. New York, D. Van Nostrand Company, Inc., 1941. 14+790 pages. $3.95.

This book covers the field of elementary high school mathematics including arithmetic, algebra, geometry, trigonometry, and elementary mechanics. The

style of writing and the nature of the contents of the thirty-seven chapters is indicated by such chapter titles as: Numerals and Numeration, Number Pygmies, Number Giants, Chain Letter Algebra, Passport for Geometric Figures, The Size of Things, The Firing Squad and Mathematics, etc.

The purpose of the book is not to develop a systematized subject matter but to show the recreational and applied phases of mathematics. The magic of the subject is emphasized by numerous well chosen applications and tricks. The text is interspersed with problems for the reader to solve, answers to which are given in the appendix. The appendix also includes a compilation of elementary algebraic processes, geometric facts, trigonometric formulas, approximation formulas, and tables of squares, square roots, logarithms, and trigonometric functions. Many figures and drawings accompany the text.

The book can serve as supplementary reading in the training of elementary and junior high school teachers. It provides interesting study for those laymen who have forgotten or omitted their high school mathematics. There is excellent material for high school mathematics club programs. Written in adult style, the book can not serve as a high school text for developing the foundation and mastery of elementary mathematics that is necessary for further mathematical study.

H. F. FEHR

Fourier Series and Orthogonal Polynomials. By Dunham Jackson. Carus Mathematical Monograph Number Six. Oberlin, Ohio, Mathematical Association of America. 1941. 12+234 pages. \$2.00. Non-members order from Open Court Publishing Co., LaSalle, Ill. \$1.25 to members.

This monograph is an exposition of the theory and applications of the classical orthogonal polynomials. In keeping with the aims of the series of Carus monographs, the discussion presupposes little background beyond the material of the first two or three terms of calculus: the variables are all real, the integrals are all Riemann, and the ϵ proofs are few and simple.

The reader is introduced to the concept of orthogonality by way of the trigonometric system, which is studied for itself alone in the first and longest chapter of the book. After the usual introductory definitions and examples, the Fourier coefficients are discussed in some detail, and their degree of convergence for a broken line function is obtained. Riemann's Theorem is proved by means of Bessel's inequality, and the main convergence theorem is referred to Riemann's Theorem by using the familiar hypothesis involving the existence of right- and left-hand derivatives for the function represented. Next, uniform convergence and degree of convergence are established for a broken line function. Weierstrass's Theorem on trigonometric approximation is then established by using the fact that any continuous function can be uniformly approximated by a broken line function, and this theorem is used in turn to establish the least square property and Parseval's Theorem. Fejér's Theorem is also proved by means of the broken line approximation. Weierstrass's Theorem is then proved

again by means of de la Vallée Poussin's integral and the broken line approximation. The chapter concludes with a treatment of the Lebesgue constants and Lebesgue's proof of uniform convergence for functions belonging to a Lipschitz class.

Except in the discussion of the Lebesgue constants, there is no use made in this chapter of the customary analysis of singular integrals by subdividing the range of integration into two or three parts. But if it be conceded that results rather than methods are the essential thing in a first treatment, then it is hard to see how the material here could be presented more coherently and elegantly within the limits of such an exposition.

The Legendre polynomials and Bessel functions are the respective subjects of the next two chapters. Here, of course, much of the available space must be taken up with formal work. The Legendre polynomials are defined through their generating function; recurrence relations, the differential equation, orthogonality, Rodrigue's formulas, and Laplace's first integral follow in that order. Analysis rears its head at the end of the chapter, where the usual bounds for $P_n(x)$ and convergence under the hypothesis used in Chapter I are established. The chapter on Bessel functions, which employs the differential equation for $J_0(x)$ as its starting point, is entirely concerned with the formal theory; of course the convergence of an expansion in Bessel functions is well beyond the scope of this work.

Two formal chapters on the classical partial differential equations of mathematical physics are inserted at this point. We pass over this material temporarily to discuss the remainder of the book, which is concerned with the development from a more general standpoint of the lines of inquiry inaugurated in the first two chapters.

Since the weight functions of the Jacobi, Hermite, and Laguerre polynomials are functions belonging to the Pearson classification of frequency functions in statistics, it is natural to study these frequency functions on their own merits in a preliminary chapter. There follows a chapter on general orthogonal polynomials, which introduces the reader to such basic concepts as Schmidt's process, general formulas of recurrence, and the least square property. A differential equation is derived for polynomials belonging to a given weight function and interval under the assumption that the weight function satisfies the Pearson differential equation. The next three chapters, each rather short, contain formal discussions of the Jacobi, Hermite, and Laguerre polynomials respectively. The Schrödinger wave equation is touched on briefly at the end of each of the last two chapters of this group. The final chapter generalizes the convergence theorems of the first two chapters. The conditions of course depend on uniform boundedness of the orthogonal polynomials, so this property is studied in some detail by means of a sequence of results culminating in an exposition of the recent theorem of Korous.

A set of exercises concludes the book. They are intended, as the author says, to illustrate and extend the text, rather than to serve for purposes of drill.

Although the author disclaims in the preface any intentions toward "rigor," the standards of precision and completeness are, on the whole, remarkably high. The pair of chapters on partial differential equations and boundary value problems constitutes somewhat of an exception to this statement. Here the author is quite obviously not attempting to write mathematically; there is scarcely any mention of the fact that continuity, differentiability, and uniqueness of the alleged solutions all need to be proved. The reviewer felt just a little let down, not because such proofs had been omitted, but because the spirit of mathematical awareness (if we may call it that) which distinguishes the rest of the book, had been so patently laid aside in these chapters. For instance, it does not seem quite right to present "derivations" of Poisson's integral in two-space and three-space which do not even mention (much less prove) that the function represented assumes its boundary values continuously for some methods of approach. But the author is certainly within an established tradition in these chapters, and perhaps greater precision, if only of statement, would disagree with the type of reader who would be most interested in this material.

The reviewer missed two topics which he rather expected to find in a modern treatment of orthogonal functions. The first, and less important, is a development of the concept of orthogonality which starts with orthogonal vectors in Euclidean space and proceeds to a consideration of functions as generalized vectors. (There is, however, a suggestion of such a development in one of the exercises.) The second is the topic of convergence in the mean, which nowhere appears except by implication in the proof of Parseval's Theorem for Fourier series. Of course, not much can be done in this connection without Lebesgue theory, but the growing recognition of the importance of this type of convergence in the applications seems to indicate that it warrants at least passing mention.

But it seems captious to criticize an author for restricting his field under the circumstances, especially when his avowed aims have been realized as effectively as in the work under review. It is easy to become very enthusiastic about the many excellent features of the book as it stands. The exposition is well organized; much of the formal work is beautifully handled; the proofs are extremely clear. It will not be surprising if this turns out to be one of the most popular of the Carus Monographs. The Association is indeed fortunate to be able to add to the Carus series an exposition by one of the world's foremost authorities on orthogonal polynomials.

J. H. CURTISS

Between Physics and Philosophy. By Philipp Frank. Cambridge, Massachusetts, Harvard University Press, 1941. 238 pages. $2.75.

It may be asserted that the last eighty years have seen the emergence of a new conception of truth. Only that is true for mankind which is verifiable in human experience. This conception, flowing from advances in the natural sciences and occupying a prominent position in the foundations of modern physics, has been developed in various forms by different schools of thought whose equiv-

alence was often not appreciated. It was advanced by the philosophcial physicist E. Mach, appeared independently in the pragmatism of William James, and constitutes the essence of the philosophical forms known as logical positivism or logical empiricism. The same basic idea appears also, in garbled and doctrinaire form, in "dialectical materialism."

These statements appear to constitute a fair summary of the views that are developed in the volume before us. Illustrative examples are drawn largely from the author's own field of physics and give a fair picture of the professional thought of physicists. The author insists, however, that, when the physicist steps outside of the strict boundaries of his science, he usually begins talking in the "illusory" language of the standard "school philosophy," which deals largely with meaningless problems. The book is worth a perusal by all who are interested in fundamental problems.

E. H. KENNARD

A Treatise on Algebra. By George Peacock. I. Arithmetical Algebra, 16+399 pages. II. Symbolical Algebra, 10+455 pages. Reprint, New York, Scripta Mathematica, 1940.

The re-publication today of a textbook on algebra, with a total of 880 pages, originally issued in London about a century ago (vol. I, 1842; vol. II, 1845) is an amazing achievement which inspires somewhat critical examination of the various circumstances which have made this possible. The suggestion of the publication was made by the authorities of St. John's College in Annapolis, Maryland, where the book is used as a text. Not only in mathematics but in other fields, also, this college has maintained the desirability of the serious use as textbooks by the young students of the college of historical classics of science. The publication of Peacock's *Treatise* was effected by the collaboration of St. John's College with Scripta Mathematica.

The recent changes in higher algebra might seem to indicate that as preparation for algebra so old a text would be useless. However, much more than half of this work, including much arithmetical computation, corresponds precisely to material presented in the elementary textbooks and in the algebra texts used widely in colleges for the freshman work in algebra, and considerable material given in a second course in college algebra for juniors.

In the first place, Peacock presents not only a large amount of work on the theory of equations but he includes what amounts to a systematic course in plane trigonometry. Peacock believes in numerical problems and computation to an extent not attempted in American texts. A student who masters these two volumes will have adequate preparation in trigonometry and algebra, including series, for the great body of elementary applications to physics. There are many teachers of mathematics who believe that this material represents much that cannot be by-passed in any sound preparation for the use of mathematics in science. The success of this text in its day and its feasibility as a text even today is

dependent upon the author's brilliant and harmonious historical development of the foundations of algebra.

George Peacock, a student and tutor of mathematics in Trinity College, Cambridge, was one of a fine group of English mathematicians who turned, in the early nineteenth century, for inspiration to the great developments of mathematics made by other Europeans. The British Association for the Advancement of Science, founded in 1831, enjoyed the support of the English reformers. Peacock at the third meeting presented a remarkable historical report on "Algebra, Trigonometry and the Arithmetic of Sines." In 1830 Peacock had published a small text on algebra incorporating many ideas new to the English, and in 1834, anonymously, a syllabus on trigonometry with algebraic material. Peacock must be associated with D. F. Gregory, Wm. Rowan Hamilton and his quaternions, and George Boole with his Laws of Thought as a precursor of many modern algebraists who work in symbolic logic, and in the foundations of mathematics.

The phrase, "principle of the permanence of equivalent forms," seems to be original with Peacock. Often on minor points Peacock gives material which is pertinent today, e.g., on p. 65, II, "it would in fact conduce very greatly to the uniformity and clearness of algebraical notation, if the use of radical signs was altogether abandoned." About Abel's researches Peacock frankly states, "the clearest understanding gets bewildered by the extreme generality and complexity of the relations which it is necessary to consider . . . and our final assent to the conclusion obtained is . . . a formal act of acquiescence in reasonings whose entire force and relevancy we can neither fully appreciate nor easily refute . . . ."

Such a work as that of Peacock helps us to understand the developments of Hamilton and Boole and that possibly some of the modern abstractions have been brewing more than a hundred years. Pedagogically too, the insistence on numerical illustration and computation seems to have a message today for many who omit this work to advance into abstract fields before the pupil is able to master the groundwork.

L. C. KARPINSKI

Higher Mathematics for Engineers and Physicists. By I. S. and E. S. Sokolnikoff. Second Edition. New York, McGraw-Hill Book Co., 1941. 11+587 pages. $4.50.

The first edition of this textbook was reviewed in this MONTHLY, vol. 41, 1934, pp. 625–627. The new edition represents a complete revision of the earlier edition. Chapter order has been changed. Several chapters have been extensively revised. A majority of the sections throughout the book have been rewritten in whole or in part. Several new sections have been added. Many new problems and illustrative examples have been added.

The principal changes in content are: (1) The chapter on Improper Integrals has been omitted from the second edition. (2) The material previously included in the chapter on Elliptic Integrals has been transferred to the chapters on Infinite Series and Ordinary Differential Equations. (3) A new chapter on Com-

plex Variable replaces the chapter on Conformal Representation. Much of the material of this chapter is new.

The authors state in the preface that the favorable reception of the first edition has sustained their belief in the need of a book on mathematics beyond the calculus, written from the point of view of the student of applied science. The reviewer believes that the second edition, representing as it does a polishing and an amplifying of the material presented in the first edition, will continue to receive the same favorable reception.

H. M. GEHMAN

Intermediate Algebra. By H. L. Rietz, A. R. Crathorne, and L. J. Adams. New York, Henry Holt and Company, 1942. 248 pages. $1.75.

The book presupposes one year's study of algebra. It begins with a rapid review of the elementary concepts; the succeeding chapters proceed to the more complex phases by gradual and continuous stages. The attempt is made to present each new step with a minimum of explanation. Great care is taken in the appropriate choice of illustrative exercises. Besides the necessary topics of linear and quadratic equations, radicals, proportion, variations and progression, the book includes mathematical induction, with application to the binomial theorem, extension of the number concept, and logarithms.

The press-work and the arrangement of the page are up to the usual high standards of the publishers.

VIRGIL SNYDER

NEW BOOKS RECEIVED

Intermediate Algebra. By N. McArthur and A. Keith. London, Methuen and Co., Ltd., 1942. 10+356 pages. 8s 6d.

Statistics for Sociologists. By Miss M. J. Hagood. New York, Reynal and Hitchcock, Inc., 1941. 8+934 pages. $4.00.

The Reading of Verbal Material in Ninth Grade Algebra. By Margaret G. McKim. (Teachers College, Columbia University. Contributions to Education, No. 850.) New York, Bureau of Publications, Teachers College, Columbia University. 8+133 pages. $1.60.

Principles of Mechanics. By J. L. Synge and B. A. Griffith. First Edition. New York and London, McGraw-Hill Book Company, Inc., 1942. 12+514 pages. $4.50.

Intermediate Algebra. By H. L. Rietz, A. R. Crathorne, and L. J. Adams. New York, Henry Holt and Company, 1942. 8+248 pages. $1.75.

Studies in Mathematical Economics and Econometrics. In Memory of Henry Schultz. Chicago, Illinois, University of Chicago Press, 1942. 292 pages. $2.50.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 516. *Proposed by W. E. Buker, Pittsburgh Public Schools*

During one week, Mr. X invited just four of his seven friends for dinner each night. The invitations were arranged so that any given pair of guests dined together on just two occasions, and for any two given nights there were two guests who were present both times. Show how this was managed. In how many different ways can it be done?

E 517. *Proposed by V. Thébault, San Sebastián, Spain*

If two tetrahedra have equal areas for corresponding faces, do they necessarily have the same volume?

E 518. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

Find three Gaussian integers x, y, z which satisfy the equation

$$x^p + y^p = z^p$$

for every prime p greater than 3.

E 519. *Proposed by Paul Brock, Brooklyn College*

If a projectile is aimed at a given point from a given origin, find the direction to which the two possible paths are equally inclined initially.

E 520. *Proposed by D. H. Browne, Buffalo, N. Y.*

Given

$$u_n = n!\sum_1^n \frac{1}{r}, \quad \text{prove that} \quad \lim_{r\to\infty} \frac{u_{r+1}}{\Delta^r u_1} = e.$$

SOLUTIONS

Trilinear polarity

E 481 [1941, 480]. *Proposed by J. A. Todd, University of Cambridge*

Let

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{pmatrix}$$

be two matrices of non-vanishing numbers, the elements of the second being the co-factors of the corresponding elements of the first. Prove that the relation

$$\begin{vmatrix} x_1^{-1} & y_1^{-1} & z_1^{-1} \\ x_2^{-1} & y_2^{-1} & z_2^{-1} \\ x_3^{-1} & y_3^{-1} & z_3^{-1} \end{vmatrix} = 0 \quad \text{implies} \quad \begin{vmatrix} X_1^{-1} & Y_1^{-1} & Z_1^{-1} \\ X_2^{-1} & Y_2^{-1} & Z_2^{-1} \\ X_3^{-1} & Y_3^{-1} & Z_3^{-1} \end{vmatrix} = 0.$$

Solution by Howard Eves, Allen Academy, Bryan, Texas

Referred to a set of rectangular coordinate axes OX, OY, OZ, let us designate the points (x_ν, y_ν, z_ν) and (X_ν, Y_ν, Z_ν) by P_ν and Q_ν, respectively, where $\nu = 1, 2, 3$. From the hypothesis of the non-vanishing elements, it follows that these points do not lie in the coordinate planes, and no two P's or Q's are collinear with the origin. The given relation implies that the points $(x_\nu^{-1}, y_\nu^{-1}, z_\nu^{-1})$ lie in a plane through the origin, say

$$fx + gy + hz = 0.$$

Therefore the points P_ν lie on the quadric cone

$$fyz + gzx + hxy = 0,$$

which contains the coordinate axes. In other words, this cone circumscribes both the trihedra $O(XYZ)$ and $O(P_1P_2P_3)$. Therefore (by one of the classic theorems due to Brianchon) there exists a cone K, inscribed in both trihedra. Since OX, OY, OZ are perpendicular to the faces of the former trihedron, while OQ_1, OQ_2, OQ_3 are perpendicular to those of the latter, it follows that K', the reciprocal cone of K, circumscribes the trihedra $O(XYZ)$ and $O(Q_1Q_2Q_3)$. Hence K' must have an equation of the form

$$Fyz + Gzx + Hxy = 0.$$

Therefore the points $(X_\nu^{-1}, Y_\nu^{-1}, Z_\nu^{-1})$ lie in the plane

$$Fx + Gy + Hz = 0$$

through the origin, and their determinant vanishes as required.

Also solved by E. P. Starke.

Editorial Note. The theorem may be proved by pure algebra as follows. The relation

$$\begin{vmatrix} x_1^{-1} & y_1^{-1} & z_1^{-1} \\ x_2^{-1} & y_2^{-1} & z_2^{-1} \\ x_3^{-1} & y_3^{-1} & z_3^{-1} \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} x_2x_3 & y_2y_3 & z_2z_3 \\ x_3x_1 & y_3y_1 & z_3z_1 \\ x_1x_2 & y_1y_2 & z_1z_2 \end{vmatrix} = 0$$

implies the existence of non-vanishing numbers a, b, c, such that

$$ax_\mu x_\nu + by_\mu y_\nu + cz_\mu z_\nu = 0 \qquad (\mu \neq \nu).$$

We deduce, in turn,

$$\frac{ax_\mu}{X_\mu} = \frac{by_\mu}{Y_\mu} = \frac{cz_\mu}{Z_\mu} \qquad (\mu = 1, 2, 3), \tag{1}$$

$$\frac{X_\mu X_\nu}{a} + \frac{Y_\mu Y_\nu}{b} + \frac{Z_\mu Z_\nu}{c} = 0 \qquad (\mu \neq \nu),$$

$$\begin{vmatrix} X_2X_3 & Y_2Y_3 & Z_2Z_3 \\ X_3X_1 & Y_3Y_1 & Z_3Z_1 \\ X_1X_2 & Y_1Y_2 & Z_1Z_2 \end{vmatrix} = 0, \qquad \begin{vmatrix} X_1^{-1} & Y_1^{-1} & Z_1^{-1} \\ X_2^{-1} & Y_2^{-1} & Z_2^{-1} \\ X_3^{-1} & Y_3^{-1} & Z_3^{-1} \end{vmatrix} = 0.$$

In terms of projective coordinates, (1) expresses that the point (x_μ, y_μ, z_μ) and line $[X_\mu, Y_\mu, Z_\mu]$ are pole and polar with respect to the conic

$$ax^2 + by^2 + cz^2 = 0 \quad \text{or} \quad \frac{X^2}{a} + \frac{Y^2}{b} + \frac{Z^2}{c} = 0.$$

This remark suggests the following modification of Eves' proof, using two conics instead of three cones.

By the vanishing of the given determinant, the lines $[x_\nu^{-1}, y_\nu^{-1}, z_\nu^{-1}]$ are concurrent, say at the point (f, g, h) or

$$fX + gY + hZ = 0.$$

Hence the conic $fx^{-1}+gy^{-1}+hz^{-1}=0$ or $fyz+gzx+hxy=0$ circumscribes the triangle (x_ν, y_ν, z_ν) as well as the triangle of reference. But two triangles inscribed in one conic are circumscribed to another. Hence the sides $[X_\nu, Y_\nu, Z_\nu]$ touch a conic of the form

$$FYZ + GZX + HXY = 0 \quad \text{or} \quad FX^{-1} + GY^{-1} + HZ^{-1} = 0.$$

Therefore the three points $(X_\nu^{-1}, Y_\nu^{-1}, Z_\nu^{-1})$ lie on the line

$$Fx + Gy + Hz = 0,$$

and their determinant vanishes as required.

Since $[x^{-1}, y^{-1}, z^{-1}]$ is the trilinear polar of (x, y, z), and (X^{-1}, Y^{-1}, Z^{-1}) is the trilinear pole of $[X, Y, Z]$, this result is equivalent to the following interesting theorem:

If the trilinear polars of the vertices of a triangle PQR with respect to a triangle ABC are concurrent, then the trilinear poles of the sides of PQR (with respect to ABC) are collinear.

This may be proved synthetically, as follows.

Let p, q, r, a, b, c denote the sides of the two triangles, and let P', Q', R', p', q', r' be the trilinear poles and polars of p, q, r, P, Q, R with respect to ABC. Let P_a, Q_a. R_a, P_a', Q_a', R_a', A_p, A_q, A_r, $A_{p'}$, $A_{q'}$, $A_{r'}$ denote the points where a meets the respective lines AP, AQ, AR, AP', AQ', AR', p, q, r, p', q', r'. Similarly, let $P_c, \cdots, C_p, \cdots$ be the points where c meets $CP, \cdots, p, \cdots$.

By the definition of trilinear polarity, the point-pairs $P_aA_{p'}$, $Q_aA_{q'}$, $R_aA_{r'}$, $P_a'A_p$, $Q_a'A_q$, $R_a'A_r$ are harmonic conjugates with respect to B and C. Thus the projectivity $P_aQ_aR_a \barwedge A_{p'}A_{q'}A_{r'}$ leaves B and C invariant. Hence, if p', q', r' are concurrent, we have

$$BP_aQ_aR_a \barwedge BA_{p'}A_{q'}A_{r'} \barwedge BC_{p'}C_{q'}C_{r'} \barwedge BP_cQ_cR_c.$$

These related ranges are joined to A and C by related pencils,

$$A(BPQR) \barwedge C(BPQR).$$

Hence A, B, C, P, Q, R lie on a conic; and a, b, c, p, q, r touch a conic. The dual of Steiner's theorem now supplies the middle link in the chain of projectivities

$$A(CP'Q'R') \barwedge CP_a'Q_a'R_a' \barwedge CA_pA_qA_r \barwedge AC_pC_qC_r \barwedge AP_c'Q_c'R_c' \barwedge C(AP'Q'R'),$$

which shows that P', Q', R' are collinear.

Squares in the denary and septenary scales

E 482 [1941, 480]. *Proposed by V. Thébault, San Sebastián, Spain*

Find a four-digit square, other than 70^2, whose last two digits are unaltered when we change from the denary to the septenary scale.

Solution by D. H. Browne, Buffalo, N. Y.

Let the four-digit number be $abcd$. Then c and d are both less than 7. Moreover, since the number is a square, d can only take the values 0, 1, 4, of which the first is ruled out by the exclusion of 70^2. Since $abcd \equiv d \pmod 7$, the three-digit number abc must be divisible by 7. The simplest way to determine this is from a table of squares, where it is found that only six meet these requirements:

3364, 4624, 4761, 5041, 6724, 9801.

Changing to the septenary scale, we find the correct penultimate digit in the second and last cases alone:

$$4624 (= 68^2) = 16324 \text{ (septenary)},$$
$$9801 (= 99^2) = 40401 \text{ (septenary)}.$$

Also solved by Paul Brock, W. E. Buker, Peter Chiarulli, and E. P. Starke.

Coaxal spheres

E 483 [1941, 480]. *Proposed by N. A. Court, University of Oklahoma*

Show that the four spheres having two points in common and each passing through a vertex and the foot of the corresponding altitude of a given orthocentric tetrahedron form a coaxal pencil.

Solution by Howard Eves, Allen Academy, Bryan, Texas

Let the given tetrahedron be $ABCD$, with altitudes AA', BB', CC', DD' and orthocenter H. Let P and Q be the given points. Since

$$AH \cdot HA' = BH \cdot HB' = CH \cdot HC' = DH \cdot HD',$$

H has the same power with respect to the four spheres. Also P and Q have the same power (namely zero) with respect to the four spheres. Moreover, P, Q, H cannot be collinear, lest the spheres degenerate into planes. Hence P, Q, H lie in a common radical plane of the four spheres, and the theorem is proved.

The corresponding theorem for the plane can be proved similarly.

Also solved by L. M. Kelly and the proposer.

Series with alternate binomial coefficients

E 484 [1941, 480]. *Proposed by David Segal, Kosow Huculski, Poland*

Defining

$$\phi_n(x) = 1 - \binom{2n}{2}x^2 + \binom{2n}{4}x^4 - \cdots \pm x^{2n},$$

$$\psi_n(x) = \binom{2n}{1}x - \binom{2n}{3}x^3 + \cdots \mp \binom{2n}{2n-1}x^{2n-1},$$

prove that, if n is even,

$$\phi_n(3) = \pm 2^n\phi_n(2), \qquad \psi_n(3) = \pm 2^n\psi_n(2),$$

and that if n is odd,

$$\phi_n(3) = \pm 2^n\psi_n(2), \qquad \psi_n(3) = \pm 2^n\phi_n(2).$$

Solution by W. R. McEwen and N. R. Amundsen, University of Minnesota

From the given relations we have at once

$$\phi_n(x) + i\psi_n(x) = (1 + ix)^{2n}.$$

Thus

$$\phi_n(2) + i\psi_n(2) = (1 + 2i)^{2n},$$

and because $\psi_n(x)$ is an odd function,

$$\phi_n(3) - i\psi_n(3) = (1 - 3i)^{2n}.$$

Dividing, we obtain

$$\frac{\phi_n(3) - i\psi_n(3)}{\phi_n(2) + i\psi_n(2)} = \left(\frac{1-3i}{1+2i}\cdot\frac{1-2i}{1-2i}\right)^{2n} = (-1 - i)^{2n} = 2^n i^n,$$

or

$$\phi_n(3) - i\psi_n(3) = 2^n i^n\{\phi_n(2) + i\psi_n(2)\}.$$

Hence, if n is even,

$$\phi_n(3) = \pm 2^n\phi_n(2) \quad \text{and} \quad \psi_n(3) = \mp 2^n\psi_n(2),$$

the sign depending on the parity of $n/2$; and if n is odd,

$$\phi_n(3) = \pm 2^n\psi_n(2) \quad \text{and} \quad \psi_n(3) = \pm 2^n\phi_n(2),$$

the sign depending on the parity of $(n+1)/2$.

Also solved by Paul Brock, J. A. Bullard, Peter Chiarulli, and E. P. Starke.

The approximate trisection of an angle

E 485 [1941, 481]. *Proposed by J. Goodfellow, West Rumney, N. H.*

Let AOB be an obtuse angle, A and B on a circle with center O. Take F and G on the minor arc AB, in directions perpendicular to OB and OA. Take D on the major arc AB so that AOD is an equilateral triangle. Take H on AD, and J on BD, so that HJ is equal and parallel to FG. Join FH, and produce to meet the circle again at K. Show that the arc AK is approximately one-third of the arc AB.

Solution by W. B. Clarke, San Jose, California

Let $\angle AOB = 180° - 2\theta$ and $\angle DOK = 2\phi$, so that $\angle AOK = 60° - 2\phi$. We have to show that, for $0° < \theta < 45°$, ϕ is approximately $\theta/3$. After a straightforward but arduous computation, it is found that

$$\cot \phi = 1 + (\sqrt{3} + 1) \cot \theta.$$

Thus $\phi = \theta/3$ when $\theta = 0°$, and again when $\theta = 45°$. Intermediate values show a very close approximation; e.g. $\theta = 36°$ makes ϕ about $11°52'$.

Editorial Note. The trisection is again exact when $\theta = 22\frac{1}{2}°$ (since then $\cot \theta = \sqrt{2}+1$ and $\cot \phi = 2+\sqrt{2}+\sqrt{3}+\sqrt{6}$). The maximum error is about $8'$, and occurs when

$$\theta = \operatorname{arc} \cot \{2 + \sqrt{3} \pm \sqrt{(2 + 2\sqrt{3})}\} = 9°21' \quad \text{or} \quad 35°39'.$$

(Then $\phi = 3°15'$ or $11°45'$, respectively.)

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4031. *Proposed by S. H. Gould, Victoria University, Toronto*

Let m be any fixed positive integer, $k=1, 2, 3, \cdots$, and $r=0, 1, 2, \cdots, m(k-1)$. Taking unity for first term when $k=1$, construct inductively the arith-

metic progression of order $m(k-1)$, the first term of whose rth difference series is the $(r+1)$st term of the A.P. of order $m(k-2)$. Prove that its $(mk+2)$nd term is k^{mk}.

4032. *Proposed by J. Findlay Paydon and H. S. Wall, Northwestern University*

In the complex z-plane $z=x+iy$, let $a_1=x_1+iy_1$ be a point on the parabola $y^2=x+1/4$, O the origin, E the point unity, and w the point on the opposite side of the real axis from a_1 where the line L through E parallel to Oa_1 cuts the circle C_1: $(x-1)^2+y^2=1$. Let C_n be the circle of radius $1/n$ tangent to C_1 on the interior at w, and K the circle with center $-i/2y_1$ and radius $|1/2y_1|$. Prove (1) that

$$w = \frac{1}{1} + \frac{a_1}{1} + \frac{\bar{a}_1}{1} + \frac{a_1}{1} + \frac{\bar{a}_1}{1} + \cdots$$

$(\bar{a}_1=x_1-iy_1)$; (2) the $(2n+1)$st approximant of this continued fraction lies at the intersection, $\neq w$, of L and C_{2n+2}; (3) the $2n$th approximant lies at the intersection, $\neq w$, of K and C_{2n+1}.

4033. *Proposed by P. D. Thomas, Norman, Oklahoma*

Points on a surface with the linear element $ds^2=[(u+a)^2+(v+b)^2](du^2+dv^2)$ correspond to points in the xy-plane by $x=u$, $y=v$. Show that geodesics on the surface correspond to equilateral hyperbolas in the xy-plane.

4034. *Proposed by V. Thébault, San Sebastián, Spain*

On the sides AB, BC, CD, DA of a convex quadrangle $ABCD$ equilateral triangles with vertices A', B', C', D' are constructed exteriorly (or interiorly). Show that the diagonals $A'C'$ and $B'D'$ of quadrangle $A'B'C'D'$ are perpendicular (or equal) according as the diagonals AC and BD of $ABCD$ are equal (or perpendicular), and conversely.

4035. *Proposed by V. Thébault, San Sebastián, Spain*

On the sides A_1A_2, A_2A_3, $\cdots$, A_6A_1 of a convex hexagon having equal principal diagonals squares with centers A_1', A_2', $\cdots$, A_6' are constructed exteriorly (or interiorly). Show that in the hexagon formed by these centers the sum of the squares of two opposite sides and of the principal diagonal which does not end in vertices of the two sides considered is a constant. Generalize for a convex polygon of $2n$ sides whose principal diagonals are equal.

SOLUTIONS

Matrices

3927 [1939, 601]. *Proposed by M. M. Flood, Princeton University*

If

$$\sum_{k=1}^{n} S_k = I_t, \qquad \sum_{k=1}^{n} p_k = t,$$

where S_k is a real square symmetric matrix of order t and rank p_k; then a real orthogonal matrix T exists such that $TS_kT^{-1}=E_k^t$, where E_k^t is a matrix with p_k consecutive units on its principal diagonal and zeros elsewhere and such that

$$\sum_{k=1}^{n} E_k^t = I_t.$$

Solution by A. T. Craig, The State University of Iowa

This solution is contained in Craig's article, *On the independence of certain estimates of variance*, in the *Annals of Mathematical Statistics*, vol. 9, no. 1, March 1938, pp. 48–55. There is first given a proof of

THEOREM I. *Let $A_1, A_2, \cdots, A_s$ be s real symmetric matrices, each of order N, such that $A_1+A_2+\cdots+A_s=I$, where I is the unit matrix of order N. Let r_v, $v=1, 2, \cdots, s$, be respectively the ranks of the matrices A_v. If $r_1+r_2+\cdots+r_s=N$, each of the non-zero roots of the characteristic equations of the matrices A_v is $+1$.*

Then follows a proof of

THEOREM II. *Let $A_1, A_2, \cdots, A_s$ be s real symmetric matrices which satisfy the conditions of Theorem* I. *Then there exist $s-1$ real orthogonal matrices of order N, say $L_1, L_2, \cdots, L_{s-1}$, such that each of the s matrices*

$$L'_{s-1}\cdots L'_1A_vL_1\cdots L_{s-1}, \qquad v = 1, 2, \cdots, s,$$

is a diagonal matrix with the r_v non-zero elements on the principal diagonal equal to $+1$. Necessarily, the sum of these s matrices is the identity matrix.

Solved also in a different manner by the proposer who remarked that the restriction that the matrices be symmetric may be weakened, for example, to the requirement that they be normal; that is, that each matrix commute with its transpose.

Geometric Probability

3948 [1940, 181]. *Proposed by Michael Goldberg, Washington, D. C.*

Suppose that n slotted discs are freely mounted on the same axis. If the portion of the circumference subtended by the slot of the ith disc is p_i, show that the probability that light, parallel to the axis, can pass through the slots is

$$p_1p_2\cdots p_n\left(\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_n}\right),$$

provided that $p_i+p_j\leqq 1$ for every i and j.

III. *Solution by R. J. Walker, Cornell University*

Let $2\pi x_i$ be the angle made by the line of symmetry of the ith disc with a fixed direction perpendicular to the axis. Each position of the set of discs is then associated with a unique point $(x_1, x_2, \cdots, x_n)$, provided we identify points whose corresponding coordinates differ by multiples of unity. The set of distinct points then constitutes an n-dimensional generalized torus T, which may be pic-

tured as the hypercube H defined by $0 \leqq x_i \leqq 1$, $i=1, 2, \cdots, n$, for which corresponding points on opposite faces are identified.

A position of the discs will allow the passage of light if and only if for some coordinates $(x_1, x_2, \cdots, x_n)$ of the corresponding point there is a constant a such that

$$|x_i - a| < p_i/2, \qquad i = 1, 2, \cdots, n. \tag{1}$$

For a fixed a, the set of points satisfying (1) shall be designated by P_a, and the set of points satisfying (1) for some a (the union of all P_a) shall be designated by P. The required probability is the (n-dimensional) volume of P, since the volume of T, or H, is unity. As a varies, P_a slides along the diagonal L of H defined by $x_1=x_2= \cdots =x_n$, and so sweeps out a prism of length $\sqrt{n}$. (In H, this "prism" includes portions surrounding each of the vertices of H.) The volume of P is therefore $\sqrt{n}$ times the ($(n-1)$-dimensional) area of the cross-section of this prism. Since P_a is a convex body, this area is half the sum of the projections of the faces of P_a on the hyperplane perpendicular to L. These $2n$ faces all make angles of arc cos $1/\sqrt{n}$ with the hyperplane, and come in pairs with areas $p_1p_2 \cdots p_n/p_i$. Hence the cross-sectional area is

$$p_1p_2 \cdots p_n(1/p_1 + 1/p_2 + \cdots + 1/p_n)/\sqrt{n},$$

and this gives the required value for the volume of P.

The computation breaks down if P intersects itself, that is, if for some values of a and b, P_a and P_b intersect in a set not containing points of L. If this occurs we can assume that $b=0$, for the situation is preserved by translation along L. In H, P_0 consists of 2^n blocks, one at each of the vertices of H. Suppose P_a, $0 \leqq a \leqq 1$, intersects, in the manner described, the block at the vertex $x_i=0$, $x_j=1$, where i runs through some subset of the numbers $1, 2, \cdots, n$ and j runs through the remaining ones. Then for a point $(x_1, x_2, \cdots, x_n)$ in both P_0 and P_a we have

$$x_i < p_i/2, \qquad 1 - x_j < p_j/2,$$
$$a - x_i < p_i/2, \qquad x_j - a < p_j/2.$$

From these we obtain

$$a < p_i, \qquad 1 - p_j < a,$$

and so

$$p_i + p_j > 1. \tag{2}$$

Hence the above value of the probability is correct whenever it is impossible to separate the integers $1, 2, \cdots, n$ into two classes such that (2) holds whenever i is in one class and j in the other—this is a considerably weaker assumption than the one proposed. If such a separation is possible the true value of the probability is less than that given by the formula.

Note. Two other solutions of 3948 were given in the December, 1941, number of this MONTHLY, pp. 705–707.

Associated Triangles

3971 [1940, 662]. *Proposed by J. R. Musselman, Western Reserve University*

If O be the circumcenter of the triangle $A_1A_2A_3$ and M_i be the other points of intersection of the circle with the lines OA_i, show that the three circles passing through O, and on M_i with centers on A_jA_k respectively, meet at the point of Feuerbach for the tangential triangle of $A_1A_2A_3$.

Solution by Li Ou, Yenching University, Peiping, China

Let the vertices of the triangle $A_1A_2A_3$ be t_i, where $|t_i|=1$. Then the coördinates of M_i will be $-t_i$, and the equation of A_iM_i is $x-t_i^2\bar{x}=0$. Hence, the perpendicular bisector of OM_1 is $x+t_1^2\bar{x}=-t_1$, which intersects A_2A_3 at

$$C_1 = \frac{t_1^2\sigma_2}{t_1^3 - \sigma_3},$$

where the σ_i's are the elementary symmetric functions of the t_i's. Then the equation of the circle with C_1 as center and OC_1 as radius will be

$$x = \frac{t_1^2\sigma_2}{t_1^3 - \sigma_3}(1 + t),$$

or

$$(t_1^3 - \sigma_3)x\bar{x} - t_1^2\sigma_2\bar{x} + t_1\sigma_1 x = 0. \tag{1}$$

Similarly for M_2, the equation of the circle is

$$(t_2^3 - \sigma_3)x\bar{x} - t_2^2\sigma_2\bar{x} + t_2\sigma_1 x = 0. \tag{2}$$

Solving for x from (1) and (2), we obtain two solutions $x=0$ and $x=\sigma_2/\sigma_1$. The former is the circumcenter, and the latter is the Feuerbach point for the tangential triangle of $A_1A_2A_3$ (see Morley and Morley, *Inversive Geometry*, p. 191).

Solved also by E. F. Allen, Frank Ayres, Jr., J. W. Clawson, and H. A. DoBell, each using inversive geometry. Allen and DoBell showed also that the point common to the three circles of the problem lies on the nine-point circle of the tangential triangle at its point of tangency with the circumcircle of $A_1A_2A_3$, thus justifying the use of the term Feuerbach point.

Apollonian Circles

3973 [1940, 662]. *Proposed by V. Thébault, San Sebastián, Spain*

The symmetrics of the Apollonian circles of a triangle with respect to the corresponding midpoints of the sides are orthogonal to the circumcircle of the anticomplementary triangle. N. A. Court has given other properties of these circles in this MONTHLY, 1926, p. 373.

Solution by L. M. Kelly, University of Missouri

Let us consider the Apollonian circle through C. Its reflection in M_c certainly passes through C' and is, as a matter of fact, the Apollonian circle of the triangle ABC', with respect to the vertex C', of course. Thus the reflection is orthogonal to the circumcircle of the triangle ABC'. But this circle is tangent to the circumcircle of the anticomplementary triangle at C'. Hence the reflection in question is orthogonal to the circle $A'B'C'$. Similarly for the others.

Solved also by J. W. Clawson, using inversive geometry, and by the proposer in a manner similar to Kelly's solution. A synthetic solution different from the above was received from Sun Nien-tseng after the preparation of the above for printing.

The Tetrahedron

3974 [1940, 662]. *Proposed by V. Thébault, San Sebastián, Spain*

If the straight lines joining the vertices A, B, C, D of a tetrahedron to the points A_1, B_1, C_1, D_1 in the planes of the corresponding opposite faces are concurrent, and also the straight lines joining the same vertices to the isogonal conjugates A_2, B_2, C_2, D_2 of A_1, B_1, C_1, D_1 with respect to the corresponding face triangles are concurrent, the tetrahedron is isodynamic, and conversely. For isodynamic tetrahedrons see N. A. Court, *Modern Pure Solid Geometry*, p. 276.

Solution by L. M. Kelly, University of Missouri

Let the lines AA_1, BB_1, CC_1, DD_1 meet in the point P and AA_2, BB_2, CC_2, DD_2 in the point Q. Further suppose the barycentric coördinates of P to be m_a, m_b, m_c, m_d and those of Q to be x_a, x_b, x_c, x_d.

By virtue of the fact that A_1 and A_2 are isogonal conjugates in the triangle BCD,

$$x_b = k_1(c'^2/m_b) \qquad x_c = k_1(b'^2/m_c) \qquad x_d = k_1(a^2/m_d)$$

where the lengths of opposite edges such as BC and DA are a and a'. Similarly since B_1, B_2 are isogonals in the triangle ADC we have

$$x_a = k_2(c'^2/m_a) \qquad x_c = k_2(a'^2/m_c) \qquad x_d = k_2(b^2/m_d)$$

and so on.

Hence

$$k_1(a^2/m_d) = k_2(b^2/m_d) \quad \text{and} \quad a^2/b^2 = k_2/k_1,$$

and

$$k_1(b'^2/m_c) = k_2(a'^2/m_c) \quad \text{and} \quad b'^2/a'^2 = k_2/k_1,$$

so that $a^2/b^2 = b'^2/a'^2$, and $aa' = bb'$. Likewise $aa' = cc'$, and the tetrahedron is isodynamic.

Orthopoles of a Line

3975 [1940, 713]. *Proposed by R. Goormaghtigh, Bruges, Belgium*

The orthopoles of a straight line parallel to one of the axes of an equilateral hyperbola as to the four triangles formed by any three of four points given on that hyperbola are on a straight line.

Solution by the Proposer

It is well known that the orthopoles of two parallel lines as to a given triangle are on a perpendicular to these lines and at the same distance from each other as these lines; hence it will only be necessary to prove the theorem for the case when the given line is one of the axes of the hyperbola.

Let then $xy=1$ be the hyperbola in rectangular coördinates and x_i, y_i $(i=1, 2, 3, 4)$ the coördinates of the given points A_i.

The projection of A_1 on the axis $x=y$ is

$$(x_1^2+1)/2x_1, \qquad (x_1^2+1)/2x_1,$$

and the perpendicular dropped from that projection on A_2A_3 is

$$y - x_2x_3x = \frac{x_1^2+1}{2x_1}(1 - x_2x_3).$$

Hence the orthopole of the considered axis as to triangle $A_1A_2A_3$ is

$$x = (x_2x_3 + x_3x_1 + x_1x_2 - 1)/2x_1x_2x_3,$$
$$y = (x_1 + x_2 + x_3 - x_1x_2x_3)/2.$$

But if

$$x_1 + x_2 + x_3 + x_4 = a,$$
$$x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2 + x_1x_2x_3 = b, \qquad x_1x_2x_3x_4 = c,$$

then

$$2x = b/c - 1/x_4 - x_4/c, \qquad 2y = a - x_4 - c/x_4,$$

and the four considered orthopoles belong to the straight line

$$2cx - 2y = b - a.$$

The Central Conic

3976 [1940, 713]. *Proposed by W. O. Pennell, Exeter, N. H.*

Through a point P in the plane of a central conic, a line CC' is drawn parallel to the diameter conjugate to the diameter located by a line joining P with the center of the conic. Draw two lines through P intersecting the conic in A and B, and A' and B', respectively. Prove that AB' and $A'B$ (extended if necessary) intersect CC' at points equidistant from P. Likewise, AA' and BB' intersect CC' at points equidistant from P.

Solution by G. A. Williams, Oregon State College

Let p be the polar of P with respect to the conic. Then $A'B$ and AB' intersect in a point O on p.

The four lines OA, OP, OB, and p form a harmonic pencil with vertex O. The line CC' cuts this pencil in four harmonic points. Then as CC' is parallel to p, P is the harmonic conjugate of the point at infinity on CC' with respect to the other two points and is consequently the mid-point of the segment joining them.

Similarly, AA' and BB' intersect in a point O' on p, *etc.*

Solved also by F. C. Hall, L. M. Kelly, F. M. Morgan, and the proposer.

Iteration

3977 [1940, 713]. *Proposed by Aaron Herschfeld, Washington, D. C.*

Denote the nth iterate of $f(x)$, a real function of a real variable x, by the symbol $f_n(x)$. Thus $f_0(x) \equiv x$, $f_1(x) \equiv f(x)$, $f_2(x) \equiv f[f(x)]$, $\cdots$, $f_n(x) \equiv f[f_{n-1}(x)]$. Prove the following:

(1) If $f(x) = x^2 - 2x$, then

$$\lim_{n\to\infty} \{f_n(x)\}^{2^{-n}} = \tfrac{1}{2}\{|x-1| + \sqrt{x^2-2x-3}\} > 1, \text{ when } |x-1| > 2,$$

$$\text{while } |f_n(x)| \leqq 3, \text{ for all } n, \text{ when } |x-1| \leqq 2.$$

(2) If $f(x) = 2x^2 - 1$, then

$$\lim_{n\to\infty} \{f_n(x)\}^{2^{-n}} = |x| + \sqrt{x^2-1} > 1, \text{ when } |x| > 1,$$

$$\text{while } |f_n(x)| \leqq 1, \text{ for all } n, \text{ when } |x| \leqq 1.$$

Solution by Fritz John, University of Kentucky

The statements in question are essentially the special cases $\alpha = 1$, $\beta = 0$ and $\alpha = 2$, $\beta = 1$ of the following theorem:

Let α and β be arbitrary real numbers ($\alpha \neq 0$); let

$$f(x) = \alpha\left[2\left(\frac{x-\beta}{\alpha}\right)^2 - 1\right] + \beta.$$

Define $f_n(x)$ by $f_1(x) = f(x)$, $f_n(x) = f(f_{n-1}(x))$. Then

$$\beta - |\alpha| \leqq f_n(x) \leqq \beta + |\alpha|,$$

if

$$\left|\frac{x-\beta}{\alpha}\right| \leqq 1 \tag{1}$$

holds, and

$$\lim_{n\to\infty}\{|f_n(x)|\}^{2^{-n}}=\left|\frac{x-\beta}{\alpha}\right|+\sqrt{\left(\frac{x-\beta}{\alpha}\right)^2-1},$$

if

$$\left|\frac{x-\beta}{\alpha}\right|>1; \tag{2}$$

(here $|f_n(x)|$ may be replaced by $f_n(x)$, if $\alpha>0$).

Proof: 1. Let x satisfy (1). Then there is a real t, such that

$$x=\alpha\cos t+\beta=\phi(t).$$

Hence, using the identity $\cos 2t=2\cos^2 t-1$,

$$f(x)=\alpha\cos 2t+\beta=\phi(2t);\qquad \left|\frac{f(x)-\beta}{\alpha}\right|\leqq 1;$$

it follows by induction that

$$f_n(x)=\phi(2^nt)=\alpha\cos(2^nt)+\beta;$$

thus $\beta-|\alpha|\leqq f_n(x)\leqq\beta+|\alpha|$.

2. Let x satisfy (2). Then there is a *positive* t, such that

$$\left|\frac{x-\beta}{\alpha}\right|=\cosh t=\tfrac{1}{2}(e^t+e^{-t}).$$

Using the identity $\cosh(2t)=2\cosh^2 t-1$, we obtain

$$1<\frac{f(x)-\beta}{\alpha}=\cosh(2t);$$

hence

$$\frac{f_n(x)-\beta}{\alpha}=\cosh(2^nt);$$

$$f_n(x)=\beta+\alpha\cosh(2^nt)=\beta+\tfrac{1}{2}\alpha(e^{2^nt}+e^{-2^nt})=e^{2^nt}[\beta e^{-2^nt}+\tfrac{1}{2}\alpha+\tfrac{1}{2}\alpha e^{-2^{n+1}t}]$$

Consequently (using $e^t>1$),

$$\lim_{n\to\infty}\{|f_n(x)|\}^{2^{-n}}=e^t=\left|\frac{x-\beta}{\alpha}\right|+\sqrt{\left(\frac{x-\beta}{\alpha}\right)^2-1},$$

where for $\alpha>0$, we may replace $|f_n(x)|$ by $f_n(x)$ under the limit sign.

Solved also by the proposer.

Editorial Note. The solution by the proposer for the two special cases is somewhat similar to the above. He remarked that the earliest published formulas for the nth iterate of a polynomial of degree greater than unity seems to be those of

E. Schröder in his article *Über iterierte Funktionen, Mathematische Annalen*, vol. 3, 1867. He was unable to find the formulas of the problem in the literature.

Spheres Associated with an Orthocentric Tetrahedron

3978 [1940, 713]. *Proposed by V. Thébault, San Sebastián, Spain*

Given an orthocentric tetrahedron and the spheres which are loci of points such that the ratio of the squares of their distances to the extremities of one of the edges of the tetrahedron is equal to the ratio of the sum of the squares of the edges of the faces containing the opposite edge. Show that the sum of the powers of the respective extremities of the latter edge with respect to either of the two spheres is constant.

Solution by Frank Ayres, Jr., Dickinson College

Associated with an orthocentric tetrahedron is a set of five mutually orthogonal spheres. The required sum is the sum of the squares of the radii of the four spheres centered at the vertices. If this is denoted by α, then the sum over all the edges is 12α. In this form the problem will be generalized to a space of n dimensions.

Let

$$P_i \equiv (x_i', x_i'', \cdots, x_i^{(n)}) \equiv (x_i), \qquad i = 1, 2, 3, \cdots, n+1,$$

be the vertices of an orthocentric $(n+1)$-point T, the origin of the coordinate system being at the center of the circumscribing hypersphere, whose radius will be denoted by R. Let r_i denote the radii of the $(n+1)$ mutually orthogonal hyperspheres centered at P_i. Then {F. Ayres, *On $n+2$ mutually orthogonal hyperspheres, National Mathematics Magazine*, vol. 10, (1936)},

$$x_i^{(2)} = R^{(2)}, \sum (x_i - x_j)^2 = r_i^2 + r_j^2 \text{ and } 2\sum x_i x_j = 2\ R^2 - r_i^2 - r_j^2,$$

where $\sum$ indicates summation with respect to the superscripts.

The hyperspheres, associated with an edge P_aP_b, which are the loci of points such that the ratio of the squares of their distances to the extremities of the edge is equal to the ratio of the sum of the squares of the edges of the n-point opposite each extremity, have equations

$$S_{A/B} = \frac{\sum (X - x_a)^2}{\sum (X - x_b)^2} = \frac{\sum_{i\neq b} r_i^2}{\sum_{i\neq a} r_i^2} = \frac{A}{B} \quad \text{and} \quad S_{B/A} = \frac{\sum (X - x_a)^2}{\sum (X - x_b)^2} = \frac{B}{A}.$$

The equation of $S_{A/B}$ may be written in the form

$$\sum\left(X - \frac{Bx_a - Ax_b}{B - A}\right)^2 = \frac{AB}{(B-A)^2}(r_a^2 + r_b^2)$$

and that of $S_{B/A}$ may be obtained from it by interchanging A and B.

The power of a vertex P_c ($c\neq a$, b) with respect to $S_{A/B}$ is $-\sum_{k\neq a,b,c}r_k^2$ and with respect to $S_{B/A}$ is $\sum_i r_i^2+r_c^2$. The sum of the powers of P_c with respect to $S_{A/B}$ and all P_j ($j\neq a$, b, c) with respect to $S_{B/A}$ is

$$(1)\qquad (n-2)\sum_i r_i^2.$$

The sum of the powers of P_c with respect to $S_{B/A}$ and all P_j ($j\neq a$, b, c) with respect to $S_{A/B}$ is

$$(2)\qquad \sum_i r_i^2-(n-3)\sum_{k\neq a,b} r_k^2.$$

The sum for all P_c ($c\neq a$, b), (i.e., the contribution for the edge P_aP_b) is $\frac{1}{2}[(n-1)^2\sum_i r_i^2-(n-1)(n-3)\sum_{k\neq a,b}r_k^2]$, and hence the sum for all edges is $n(n-1)^2\sum_i r_i^2$.

For $n=3$, the sums (1) and (2) are equal and independent of the edge; hence, the given theorem.

Editorial Note. The proposer gave the following indications of his solution: Given the tetrahedron $ABCD$, we denote the lengths of its edges BC, CA, AB, DA, DB, DC by a, b, c, a', b', c'. The locus of the points M such that $(MB)^2:(MC)^2=m^2:n^2$ is a sphere (ω_a) whose radius is $\rho_a=mna/(m^2-n^2)$ and whose center ω_a is on BC. The locus of points N such that $(NB)^2:(NC)^2=n^2:m^2$ is a sphere (ω_a') whose center is the symmetric of ω_a with respect to the midpoint of BC. We then have

$$(A\omega_a)^2-\rho_a^2=(m^2b^2-n^2c^2)/(m^2-n^2);$$
$$(D\omega_a')^2-\rho_a^2=(n^2c^2-m^2b^2)/(n^2-m^2)$$

where $m^2=a'^2+b^2+c'^2$, $n^2=a'^2+b'^2+c^2$. We now have

$$(P_a)=[b^2(a'^2+b^2+c'^2)-c^2(a'^2+b'^2+c^2)]/[(m^2-n^2),$$
$$(P_d')=[b'^2(a'^2+b^2+c'^2)-c'^2(a'^2+b'^2+c^2)]/(m^2-n^2);$$

and then it follows that

$$(P_a)+(P_d')=[(b^2+b'^2)(a'^2+b^2+c'^2)-(c^2+c'^2)(a'^2+b'^2+c^2)]/(m^2-n^2).$$

This is for any tetrahedron; if it is orthocentric, $a^2+a'^2=b^2+b'^2=c^2+c'^2=K^2$. The above sum of powers then reduces to K^2, and this completes the proof.

The expressions for (P_a) and (P_d'), the powers of A and D with respect to the corresponding spheres, may be written immediately using vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ for the vertices with any given origin. For, we have

$$(m^2-n^2)(P_a)=m^2(\mathbf{a}_1-\mathbf{a}_3)^2-n^2(\mathbf{a}_1-\mathbf{a}_2)^2=m^2b^2-n^2c^2;$$

since, if we replace $\mathbf{a}_1$ by the variable vector $\mathbf{x}$ in the middle expression and then set it equal to zero, we obtain the equation of the sphere corresponding to A;

similarly for (P_d'). The constant K^2 is $4(R^2-m)$, where R is the circumradius and m is the square of the radius of the polar sphere with center at H, the orthocenter. It is also $\sum r_i^2$ in Ayres' solution. It is assumed in the above that $m^2-n^2 \neq 0$.

Equation of the Circle through Four Points

3979 [1941, 69]. *Proposed by W. V. Parker, Louisiana State University*

If A_i, $(i=1, 2, 3, 4)$, are four points on a circle in the order of the subscripts with the rectangular coördinates x_i, y_i, prove that the equation of the circle is

$$a_{23}a_{14}\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}\begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} + a_{12}a_{34}\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\begin{vmatrix} x & y & 1 \\ x_4 & y_4 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0,$$

where a_{ij} is the length of the side A_iA_j.

I. *Solution by Eduardo Gaspar, Rosario, Argentina*

We will apply the so called "Pappus theorem" that says: "If M is any one point on the circle that passes through A_1, A_2, A_3, A_4, and h_{ij} represents the distance from M to the side A_iA_j, then

$$(1) \qquad h_{41}h_{23} = h_{12}h_{34}.$$

Using this theorem, which is not difficult to demonstrate, let us consider now the triangle $M\,A_iA_{i+1}$ $(i=1, 2, 3, 4, 1)$; its area will be

$$F_{i,i+1} = \tfrac{1}{2}h_{i,i+1}a_{i,i+1}$$

and (1) gives us

$$(2) \qquad a_{12}a_{34}F_{41}F_{23} = a_{41}a_{23}F_{12}F_{34}.$$

Given three ordered points $M(x, y)$, $A_i(x_i, y_i)$, $A_{i+1}(x_{i+1}, y_{i+1})$ on a circle, the area of the triangle $M\,A_iA_{i+1}$ is

$$F_{i,i+1} = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ x_i & y_i & 1 \\ x_{i+1} & y_{i+1} & 1 \end{vmatrix}.$$

This area will be negative if A_i, M, A_{i+1} are in the positive sense of rotation, and positive in the other case. Consequently among the areas $F_{i,i+1}$ only one is negative (for example, if M lies between A_2, A_3 the single negative triangle will be $M\,A_2A_3$).

Hence the equation (2) is the same as that which appears in the proposed problem.

II. *Solution by Kwan Chao-Chih, Yenching University, Peiping, China*

The equation

$$
(1) \qquad K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} + L\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ x_4 & y_4 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0
$$

represents a conic passing through the points A_i, $i=1, 2, 3, 4$. We want to choose K and L such that (1) will represent the circle passing through the four points. We first change to homogeneous coördinates by replacing $x:y:1$ by $x:y:t$. By making use of one of the two circular points, say $(1, i, 0)$, we get, after changing back again to rectangular coordinates,

$$
(2) \qquad \begin{aligned} K/L = &- [(y_2 - y_3) - i(x_2 - x_3)][(y_4 - y_1) - i(x_4 - x_1)] \\ &\div [(y_1 - y_2) - i(x_1 - x_2)][(y_3 - y_4) - i(x_3 - x_4)]. \end{aligned}
$$

Writing $z=x+iy$, we have

$$
K/L = (z_2 - z_3)(z_1 - z_4)/(z_1 - z_2)(z_3 - z_4) = a_{23}a_{14}e^{i(\alpha-\beta)}/a_{12}a_{34},
$$

where $|z_h - z_k| = a_{hk}$, and α is the angle from the vector $z_1 - z_2$ to $z_2 - z_3$, and β the angle from $z_1 - z_4$ to $z_3 - z_4$. From the concyclic property and the given order of the A's we can immediately see that $\alpha = \beta$, and thus K/L is completely determined as stated in the problem.

Solved also by Frank Ayres, Jr., E. H. Clarke, J. W. Clawson, H. A. DoBell, H. W. Eves, G. B. Huff, C. R. Phelps, O. J. Ramler, C. W. Trigg, and G. A. Williams.

Editorial Note. Most of the solvers used as a basis the above Pappus theorem and gave proofs of the latter. Trigg deduced this basic theorem from the theorem: The perpendicular from a point on a circle to a chord is the mean proportional between the perpendiculars from the point to the tangents at the extremities of the chord; see 3726 [1937, 58]. Williams deduced it from the theorem: The product of two sides of a triangle is equal to the product of the altitude on the third side and the circumdiameter. Phelps at the end of his solution deduced this basic theorem as an incidental result. Some of the solvers first replaced the given determinants by areas of triangles expressed in terms of an angle and the two including sides, and concluded the proof by the use of the law of sines, noting equalities of angles in the figure.

PRE-TRAINING OF AVIATION CADETS

In January the War Department appointed a committee consisting of Professors W. L. Hart (Department of Mathematics, University of Minnesota), W. M. Whyburn (Department of Mathematics, University of California, Los Angeles), and C. C. Wylie (Department of Astronomy, University of Iowa) "to make a survey of the ground school courses offered in pilot and non-pilot courses in the Air Corps Flying Training System, with a view to outlining preparatory courses to be given in colleges and universities."

The committee made a study of various Air Corps schools and recommended a pre-training program for Aviation Cadets. These recommendations are incorporated in the following communication issued by Major General Yount:

War Department
Headquarters Air Corps Flying Training Command
Washington

In order to man our ever increasing armada of combat planes, the Air Corps has need for a continuous flow of well prepared, intelligent aviation cadets. It is definitely our opinion that the typical young man who satisfies the recently announced requirements for enlistment as an aviation cadet will satisfactorily pass one of the curricula of the Air Corps for which he is eligible. However, certain subject matter which can be studied in high school or in college would widen the possible range of a cadet's usefulness to the Air Corps and might decrease the time required for him to arrive at maximum combat efficiency. Consequently, in the case of a young man who does not intend to enlist immediately as an aviation cadet, but who plans such action later, the Air Corps recommends whichever of the following arrangements for pre-training is most appropriate for him:

Plan I. Pre-training through Regular High School and College Courses

If time limitations permit, it is recommended that a student get his pre-training through regular high school and college courses, including the following: advanced high school algebra; at least twenty-five lessons in solid geometry including the geometry of the sphere; plane and spherical trigonometry; descriptive astronomy; a college course in general physics; a course including a substantial amount of cartography. Additional courses in mathematics and the physical sciences would be useful for particular objectives within the Air Corps. It should be noticed that many of the courses in the preceding program can be taken in high school.

Plan II. A Special College Curriculum

Prerequisites: Elementary high school algebra and plane geometry, as taken normally in grades 9 and 10.

Extent: Fifteen semester-units or the equivalent in college quarter-units, divided between three courses. This amounts to approximately 260 class hours of recitations, lectures, and examinations; *two* hours of laboratory work are to be rated as the equivalent of *one* hour of recitation or lecture.

Time Allowance: One semester, or at most two quarters.

Course A: Mathematics; 6 Semester Units

General Features: The emphasis on theory should be limited to that minimum amount which is essential if the student is to appreciate the content of the course. Numerical applications should be emphasized whenever possible.

Part 1. Algebra: Approximately 25 class hours; the content should be selected from any reputable college text to emphasize the manipulative skills needed for numerical trigonometry, physics, and the most elementary technical fields; graphical methods should be introduced.

Part 2. Plane Trigonometry and Logarithms: Approximately 40 class hours; the content should be selected from any reputable college text including both plane and spherical trigonometry; primarily a course in the numerical aspects of trigonometry with only that amount of analytical trigonometry which is essential for the major purpose of the course and for the similar course in spherical trigonometry; substantial emphasis on slide rule computation with each student possessing a cheap slide rule; stress on applications of all sorts, particularly those involving vector forces and velocities, and army or navy terminology; only simple aspects of graphing need be included.

Part 3. Solid Geometry: Approximately 25 class hours; the course is designed to create accurate space intuitions on the part of the student and to prepare him for spherical trigonometry and certain aspects of astronomy; the content should be selected from any standard text for high school solid geometry and should include a treatment of straight lines, planes, dihedral and trihedral angles, and the geometry of the sphere; other major parts of the usual course may be practically omitted; proofs should be held to a bare minimum; great emphasis should be placed on the drawing

of figures and the making of simple paper models for three dimensional situations; the items of con tent which will be used in spherical trigonometry should receive particular attention.

Part 4. Spherical Trigonometry: Approximately 10 hours; introduction to the formulas for the solution of right triangles and general triangles; emphasis on problems relating to latitude, longitude, and the astronomical triangle on the celestial sphere; examinations should be of the "open book" type; a major object of the course is to give the student confidence later in the use of navigation tables which frequently make it unnecessary for the navigator to carry out the solution of spherical triangles.

Course B: Astronomy, Maps, and Weather: 4 Semester Units

General Characteristics: The object of this course is to give a thorough familiarity with those features of astronomy which are essentials for navigation. It would be sufficient to give merely a few lectures of popular type concerning the topics stressed in the usual course in descriptive astronomy, but not included below. The textbooks on descriptive astronomy, now available for this course, will have to be supplemented by material on map projections and weather phenomena.

Topical Outline: Coordinates on the celestial sphere; motions of the earth; rough determination of time; star charts and maps; the atmosphere; seasons and climates; the planets; identification of stars and planets in evening laboratory hours, not necessarily using a telescope.

Course C: Physics: 5 Semester Units

General Features: Numerical problems, vector methods, and applications of trigonometry should be stressed at every opportunity. The course should not be of theoretical type. It should include from two to four hours of laboratory work per week. The teacher should employ a standard college textbook from which most of the indicated topics can be selected.

Topical Outline: Mechanics; heat, light; sound; electricity and magnetism. At the appropriate places, the following topics should be given special attention because of their importance in meteorology: saturation; vapor pressure; humidity; latent heat on condensation; evaporation; sublimation; fusion; super-saturation; super-cooling.

Note: The preceding special Plan II could be telescoped into eleven or twelve weeks for students who have already had advanced high school algebra and some solid geometry.

BARTON K. YOUNT
Major General, U. S. Army
Commanding

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-seventh Annual Meeting, New York, N. Y., December 30–31, 1942.

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, April 25
ILLINOIS, Decatur, May 8–9, 1942
INDIANA, Crawfordsville, April 24–25, 1942
IOWA, Mt. Pleasant, April 17–18, 1942
KANSAS
KENTUCKY, Lexington, April 11, 1942
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Ashland, Va., May 2, 1942
METROPOLITAN NEW YORK, New York, April 18, 1942
MICHIGAN
MINNESOTA, Northfield, May 9, 1942
MISSOURI, fall, 1942
NEBRASKA, Omaha, May 9, 1942
NORTHERN CALIFORNIA, San Francisco, Jan. 30, 1943
OHIO, Columbus, April 2, 1942
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 28, 1942
ROCKY MOUNTAIN, Golden, Colo., April 17–18, 1942
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN, State College, N. M., April 27–28, 1942
TEXAS, Lubbock, April 3–4, 1942
UPPER NEW YORK STATE, fall, 1942
WISCONSIN, Oshkosh, May 2, 1942

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EMILE PICARD, 1856–1941

S. MANDELBROJT, Rice Institute

Since the last quarter of the eighteenth century mathematicians have appeared in France in groups. The first of these would include such figures as Lagrange, Monge, Laplace, Legendre, Fourier, and Cauchy.

Picard was the last survivor of an epoch which rose to its apogee with the researches of Poincaré and was initiated by Hermite. These men might well have chosen as their symbol the modular function. Hermite introduced it, and for Poincaré it constituted the beginning of an admirable chapter in mathematics—the theory of automorphic functions. It was given to Picard to make its most impressive application in the theory of functions.

The proofs of both theorems of Picard on entire functions: "the first theorem" and "the great theorem"—both great—are so astonishing that one need not hesitate to describe them as dramatic.

Generalizations of these theorems (Borel, Landau, Schottky, Carathéodory, Julia, Bloch), theories connected with them (normal families of Montel, characteristic functions of Nevanlinna) and their applications are so numerous and important, that scores of well-known mathematicians have devoted and still devote their lives to them. The results obtained constitute a large autonomic field comparable in importance only to the great classical domains of modern mathematics.

In relation to Picard's theorems on entire functions (or more generally on the behavior of analytic functions in the neighborhood of isolated singularities) one should remember the rôle, in the theory of discontinuous groups, of the "group of Picard."

Followers of Picard attempted to avoid the use of the modular function in the proofs of his theorems, in an effort to provide an "elementary proof" of them. Certain important results were obtained in this direction, but it is remarkable that other not less important contributions have demonstrated that this function is intrinsically related to the field created by Picard.

It should be pointed out, perhaps, that the idea of the Riemann surface appears in its true light through the topological aspects of Picard's theorem.

A capital contribution to the understanding of general existence theorems in differential equations is given by Picard's method of successive approximations. It furnished its author, and many of his followers, with a speedy proof of convergence of a series which gives the integral, replacing advantageously the older methods of Cauchy-Lipschitz. Its constructive character permits a new outlook on questions of existence, and, in many ways has proved to be of great utility.

We owe to Picard the development of the theory of algebraic functions of many variables, a study which was originated by Riemann. In this connection should be mentioned Picard's work on algebraic surfaces with rational plane sections.

It is of course impossible to do more than mention Picard's work on partial differential equations, for instance, that concerning the methods of integration of equations of hyperbolic type (related to the propagation of electrical waves through a homogeneous telegraphic wire), or methods employed in various forms of the solution of the Dirichlet problem.

The "method of Schwartz-Picard-Poincaré" in the theory of integral equations will recall Picard's interest in this subject which was, for many years, his favorite field of investigation and instruction.

Theory of functions, theory of groups, algebraic functions of many variables, theoretical physics, differential equations were perhaps the branches of mathematics which Picard enriched most, but he was not one of those whom one may describe as "a specialist" in one or another field—he was one of the few Masters who formed the very basis of our mathematical science.

There is not a single French mathematician over thirty years of age who was not a student of his. But not only French students enjoyed his Cours d'Analyse Supérieure. The Amphitheatre at the Sorbonne where be taught for 50 years was the meeting place of young mathematicians from all over the world. Even those who had some difficulties in understanding French could admire the simplicity and directness of his teaching. Sometimes, what is generally called "an elegant exposition" seems unpleasant in mathematics, since elegance is often won by hiding the real difficulties of the subject. Is it that simplicity was inherent in the subjects he used to teach, or is it that he made these subjects susceptible of simplification? At any rate students could but admire the Master as well as the subject. One can form an idea of his manner of teaching by reading his famous "Traité d'Analyse."

Picard has had, of course, all the distinctions a great mathematician can have: membership in French and foreign Academies, Scientific Societies, etc. But in France his prestige surpassed even that of some of the greatest representatives of our science. His position in the mathematical world, his profound interest in the history of French thought, exemplified in the beautifully written essay on Pascal, have made him recognized representative of French Science. And if the official consecration of his prestige was his election as Secrétaire Perpétuel de l'Académie des Sciences, and—a rare honor for a scientist—as "Immortel," Member of the Académie Française, his olympian attitude reflected a sense of the rôle he played in his time. He was a worthy representative of a proud epoch.

ANNUAL MEETING OF NORTHERN CALIFORNIA SECTION

The fourth annual meeting of the Northern California Section was held at the University of California, Berkeley, on Saturday, January 31, 1942. Professor F. R. Morris, chairman of the Section, presided at both morning and afternoon sessions. During the noon recess luncheon was served at the Men's Faculty Club.

The attendance at the two sessions was sixty-five, including the following twenty members of the Association: H. M. Bacon, G. A. Baker, T. J. Bass, B. A. Bernstein, G. C. Evans, Emma Hesse, Einar Hille, D. H. Lehmer, Sophia H. Levy, A. L. McCarty, E. D. Miller, F. R. Morris, W. H. Myers, E. B. Roessler, Ethel Spearman, Pauline Sperry, Ruth G. Sumner, Gabor Szegö, A. R. Williams, Fredrick Wood.

The following officers were elected for the coming year: Chairman, Fredrick Wood, University of Nevada; Vice-Chairman, E. B. Roessler, University of California at Davis; Secretary-Treasurer, H. M. Bacon, Stanford University. Mrs. Ruth G. Sumner, Oakland High School, was re-elected to represent the Section as associate editor of the *California Journal of Secondary Education*. Professor Sophia H. Levy reported briefly on the activities of the Committee on Mathematical Education of the two California Sections which had been authorized by a mail vote of the two Sections and appointed by the Regional Governor; the membership of the Committee is as follows: Sophia H. Levy, Chairman; C. G. Jaeger, E. B. Roessler, S. E. Urner, W. M. Whyburn, H. M. Bacon (ex-officio), G. C. Evans (ex-officio).

By invitation of the Section, Professor H. C. Burbridge of Fresno State College gave an hour's address during the morning session.

The following papers were read:

1. "Theory of budgets based on parabolic Engel curves" by Dr. G. A. Baker, University of California at Davis.

2. "Concerning the altitudes of a tetrahedron" by Professor J. H. McDonald, University of California, introduced by Professor Levy.

3. "Recent developments in engineering" by Professor H. C. Burbridge, Fresno State College, by invitation.

4. "Senior mathematics: a semester course in algebra and geometry designed for students without previous high school mathematics" by L. J. Hill, San Jose High School, introduced by Professor Myers.

5. "On conjugate trigonometric polynomials" by Professor Gabor Szegö, Stanford University.

6. "On unbounded solutions of linear second order differential equations" by Professor Einar Hille, Yale University.

Abstracts of the papers follow, numbered in accordance with their listing:

1. Dr. Baker showed that, considering averages, expenditures on groups of items are parabolic functions of the total income or expenditure. If total income is divided into expenditure on savings and expenditures on all other

items, then the preference curve is parabolic and a quadratic utility surface is determined except for an additive constant. The indifference curves are arcs of ellipses which are orthogonal to the preference curve.

2. Professor McDonald showed that the altitudes of a tetrahedron in general are four lines of a regulus; the quadric of this regulus contains eight other lines. These are the four perpendiculars to the faces of the tetrahedron erected at their orthocenters, and four other lines. Any one of the latter is determined as the line of intersection of three planes drawn through three concurrent edges of the tetrahedron and perpendicular, in each case, to the face of the other two edges. It may be shown that the quadric is a rectangular one, namely, such that when given by an equation in rectangular coordinates the sum of the coefficients of the squares is zero, and characterized geometrically by the property that each of its reguli contains sets of three mutually perpendicular lines.

3. Professor Burbridge described developments in aviation including testing devices, design of planes, development of fuels, methods of testing the smoothness of surfaces and their relation to the design and construction of machines. A particularly interesting illustration described the securing of the same result by two widely different methods of research. Several developments in the fields of radio and telephony were discussed, especially the increase in the number of telephone meassages which can be carried by a single cable. The role of scientists in saving England during the present war was mentioned, together with a brief account of recent developments in astronomy.

4. At the present time many senior students in San Jose High School wish to change from fields requiring little or no mathematics to fields requiring a mathematical background. Mr. Hill described a semester course to cover briefly the fundamentals of algebra, geometry, and plane trigonometry which would attempt to meet as well as possible the needs of these senior students in the very limited time available.

5. Professor Szegö considered the set of all polynomials $f(z)$ of degree n whose real part lies between preassigned bounds, say between -1 and $+1$ as $|z| \leqq 1$; let $f(0)$ be real. The problem is to find the greatest possible value for the imaginary part of $f(z)$, $|z| \leqq 1$. After transforming this problem into one in terms of conjugate trigonometric polynomials the solution follows by means of a proper interpolation formula. The bound in question is $\cong (2/\pi) \log n$ as $n \to \infty$, attained for certain polynomials having the form $\pm f_0(\epsilon z)$, $|\epsilon| = 1$; here $f_0(z)$ is a special polynomial of the given set with pure imaginary coefficients which assumes the values $+1$ and -1 at the points $m\pi/(n+1)$ and $-m\pi/(n+1)$, respectively, $1 \leqq m \leqq n$, m odd. These are the only extremum polynomials if n is odd, whereas for even n besides these some other extremum polynomials exist.

6. Professor Hille assumed the given linear second order differential equation to be self-adjoint and the solutions to be non-oscillatory in an interval (a, b). A necessary and sufficient condition was given in order that the solutions be unbounded in (a, b). It was shown by examples that the inequalities obtained are the best possible.

H. M. BACON, *Secretary*

WHAT IS THE JORDAN CURVE THEOREM?

J. R. KLINE, University of Pennsylvania

In non-technical terms we may define a simple closed curve or Jordan curve as the most general set which can be obtained from a circle by bending and stretching without breaking or crossing. More precisely, a simple closed curve is the image of a circle under a homeomorphism, *i.e.*, under a (1–1) continuous transformation with a continuous inverse, while a simple continuous arc is the image of a straight line interval under a transformation of the same type. While such simple figures as a triangle or an ellipse are simple closed curves, we can also obtain simple closed curves the structure of which is extremely complex. We shall exhibit two curves of the latter type to show that, under a homeomorphism, a circle does not necessarily retain any one of the following properties:

(a) that of having a tangent at every point;
(b) that of having a length;
(c) that of having its two-dimensional area (measure) equal to zero.

That (a) and (b) may be lost can easily be seen from the following well-known simple closed curve: let C_0 be an equilateral triangle while C_1 is the outer boundary of the figure obtained if, on the middle third of each straight line segment of C_0, we erect an equilateral triangle whose interior lies wholly without C_0. Assuming that C_n has been constructed, we obtain C_{n+1} as the outer boundary of the figure reached by erecting, on the middle third of each straight line segment of C_n, an equilateral triangle whose interior lies wholly in the exterior of C_n. Let C be the limit of $C_0, C_1, C_2 \cdots$. This limit is a simple closed curve. Clearly at no point does C possess a tangent while the length of the perimeter of C_n becomes infinite with n.

To show that property (c) is not necessarily retained, we have recourse to a scheme based on the well-known Cantor middle third process. The Cantor process provides us with a nowhere dense perfect set on a straight line interval. Our modification leads to a perfect set C which contains no connected subsets other than single points and has plane measure different from zero.* Now with the use of a theorem due to R. L. Moore and the author [1] we are able to construct a simple closed curve J which contains every point of C and hence must

* In this footnote we shall describe the process by which we obtain C. Let C_0 denote the square $ABCD$ plus its interior. To obtain C_1, we remove from C_0 all of those points which lie between two perpendiculars to BC which (1) are symmetric to the perpendicular bisector of BC and (2) include between them 1/3 of the square.

Next, remove from C_1 that portion of C_1 which lies between two lines which (1) are perpendicular to AB and are symmetric to the perpendicular bisector of AB and (2) are such that the parts of the two rectangles, which comprise C_1, included between them, have together an area equal to 1/9 the original square. Call the four remaining rectangles C_2.

Next on *each* of the intervals of BC which remain erect a pair of lines which are (1) perpendicular to BC and symmetric to the perpendicular bisector of the interval and (2) are such that

have plane area different from zero. These examples contradict many of our more intuitive notions of a curve.

In view of these examples, one naturally wonders what properties, if any, of the circle C must remain after it has been subjected to so general a transformation as a homeomorphism. We do find that our conditions will be sufficient to require the resulting set J still to possess properties such as (1) that of having any two distinct points separate J into two mutually separated connected pieces of which they are the common boundary and (2) that of remaining connected after the removal of any connected piece. Both of these properties are, of course, possessed by the original circle. Much more interesting, however, is the relation of the resulting simple closed curve J to the plane S in which it may be imbedded. From elementary geometry we remember that a circle divides the remaining points of the plane into two sets, the interior and the exterior, the former consisting of points at a distance less than the radius from the center, while the latter contains the points whose distance from the center is greater than the radius. These sets have the property that, in order to join a point of one set to a point of the other by any simple continuous arc, one must pass through a point of the circle. Intuitively one feels certain that a separation of the remaining points into distinct sets must still persist after the circle has been carried by a homeomorphism into any simple closed curve in the plane. Before the introduction of modern standards of mathematical rigor, mathematicians generally accepted the existence of such a separation without proof. However, in 1865 C. Neumann in his *Vorlesungen über die Riemannschen Theorie der Abelschen Integrale* explicitly asked for a proof of the existence of such a division of the plane.

In 1887, C. Jordan gave the first so-called proof of the theorem that now bears his name, *i.e., if a simple closed curve J is imbedded in a plane S, then $S-J$ is the sum of exactly two mutually exclusive domains and every point of J is a boundary point of each domain.* Jordan's argument did not suffice even for the case of a polygon. The first complete proof of the theorem in its most general form was given by Veblen [2] in the *Transactions of the American Mathematical Society* in 1905. Subsequently many other proofs were given. Prominent among

the parts of the rectangles of C_2 included between the two pairs of perpendiculars have together an area equal to 1/27 of the whole square. Call the eight rectangles which remain C_3.

Next we turn to the invervals of AB which remain and perform a similar construction, removing in all from C_3 an amount whose area is 1/81 of the original square. We proceed in this manner, removing alternately strips symmetric to the perpendicular bisectors of all the intervals of BC that survive, and then strips symmetric about perpendicular bisectors of all intervals of AB that survive. We arrange the width of the strips in such a manner that at the end of the nth step we remove from C_{n-1}, a total area equal to $(1/3)^n$ of the original square.

Let C be the set of points common to C_0, C_1, C_2, $\cdots$. It is clear that this set is (1) closed (2) contains no connected subset other than a single point. As the part of the square which has been removed has an area equal to 1/2 of the square, it is clear that C has positive measure equal to 1/2 the square.

By a suitable modification of this process we can obtain a totally disconnected closed set whose area is any *proper* fraction of the area of the square.

these are the proofs by Brouwer, Alexander, Kerekjarto, Schoenflies, and Winternitz. A very satisfactory recent proof is that contained in the chapter on separation theorems in a recent book by M. H. A. Newman [3]. The principal steps in the proof are the proving of (1) the fact *that a simple continuous arc does not separate the plane* S and (2) the fact that if F_1 and F_2 are two bounded continua, neither of which separates S, then a necessary and sufficient condition that F_1+F_2 shall separate S is that the common part $F_1 \cdot F_2$ be not connected.

One naturally inquires as to the truth of the converse of the Jordan Curve Theorem. If we apply the term "closed curve" to any closed and bounded plane set which divides its plane into exactly two parts of which it is the common boundary, then our question is whether every closed curve is necessarily a simple closed curve. That this is not true may easily be seen by considering the example which is so frequently used by topologists, *i.e.*, the set composed of (1) $y=\sin\pi/x$ $(0<x\leqq 1)$, (2) the interval of the y-axis from $(0, -2)$ to $(0, 1)$, (3) the interval of $y=-2$ from $(0; -2)$ to $(1, -2)$, and (4) the interval of $x=1$ from $(1, -2)$ to $(1, 0)$. We mention three conditions which are such that, if any one of them is added to the hypothesis that our set C is a *closed curve*, our set then becomes a simple closed curve; and conversely each simple closed curve in a plane has these properties. They are:

(1) Every point of C is accessible from both of the complementary domains $D_i (i=1, 2)$ where a point P of C is said to be accessible from D_i, if, for every point Q of D_i, there exists a simple continuous arc from Q to P which lies, except for P, entirely in D_i. This is the condition which was used by Schoenflies [4] in obtaining a proof of the converse of the Jordan Curve Theorem.

(2) The set C is locally connected at every point. This condition was exhibited by the author [5] of this paper as a necessary and sufficient condition that a closed curve be a simple closed curve.

(3) The set C at each of its points separates the plane locally into the same finite number, n, of domains. The plane is cut locally at P into n parts by C if, for every sufficiently small neighborhood U_P of P, the set $U_P - C$ has at least n components and there exist arbitrarily small neighborhoods for which the number of components is exactly n. This is the condition of Zarankiewicz [6].

The simple closed curve J is obtained from the circle by a homeomorphic transformation π which is concerned only with the points of the circle C and their image points on J. Nothing whatsoever is assumed as to what happens to the remaining points of the plane S. There is, however, a powerful theorem, due originally to Schoenflies [7], according to which whenever we have been given π, then we can define a new homeomorphism π' of the entire plane S into itself which has the property that, insofar as points of C are concerned, their image under π' is exactly the same as their image under the original π. When we turn to plane figures which are but slightly more complicated, such an extension of the homeomorphism is no longer always possible. Consider, for instance, the set consisting of three simple continuous arcs AB_iC $(i=1, 2, 3)$ in which (1) no two arcs have a point, other than A and C, in common and (2) except for its

endpoints AB_2C lies wholly in the interior of the simple closed curve AB_1CB_3A. Let π be a homeomorphism such that A and C remain fixed and $\pi\ (AB_iC) = AB_{i+1}C$ when subscripts are reduced modulo 3. Clearly this homeomorphism cannot be extended, for, were the extension possible, the exterior of AB_1CB_3A would necessarily be transformed into the interior of AB_1CB_3A under the extended transformation (Figure 1). Much interesting research has been done on the problem of when a transformation of a subset of the plane can be extended to a transformation of the entire plane into itself.

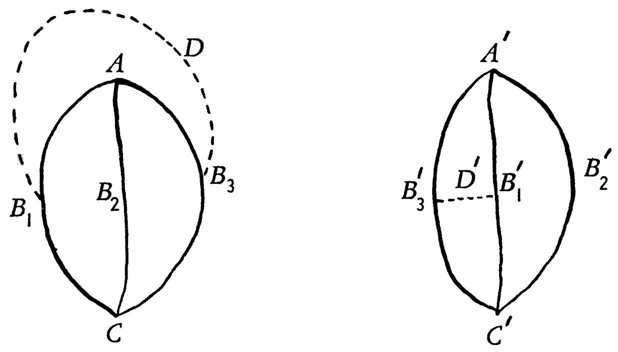

Fig. 1

When we turn to three or more dimensions, the situation becomes more interesting and also much more complicated. That the homeomorph of a Euclidean n-sphere divides Euclidean space of $(n+1)$ dimensions was first proved by Brouwer [8]. In 1924 Alexander [9] gave an example of a simple closed surface S, *i.e.*, the homeomorph of the sphere $x^2+y^2+z^2=r^2$ which has in its exterior a simple closed curve which cannot be shrunk to a point without cutting S. Of course every simple closed curve in the exterior of $x^2+y^2+z^2=r^2$ can be shrunk to a point while remaining entirely in the exterior. By virtue of this example, we see immediately that a theorem analogous to Schoenflies' extension theorem is not true for simple closed surfaces in space of three dimensions.

In addition, in space of three or more dimensions, there are offered for our consideration sets much more general than the homeomorph of a sphere of one less dimension. As an illustration in three dimensions, we take the torus T with the system of meridianal and longitudinal simple closed circles thereon. When the torus is removed from three space, not only do we have a pair of points which cannot be joined by an arc without crossing T, but we have also a pair of simple closed curves, one in the bounded domain cut off by T and the other in the unbounded domain, neither of which is the boundary of a two-cell which does not cut T. The number of independent non-bounding simple closed curves in E_3-T is and must be the same as the number of independent non-bounding simple closed curves on T. This is a very simple case of the consequences of

Alexander's Duality Theorem [10]; this powerful theorem was published in 1922. In contains the Jordan Curve Theorem as a corollary. The duality theorem has been the subject of a long series of important investigations culminating in Pontrjagin's general topological theorem of duality for closed sets [11].

The Jordan Curve Theorem and ideas connected therewith play an important role in R. L. Moore's work on the *Foundations of Plane Analysis Situs* [12] and in the subsequent far-reaching research which he and his students have done. While it is impossible even to indicate all the principal directions that this theorem has given to research in Topology, I wish to call attention to two of its important applications. In the first place, it has been of fundamental importance in determining those continuous curves which are such that on the surface of a sphere there can be found a set which is the homeomorphic image of the original curve. Here fundamental work has been done by Kuratowski and Claytor [13]. The Jordan Curve Theorem also plays a very important role in setting up necessary and sufficient conditions which must be satisfied in order that a continuum shall be a simple closed surface. Here Zippin [14] proved that any compact locally-connected continuum which satisfies the Jordan Curve Theorem *non-vacuously* must be a simple closed surface. Other interesting work in applying the Jordan Curve Theorem to the definition of simple closed surfaces and two-dimensional manifolds has been done by Miss Gawehen, van Kampen [15], J. H. Roberts, Hassler Whitney, and R. L. Wilder. If, in addition to the direct applications of the Jordan Curve Theorem, we consider also the research which is based on the ideas and methods developed in connection with it and its various extensions, we find that it has had an influence on the mathematics of the past fifty years which is surpassed by few, if any, other theorems.

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REMARKS ON DIVISORS OF ZERO*

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1. Introduction. This paper deals with certain elements of those important algebraic systems which are called *rings*. For the sake of completeness, we shall first give definitions of some of the fundamental concepts with which we shall be concerned.†

A *ring* R is a set of elements $a, b, c, \cdots$, with the property that for any two elements a and b of R there is a uniquely defined sum $a+b$ and product ab which are elements of R, and such that the following are always true:

$$\begin{aligned} a+(b+c) &= (a+b)+c, \\ a+b &= b+a, \\ a+x &= b \text{ has a solution } x \text{ in } R, \\ a(bc) &= (ab)c, \\ a(b+c) &= ab+ac, \\ (b+c)a &= ba+ca. \end{aligned}$$

If, for every a and b in R, $ab=ba$, the ring R is said to be a *commutative* ring, otherwise it is a *non-commutative* ring. We shall, for the most part, be concerned with commutative rings.

The above properties are certainly satisfied by all the number systems of elementary algebra with the ordinary definitions of sum and product. Thus, as

* An address delivered at the Bethlehem meeting of the Mathematical Association of America on January 1, 1942.

† The terms and ideas introduced in this section are to be found in van der Waerden [11], and also in most other texts on abstract algebra.

simplest examples, we have the ring of real numbers, the ring of rational numbers or the ring of integers. Another illustration of this concept, and one of a different nature, is the so-called *ring of integers modulo* 6, that is a ring of six elements 0, 1, 2, 3, 4, 5 with $a+b$ defined as the least non-negative remainder when the ordinary sum of a and b is divided by 6, and a similar definition of products. Thus, for example, in this ring we have $4+5=3$ and $2\cdot 4=2$. Naturally, any positive integer m may be used in place of the 6 in this example. Throughout, the ring of integers modulo m will be denoted by I_m.

All the rings introduced so far are commutative rings. As a familiar example of a non-commutative ring, consider the set of all matrices of order two with elements in a ring R, with addition and multiplication defined in the usual way as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix},$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

It is easily verified that the set of all such matrices is a non-commutative ring, even if R is commutative, which we may denote by R_2. In like manner, the ring R_n may be used to denote the ring of all matrices of order n with elements in R.

From the definition of a ring, it is easy to prove that there is always a unique element 0 with the property that, for all a in R, $a+0=a$. This element is naturally called the *zero* of R. It is quite easy to show, as in ordinary algebra, that $a0=0a=0$ for all a in R. However, in the ring I_6, we see that $2\cdot 3=0$. Thus it may happen, in a ring, that a product is zero without either factor being zero. We thus come to the following definition. An element a of a ring R is a *divisor of zero* in R if there exists a non-zero element b of R such that $ab=0$, or a non-zero element c of R such that $ca=0$. If $ab=ba=0$, $b\neq 0$, we may say also that b is an *annihilator* of a, or that a is *annihilated* by b. To avoid confusion later on, we may emphasize that an annihilator is necessarily different from zero.

According to the definition, zero is always a divisor of zero, but it will be convenient and also cause no misunderstanding if, instead of saying that a ring has no divisors of zero except zero, we merely say that a ring has no divisors of zero. Thus, for example, the ring of rational numbers clearly has no divisors of zero.

In all the examples of rings mentioned so far there is an element 1 with the property that $1\cdot a=a\cdot 1=a$ for every element a of the ring. Such an element is called a *unit element* of the ring. We shall assume throughout that all rings considered have unit elements.

Now the possible presence of divisors of zero in a ring accounts for a large part of the differences which arise in algebraic manipulation of elements of a ring, at least in the commutative case, as compared with operations on ordinary numbers. In this paper, we shall present several miscellaneous theorems having to do with divisors of zero. Inasmuch as any calculation with elements of a ring is likely to lead to considerations of divisors of zero, the literature is extensive and the few topics presented here are to be considered merely as a sample and not as an exhaustive treatment of the subject.

2. Linear homogeneous equations. Let us begin by a consideration of the system of linear homogeneous equations,

$$\sum_{j=1}^{n} a_{ij}x_j = 0 \qquad (i = 1, 2, \cdots, m), \tag{1}$$

where the a_{ij} are elements of a *commutative* ring R and solutions are sought in R. Let M denote the matrix of coefficients of the unknowns in this system of equations. The matrix M may be said to be of *rank* r if the determinant* of every square minor of M of order $r+1$ is annihilated by some one element of R, and the same is not true for minors of order r. M is of rank zero if all elements of M are annihilated by a fixed element of R. Note that if the ring R has no divisors of zero, these definitions agree with the familiar ones.

Obviously, the equations (1) always have the *trivial* solution $(0, 0, \cdots, 0)$. We shall now prove the following.

THEOREM 1. *The system* (1) *has a non-trivial solution if and only if the rank of the matrix of the coefficients is less than the number of unknowns.*

The proof is essentially that to be found in most texts on the theory of equations, with simple modifications to take care of the possible presence of divisors of zero in R. Suppose first that there exists a non-trivial solution $(x_1, x_2, \cdots, x_n)$ with $x_k \neq 0$. If $m<n$, clearly the rank r of M is less than n. If $m \geqq n$, consider a fixed determinant D of order n which can be found in M. For simplicity of statement suppose it comes from the first n rows of M. Multiply the first equation by the co-factor of a_{1k} in D, the second by the co-factor of a_{2k} in D, $\cdots$, the nth by the co-factor of a_{nk} in D, and add. There results, $x_kD=0$. Similarly, it will be seen that x_k annihilates the determinant of every minor of M of order n, and thus the rank of M is less than n.

Now let us assume that the rank of M is $r<n$, and let $d\neq 0$ be an element of R which annihilates the determinants of all minors of M of order $r+1$. If $r=0$, then clearly $x_j=d (j=1, 2, \cdots, n)$ is a non-trivial solution of equations (1). If $r>0$, then the product of d by the determinant of some minor of M of order r is not zero. Let us assume for convenience of notation that this minor is in the

* We shall not go into the question here, but it is not difficult to show that the usual theorems on expansions of a determinant are valid, if the elements lie in an arbitrary commutative ring. See, e.g., McCoy [6], p. 281.

upper left-hand corner of M. Let $c_1, c_2, \cdots, c_{r+1}$ be the co-factors of the elements in the last row of the determinant of order $r+1$ in the upper left-hand corner.* Then we assert that

$$x_j = dc_j \qquad (j = 1, 2, \cdots, r+1),$$
$$x_j = 0 \qquad (j = r+2, \cdots, n),$$

in a solution of equations (1), and is certainly non-trivial as $x_{r+1} \neq 0$. That these values satisfy the first r equations comes from the fact that the sum of the products of the elements of any row of a determinant by the co-factors of the corresponding elements of another row, is always zero. The remaining equations are also satisfied since when these values are substituted for the unknowns, we get the product of d by the determinant of a square minor of M of order $r+1$, which vanishes by hypothesis. The theorem is therefore established.

We shall obtain one further result before proceeding to a different topic. Let A be a given element of the matrix ring R_n, that is, A is a matrix with n rows and n columns and with elements in the commutative ring R. Let us seek an element X of R_n such that $AX=0$. It is clear from the definition of multiplication of matrices that each column of X must be a solution of a system of linear homogeneous equations, the matrix of whose coefficients is precisely A. Thus, by the theorem just established, there exists a matrix X, other than the zero matrix, such that $AX=0$ if, and only if, the determinant of A is a divisor of zero in R. A similar conclusion follows from a consideration of the equation $XA=0$. We thus have the

COROLLARY.† *An element A of R_n is a divisor of zero in R_n if and only if the determinant of A is a divisor of zero in R.*

3. Polynomial rings. If R is a given ring, we may construct a new ring in the following almost obvious way. Let λ be a new symbol or *indeterminate* and consider the totality of polynomials,

$$f(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n, \tag{2}$$

with coefficients a_i in R. If we define addition and multiplication of polynomials by the usual formal rules, assuming that λ is commutative with elements of R, it will be seen that the set of all such polynomials is a ring, which may be denoted by the symbol $R[\lambda]$. The ring $R[\lambda]$ contains all elements of R, the polynomials consisting only of "constant" terms, and we say therefore that R is a *subring* of $R[\lambda]$. We may point out that two polynomials are considered equal if, and only if, the coefficients of the different powers of λ are the same in both polynomials.

* If $r=m<n$ we may add to our system of equations another equation in which all coefficients are zero.

† This result, as well as some others having to do with divisors of zero in rings of matrices, will be found in [7].

If the element $f(\lambda)$ of $R[\lambda]$ defined by (2) has $a_0 \neq 0$, we say that $f(\lambda)$ has *degree n*, and call a_0 the *leading coefficient* of $f(\lambda)$. It does not follow, however, that the degree of a product of two polynomials is always equal to the sum of the degrees of the factors. As an illustration of an extreme case, but one in which we are particularly interested, suppose for the moment that R is the ring I_{12} of integers modulo 12. Then clearly

$$(4\lambda^2 + 8\lambda + 4)(3\lambda + 6) = 0.$$

Thus $f(\lambda) = 4\lambda^2 + 8\lambda + 4$ is annihilated by a polynomial of the first degree, namely $3\lambda + 6$. But it is obvious that $f(\lambda)$ is in fact annihilated by an element of I_{12}, e.g. the element 3. This is an almost trivial illustration of the following general result.

THEOREM 2. *If R is a commutative ring and an element $f(\lambda)$ of $R[\lambda]$ is a divisor of zero in $R[\lambda]$, then $f(\lambda)$ is annihilated by an element of R.*

One method of proof of this theorem will be illustrated by a special case. Let

$$f(\lambda) = a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \qquad a_0 \neq 0,$$

and

$$g(\lambda) = b_0\lambda^2 + b_1\lambda + b_2, \qquad b_0 \neq 0,$$

be elements of $R[\lambda]$ such that $f(\lambda)g(\lambda) = 0$. Multiplying, and collecting coefficients of the different powers of λ, we see therefore that

$$\begin{aligned} a_0b_0 \qquad\qquad\quad &= 0, \\ a_1b_0 + a_0b_1 \qquad\quad &= 0, \\ a_2b_0 + a_1b_1 + a_0b_2 &= 0, \\ a_3b_0 + a_2b_1 + a_1b_2 &= 0, \\ a_3b_1 + a_2b_2 &= 0, \\ a_3b_2 &= 0. \end{aligned} \tag{3}$$

This says, since $b_0 \neq 0$, that (b_0, b_1, b_2) is a non-trivial solution of a certain system of linear homogeneous equations. The first part of the proof of Theorem 1 then shows that b_0 annihilates the determinant of every minor of order three which can be chosen from the matrix of a's. In particular,

$$0 = b_0 \begin{vmatrix} a_1 & a_0 & 0 \\ a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 \end{vmatrix} = b_0a_1^3, \tag{4}$$

since a_1^3 is the only term in the expansion of this determinant which does not contain a_0 as a factor, and $a_0b_0 = 0$ by the first equation of the system (3). Thus

there exists an integer j_1 with $0 \leqq j_1 \leqq 2$ such that $b_0 a_1^{j_1} \neq 0$ but $b_0 a_1^{j_1+1} = 0$.* Then $b_0 a_1^{j_1}$ annihilates both a_0 and a_1. Now applying the same argument as was used to get (4), we see that

$$0 = b_0 a_1^{j_1} \begin{vmatrix} a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 \\ 0 & a_3 & a_2 \end{vmatrix} = b_0 a_1^{j_1} a_2^3. \tag{5}$$

Now choose an integer j_2 with $0 \leqq j_2 \leqq 2$ such that $b_0 a_1^{j_1} a_2^{j_2} \neq 0$ but $b_0 a_1^{j_1} a_2^{j_2+1} = 0$. Thus $b_0 a_1^{j_1} a_2^{j_2}$ annihilates a_0, a_1 and a_2. Then finally, by using the last three equations of (3), we see

$$0 = b_0 a_1^{j_1} a_2^{j_2} \begin{vmatrix} a_3 & a_2 & a_1 \\ 0 & a_3 & a_2 \\ 0 & 0 & a_3 \end{vmatrix} = b_0 a_1^{j_1} a_2^{j_2} a_3^3.$$

Again choose an integer j_3 such that $b_0 a_1^{j_1} a_2^{j_2} a_3^{j_3} \neq 0$, but $b_0 a_1^{j_1} a_2^{j_2} a_3^{j_3+1} = 0$. This element $c = b_0 a_1^{j_1} a_2^{j_2} a_3^{j_3}$ then is an element of R which annihilates all coefficients in $f(\lambda)$, as required by the theorem. A general proof can be given along lines of this calculation, the only difficulty being one of notation. An inductive proof of this theorem has also been obtained by Mrs. Alexandra Illmer Forsythe.

We note that the method of proof of this theorem outlined above shows that if b_0 is the leading coefficient of *any* annihilator of $f(\lambda)$, then $f(\lambda)$ is annihilated by cb_0 for proper choice of c in R. This observation will be used presently.

If now μ is another indeterminate, we may introduce the ring of polynomials in λ and μ, with coefficients from R, and this ring may be denoted by $R[\lambda, \mu]$. Elements of $R[\lambda, \mu]$ may also be considered as polynomials in μ with coefficients from $R[\lambda]$, or as polynomials in λ with coefficients from $R[\mu]$. It is now easy to prove

THEOREM 3. *If R is a commutative ring and an element $f(\lambda, \mu)$ of $R[\lambda, \mu]$ is a divisor of zero in $R[\lambda, \mu]$, then $f(\lambda, \mu)$ is annihilated by an element of R.*

Let

$$f(\lambda, \mu) = f_0(\mu)\lambda^n + f_1(\mu)\lambda^{n-1} + \cdots + f_n(\mu), \qquad f_0(\mu) \neq 0,$$

where the $f_i(\mu)$ are elements of $R[\mu]$. Since $f(\lambda, \mu)$ is a divisor of zero in $R[\lambda, \mu]$, the preceding theorem, with R replaced by $R[\mu]$, states that there exists an element $h(\mu)$ of $R[\mu]$ which annihilates $f(\lambda, \mu)$, thus

$$f_i(\mu)h(\mu) = 0 \qquad (i = 0, 1, \cdots, n). \tag{6}$$

Suppose $h(\mu)$ has leading coefficient b_0, and consider the first of the equations (6). Then the observation made above shows that there is an element c_0 of R such that $c_0 b_0$ annihilates $f_0(\mu)$. Then clearly

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* By a_1 we shall mean the unit element 1 of R.

$$f_1(\mu)[c_0 h(\mu)] = 0, \qquad c_0 b_0 \neq 0,$$

and again since $c_0 b_0$ is the leading coefficient of an annihilator of $f_1(\mu)$, there is an element c_1 of R such that $c_1 c_0 b_0$ annihilates $f_1(\mu)$, and it clearly also annihilates $f_0(\mu)$. A repetition of this argument finally leads to the existence of elements $c_0, c_1, \cdots, c_n$ of R such that $c_0 c_1 \cdots c_n b_0$ annihilates all $f_i(\mu)$ and therefore annihilates $f(\lambda, \mu)$. This argument can naturally be just as well applied to prove the corresponding theorem for the case of polynomials in more than two indeterminates.

4. Rings without divisors of zero. So far we have been considering rings with divisors of zero; we may now mention briefly rings without divisors of zero. First we need another definition. A ring R, not necessarily commutative, in which the equations $ax=b$ and $ya=b$ always have solutions for arbitrary b in R and arbitrary $a\neq 0$ in R, is called a *quasi-field.* A commutative quasi-field is called simply a *field.* The ring of rational numbers is a field, as is also the ring of real numbers or the ring of complex numbers. It can also be shown that the ring of integers modulo p, where p is a prime, is a field with exactly p elements. These are simple examples but there are many others, and a careful study of fields is one of the most important subjects of study of modern abstract algebra. For the moment we are, however, principally interested in the almost obvious observation that a quasi-field can have no divisors of zero. For if $cd=0$, $c\neq 0$, multiplication on the left by the solution x of the equation $xc=1$, shows that $d=0$. It is furthermore clear that no subring of a quasi-field can have divisors of zero or, in other words, no ring which can be imbedded in a quasi-field can have divisors of zero. The following converse of this statement, for the commutative case only, is well known.

THEOREM 4. *A commutative ring without divisors of zero is a subring of a field.*

We shall not give the proof of this theorem, as it is to be found in all texts on abstract algebra.* Suffice it to say that the proof is entirely analogous to the method by which the rationals may be logically obtained from the integers by introduction of *formal quotients, i.e.*, pairs of integers (a, b), $b\neq 0$, having assigned properties of the familiar quotients a/b.

When we pass to non-commutative rings, the situation is much more complicated. Malcev [3] has given an example of a ring without divisors of zero which is not a subring of any quasi-field, so that we cannot hope to establish Theorem 4 for the non-commutative case. However, Ore [8] has made some progress in this direction by a careful study of formal quotients. The main result of Ore in this connection is the following

THEOREM 5. *A ring R, without divisors of zero, can be imbedded in a quasi-field by use of formal quotients if, and only if, for every pair a, b of non-zero elements of R, there exist non-zero elements m, n of R such that*

* See, *e.g.*, van der Waerden [11], p. 47.

$$am = bn. \tag{7}$$

It will be noticed that, if R is commutative, we may choose $m=b$, $n=a$, so that Theorem 5 is actually a direct generalization of Theorem 4. Theorem 5 states what can be accomplished by use of formal quotients, but it does not exclude the possibility of some other imbedding process. Thus, whether condition (7) is a necessary property of every subring of an arbitrary quasi-field is an interesting and unsolved problem.

5. Rings without nilpotent elements. We henceforth restrict our attention to commutative rings. It may happen, as an extreme case, that a non-zero element a of a ring R has the property that for a suitably chosen positive integer n, $a^n=0$, in which case a is said to be *nilpotent*. Thus, for example, in the ring I_4, we have $2^2=0$. Clearly a nilpotent element is a special kind of a divisor of zero. Now it happens that rings without nilpotent elements are, on the whole, much better behaved than rings with nilpotent elements. We shall presently state a theorem giving a certain characterization of commutative rings without nilpotent elements. It should be emphasized, however, that a ring without nilpotent elements may have divisors of zero. Thus in the ring I_6 there are no nilpotent elements, but $2\cdot 3=0$.

We now need a further definition, Let R_1 and R_2 be two rings, not necessarily distinct, and consider the set of all pairs (a_1, a_2) where a_1 is an element of R_1 and a_2 an element of R_2. Define addition and multiplication of pairs as follows:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2),$$
$$(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2),$$

it being naturally understood that a_1+b_1 is addition in R_1, a_2+b_2 is addition in R_2, and so on. With these definitions, it is found that the set of all such pairs is a ring which is called the *direct sum* of the rings R_1 and R_2, and written as $R_1\dot{+}R_2$. In this direct sum, the zero element is the pair $(0, 0)$ and the unit element is $(1, 1)$. Clearly $R_1\dot{+}R_2$ will have divisors of zero even if R_1 and R_2 have none, for example, we have $(1, 0)\ (0, 1)=(0, 0)$. But, for our purposes, it is significant and easily verified that if R_1 and R_2 have no nilpotent elements neither does their direct sum. We may pause to illustrate these concepts, as well as some other points, by an example. The ring $I_2\dot{+}I_3$ consists of the six elements $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$ and $(1, 2)$. Let us set up a one-to-one correspondence between these six elements and the elements $0, 1, 2, 3, 4, 5$ of the ring I_6 as follows:

$$\begin{array}{ll} 0 \leftrightarrow (0, 0) & 3 \leftrightarrow (1, 0) \\ 1 \leftrightarrow (1, 1) & 4 \leftrightarrow (0, 1) \\ 2 \leftrightarrow (0, 1) & 5 \leftrightarrow (1, 2) \end{array}$$

It may now be observed that if, in this correspondence, $a\leftrightarrow(b, c)$ and $d\leftrightarrow(e, f)$, then $a+d\leftrightarrow(b, c)+(e, f)$ and $ad\leftrightarrow(b, c)\ (e, f)$. Thus the ring I_6 is the same ring

as $I_2 \dotplus I_3$ except for the notation employed. It is customary to say that these rings are therefore *isomorphic*. Since the ring of integers modulo a prime is actually a field, we have shown that the ring I_6 is isomorphic to a direct sum of two fields.* It is this kind of result which we now propose to generalize. But first we need to extend our definition of direct sum.

In the definition given above, we limited ourselves to the direct sum of two rings, but it is almost obvious that a corresponding definition can be given for the direct sum of any finite, or even infinite, set of rings. In order to illustrate the general notation to be used presently, let us re-formulate the definition of $R_1 \dotplus R_2$ in a different, but equivalent, way. Let $\mathfrak{N}$ be the set consisting of two elements α_1 and α_2. Then $R_1 \dotplus R_2$ may be considered as the set of all functions $f(x)$ defined on $\mathfrak{N}$, with $f(\alpha_i)$ having values in $R_i (i=1, 2)$, sums and products of functions being defined in the usual way. It is now easy to formulate a general definition of direct sum. Let $\mathfrak{M}$ be an arbitrary set such that to each element α of $\mathfrak{M}$ there corresponds a ring R_α. The set of all functions $f(x)$, defined on $\mathfrak{M}$, such that for every element α of $\mathfrak{M}$, $f(\alpha)$ is in R_α, is a ring which we define to be the *direct sum of the rings* R_α (α *in* $\mathfrak{M}$). It is not assumed that the rings R_α are all distinct. In fact, as a special case of some importance, they may all be identical.

We now state without proof our final theorem.

THEOREM 6. *A commutative ring without nilpotent elements is isomorphic to a subring of a direct sum of fields.*

This theorem was explicitly stated in [5], it being an almost immediate consequence of a theorem of Krull [2] and a principle introduced by Montgomery and the author in [4].

We may remark that, in general, it is not necessarily true that a commutative ring without nilpotent elements is isomorphic to a full direct sum of fields, but only to a subring of such a direct sum. However, it may happen as a special case that the subring consists of the entire ring. This was true in the example of I_6, as we showed it to be isomorphic to $I_2 \dotplus I_3$.

We conclude with an application of Theorem 6 of some interest. Stone [10] has defined a *Boolean ring* as a ring B such that $a^2=a$ for every element a of B. It is easy to show that B is necessarily commutative and also that $a+a=0$ for every a in B. Henceforth, let B denote a fixed Boolean ring. Since $a^2=a$, no element can be nilpotent and thus, by the preceding theorem, there exists a set $\mathfrak{M}$ and fields $F_\alpha(\alpha$ in $\mathfrak{M})$ such that B is isomorphic to a subring of the direct sum of the fields $F_\alpha(\alpha$ in $\mathfrak{M})$. Hence to each element a of B we may make correspond a unique element $f_a(x)$ of this direct sum, and the ring of functions $f_a(x)$ (a in B) is isomorphic to B. Now it can be shown, although we shall omit the proof, that in the case of Boolean rings each of the fields F_α may be taken to be the field

* We may remark that $R_1 \dotplus R_2$ and $R_2 \dotplus R_1$ are clearly isomorphic. Hence in speaking of the direct sum of rings we do not need to specify any definite order in which the rings occur.

I_2 of integers modulo 2; thus it has only two elements 0 and 1. To each element a of B let us now associate the subset $\mathfrak{M}_a$ of $\mathfrak{M}$ consisting of all points α of $\mathfrak{M}$ such that $f_a(\alpha)=1$. If $a\neq b$, clearly $f_a(x)\neq f_b(x)$ and therefore $\mathfrak{M}_a\neq\mathfrak{M}_b$. This correspondence, which we may indicate by $a\leftrightarrow\mathfrak{M}_a$ is clearly a one-to-one correspondence between the elements of B and a certain class of subsets of $\mathfrak{M}$. Since now in I_2, $1+1=0$, it is easy to verify that if $a\leftrightarrow\mathfrak{M}_a$ and $b\leftrightarrow\mathfrak{M}_b$, then ab corresponds to the intersection of $\mathfrak{M}_a$ and $\mathfrak{M}_b$ while $a+b$ corresponds to the set of points of $\mathfrak{M}$ which are in $\mathfrak{M}_a$ or in $\mathfrak{M}_b$ but not in both. A correspondence of this type has been called by Stone a *representation* of the Boolean ring B. We have therefore sketched a proof that every Boolean ring has a representation. A number of different proofs of this fact are to be found in the literature.* For applications of the representation theory, as well as the relation of Boolean rings to the Boolean algebras of logic, reference may be made to a series of papers by Stone, particularly [9] and [10] listed below.

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* See Stone [9], Frink [1] and the further references given in this latter paper.

THE CESÀRO KERNEL TRANSFORMATION

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1. Introduction. In this paper we are concerned with transformations of the type

$$z(s) = \int_0^s k(s, t)x(t)dt, \tag{1.1}$$

where $x(s)$ is bounded and integrable, $0 \leqq s \leqq s_1$, and the *kernel* $k(s, t)$ is integrable in t for each t, $0 \leqq t \leqq s$. The transformation or the kernel is said to be *regular* if $\lim_{s\to\infty} x(s)$ implies the existence of $\lim_{s\to\infty} z(s)$ and the equality of the two limits [1]. In particular, the kernel

$$k(s, t) = \alpha(1 - t/s)^{\alpha-1}/s, \qquad \alpha > 0,$$

defines Cesàro summability (C, α), $\alpha > 0$.

To illustrate the use of regular kernel transformations in assigning values to functions $x(t)$, where $\lim_{t\to\infty} x(t)$ fails to exist, we show that summability $(C, 1)$ assigns the value 0 to the function $x(t) = \sin t$. Indeed, for the example at hand, we have simply

$$z(s) = \frac{1}{s}\int_0^s \sin t\, dt = \frac{1}{s}(1 - \cos s) \to 0 \quad \text{as} \quad s \to \infty.$$

Various sets of regularity conditions for the transformation (1.1) have been given by Agnew [2], Knopp [3], and Silverman [1]. We reproduce here a set of regularity conditions written by Silverman which we find convenient to use in this paper.

THEOREM 1. *Let $k(s, t)$ be defined, $0 < s$, $0 < t < s$, and integrable in t for each s; then sufficient conditions that $k(s, t)$ shall be a regular kernel are*

$$\lim_{s\to\infty} \int_0^s k(s, t)dt = 1, \tag{i}$$

$$\lim_{s\to\infty} k(s, t) = 0 \textit{ uniformly in } t,\ 0 \leqq t \leqq q, \tag{ii}$$

$$\int_0^s k(s, t)dt < A, \qquad 0 < s, \tag{iii}$$

where q is an arbitrary constant and A is a positive constant.

Let us designate the method of summation defined by (1.1) by the symbol (k). It is the object of this paper to determine conditions on a regular kernel $k(s, t)$ in order that $(k) \supset (C, \alpha)$, $\alpha > 0$. It will be evident from the discussion which follows that two cases have to be considered according as α is an integer or is not an integer. Two theorems are obtained and examples given in illustration of them.

2. The non-integral case. The main result of this section is contained in the following theorem.

THEOREM 2. $(k)\supset(C, \alpha)$, $\alpha=n+\beta$, $0<\beta<1$, $(n=0, 1, 2, \cdots)$, *provided that the kernel*

$$K(s, t) = \frac{(-1)^{n+1}t^{n+\beta}}{\Gamma(1-\beta)\Gamma(n+\beta+1)}\frac{\partial^{n+1}}{\partial t^{n+1}}\omega(s, t), \tag{2.1}$$

where

$$\omega(s, t) = \int_t^s \frac{k(s, u)}{(u-t)^\beta}du,$$

is regular, and provided that

$$\frac{(-1)^{n+1}}{\Gamma(1-\beta)\Gamma(n+\beta)}\int_t^s (u-t)^{n+\beta-1}\frac{\partial^{n+1}}{\partial u^{n+1}}\omega(s, u)du = k(s, t) \tag{2.2}$$

is an identity.

The Cesàro transformation for summability (C, α), $\alpha>0$, is

$$y(t) = \frac{\alpha}{t}\int_0^t\left(1-\frac{u}{t}\right)^{\alpha-1}x(u)du. \tag{2.3}$$

We find it convenient to set $\alpha=n+\beta$, $0<\beta\leqq 1$, $(n=0, 1, 2, \cdots)$. There are two cases to be considered according as $\beta=1$ or not. The case $\beta=1$ is the simplest case and will be considered in section 4.

For the case $\beta\neq 1$ we wish to determine conditions on the regular kernel $k(s, t)$ so that $\lim_{s\to\infty}y(s)=l$ implies $\lim_{s\to\infty}z(s)=l$. In this connection the problem of expressing z in terms of y presents itself. We do this formally at first, making all necessary assumptions to determine the *form* of the required transformation. Our formal method of procedure will be to solve (2.3) for x in terms of y and then substitute in (1.1), obtaining z in terms of y.

To solve (2.3) for x in terms of y we begin by writing the equation (2.3) in the form

$$t^{n+\beta}y(t) = (n+\beta)\int_0^t (t-u)^{n+\beta-1}x(u)du, \tag{2.4}$$

and setting

$$v_1(u) = \int_0^u x(w)dw,\ v_2(u) = \int_0^u v_1(w)dw,\ \cdots,\ v_n(u) = \int_0^u v_{n-1}(w)dw.$$

Integrating by parts in (2.4) we get

$$t^{n+\beta}y(t) = (n+\beta)(n+\beta-1)\int_0^t (t-u)^{n+\beta-2}v_1(u)du.$$

After n successive integrations by parts we obtain

$$\frac{\Gamma(\beta)t^{n+\beta}y(t)}{\Gamma(n+\beta+1)} = \int_0^t (t-u)^{\beta-1}v_n(u)du.$$

This equation is now in the form of an Abel's integral equation which we can solve for v_n in terms of y [4] to obtain

$$v_n(t) = \frac{1}{\Gamma(1-\beta)\Gamma(n+\beta+1)} \frac{d}{dt}\int_0^t \frac{u^{n+\beta}y(u)}{(t-u)^\beta}du.$$

Differentiating this equation n times with respect to t we get

$$x(t) = \frac{1}{\Gamma(1-\beta)\Gamma(n+\beta+1)} \frac{d^{n+1}}{dt^{n+1}}\int_0^t \frac{u^{n+\beta}y(u)}{(t-u)^\beta}du. \tag{2.5}$$

Now, eliminating x between (1.1) and (2.5), we have

$$z(s) = \frac{1}{\Gamma(1-\beta)\Gamma(n+\beta+1)}\int_0^s k(s,t)\frac{d^{n+1}}{dt^{n+1}}\int_0^t \frac{u^{n+\beta}y(u)}{(t-u)^\beta}du\,dt,$$

or

$$z(s) = \frac{1}{\Gamma(1-\beta)\Gamma(n+\beta+1)}\int_0^s k(s,t)\int_0^t \frac{D_u^{n+1}\{u^{n+\beta}y(u)\}}{(t-u)^\beta}du\,dt.$$

In order to find the kernel of this transformation we first interchange the order of integration to obtain

$$z(s) = \frac{1}{\Gamma(1-\beta)\Gamma(n+\beta+1)}\int_0^s D_u^{n+1}\{u^{n+\beta}y(u)\}\omega(s,u)du,$$

with $\omega(s, u)$ as previously defined. Integrating by parts $n+1$ times we have finally

$$z(s) = \frac{(-1)^{n+1}}{\Gamma(1-\beta)\Gamma(n+\beta+1)}\int_0^s \frac{\partial^{n+1}}{\partial u^{n+1}}\omega(s,u)\cdot u^{n+\beta}y(u)du. \tag{2.6}$$

It is clear that this is not a valid transformation from y to z unless we impose restrictions of unwarranted severity on the class of functions x which we are considering. Accordingly, we shall look for a means of obtaining (2.6) directly without further limiting the class of bounded and integrable functions x. To this end we eliminate y between (2.3) and (2.6) and attempt to retrace our way to (1.1) without adding to the original restrictions on x. Thus, we have

$$\begin{aligned} z(s) &= \frac{(-1)^{n+1}}{\Gamma(1-\beta)\Gamma(n+\beta)}\int_0^s \frac{\partial^{n+1}}{\partial u^{n+1}}\omega(s,u)\int_0^u (u-t)^{n+\beta-1}x(t)dt\,du \\ &= \frac{(-1)^{n+1}}{\Gamma(1-\beta)\Gamma(n+\beta)}\int_0^s x(t)\int_t^s (u-t)^{n+\beta-1}\frac{\partial^{n+1}}{\partial u^{n+1}}\omega(s,u)du\,dt. \end{aligned}$$

If this transformation is to reduce to (1.1) we must have

$$(2.7)\qquad \frac{(-1)^{n+1}}{\Gamma(1-\beta)\Gamma(n+\beta)}\int_t^s (u-t)^{n+\beta-1}\frac{\partial^{n+1}}{\partial u^{n+1}}\,\omega(s,u)du = k(s,t).$$

Accordingly, to obtain (2.6) directly we must establish (2.7) as an identity.

Finally, the kernel $K(s, t)$ of (2.6) must be regular in order that $\lim_{s\to\infty}y(s)=l$ shall imply $\lim_{s\to\infty}z(s)=l$.

3. An illustration. To illustrate the use of Theorem 2 we shall prove the known result, $(C, \gamma)\supset(C, n+\beta)$, $\gamma>n+\beta$, $0<\beta<1$, $(n=0, 1, 2, \cdots)$.

To prove the identity (2.2) we replace $k(s, t)$ in the left member by the kernel for summability (C, γ) and write

$$\begin{aligned}
&\frac{\gamma(-1)^{n+1}}{s^\gamma\Gamma(1-\beta)\Gamma(n+\beta)}\int_t^s (u-t)^{n+\beta-1}\frac{\partial^{n+1}}{\partial u^{n+1}}\int_u^s (s-t)^{\gamma-1}(t-u)^{-\beta}dt\,du\\
&\qquad=\frac{\Gamma(\gamma+1)(-1)^{n+1}}{s^\gamma\Gamma(\gamma+1-\beta)\Gamma(n+\beta)}\int_t^s (u-t)^{n+\beta-1}\frac{\partial^{n+1}}{\partial u^{n+1}}(s-u)^{\gamma-\beta}du\\
&\qquad=\frac{\Gamma(\gamma+1)}{s^\gamma\Gamma(\gamma-\beta-n)\Gamma(n+\beta)}\int_t^s (u-t)^{n+\beta-1}(s-u)^{\gamma-\beta-n-1}du\\
&\qquad=\frac{\gamma}{s^\gamma}(s-t)^{\gamma-1}.
\end{aligned}$$

It remains to prove that the kernel $K(s, t)$ of the transformation (2.6), with $k(s, t)$ replaced by the kernel for summability (C, γ), is regular. We have

$$\begin{aligned}
K(s,t)&=\frac{\gamma(-1)^{n+1}t^{n+\beta}}{s^\gamma\Gamma(1-\beta)\Gamma(n+\beta+1)}\frac{\partial^{n+1}}{\partial t^{n+1}}\int_t^s (s-u)^{\gamma-1}(u-t)^{-\beta}du\\
&=\frac{(-1)^{n+1}\Gamma(\gamma+1)t^{n+\beta}}{s^\gamma\Gamma(n+\beta+1)\Gamma(\gamma+1-\beta)}\frac{\partial^{n+1}}{\partial t^{n+1}}(s-t)^{\gamma-\beta}\\
&=\frac{\Gamma(\gamma+1)t^{n+\beta}(s-t)^{\gamma-\beta-n-1}}{\Gamma(\gamma-\beta-n)\Gamma(n+\beta+1)s^\gamma}\\
&=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\beta-n)\Gamma(n+\beta+1)}\frac{t^{n+\beta}}{s^{n+\beta+1}}\left(1-\frac{t}{s}\right)^{\gamma-\beta-n-1}.
\end{aligned}$$

It is clear that $\lim_{s\to\infty}K(s, t)=0$ uniformly in t, $0\leqq t\leqq q$, where q is an arbitrary constant. Accordingly, condition (ii) of Theorem 1 is satisfied. Moreover, since

$$\int_0^s K(s,t)dt=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\beta-n)\Gamma(n+\beta+1)s^\gamma}\int_0^s t^{n+\beta}(s-t)^{\gamma-\beta-n-1}dt = 1,$$

conditions (i) and (iii) of Theorem 1 are also fulfilled. This completes the proof of this section.

4. The integral case. The theorem with which this section is occupied is much easier to establish and to apply than Theorem 2.

THEOREM 3. $(k)\supset(C, n)$, $(n=1, 2, 3, \cdots)$, *provided that the kernel*

$$K(s, t) = \frac{(-1)^n}{\Gamma(n+1)} t^n \frac{\partial^n}{\partial t^n} k(s, t) \tag{4.1}$$

is regular, and provided that

$$\left.\frac{\partial^i}{\partial t^i} k(s, t)\right]_{t=s} = 0, \qquad i = 0, 1, 2, \cdots, n-1. \tag{4.2}$$

To prove this theorem we follow almost exactly the procedure of section 2. First of all we set $\alpha=n$ in (2.4) to get

$$t^n y(t) = n \int_0^t (t-u)^{n-1} x(u) du. \tag{4.3}$$

Using the notation of section 2 we obtain, after $n-1$ integrations by parts,

$$t^n y(t) = \Gamma(n+1) \int_0^t v_{n-1}(u) du.$$

After n successive differentiations with respect to t we get

$$x(t) = \frac{1}{\Gamma(n+1)} D_t^n \{t^n y(t)\}. \tag{4.4}$$

Now, eliminating x between (1.1) and (4.4), we have

$$z(s) = \frac{1}{\Gamma(n+1)} \int_0^s k(s, t) D_t^n \{t^n y(t)\} dt.$$

In order to find the kernel of this transformation we first integrate by parts n times; then with the aid of (4.2) we have

$$z(s) = \frac{(-1)^n}{\Gamma(n+1)} \int_0^s t^n \frac{\partial^n}{\partial t^n} k(s, t) y(t) dt. \tag{4.5}$$

Thus, once again, we obtain formally the transformation from y to z. As in section 2 we now look for a direct means of obtaining (4.5). To this end we eliminate y between (4.3) and (4.5) and attempt to retrace our way to (1.1). We have

$$z(s) = \frac{(-1)^n}{\Gamma(n)} \int_0^s \frac{\partial^n}{\partial t^n} k(s, t) \int_0^t (t-u)^{n-1} x(u) du\, dt.$$

Interchanging the order of integration we obtain

$$z(s) = \frac{(-1)^n}{\Gamma(n)} \int_0^s x(u) \int_u^s (t-u)^{n-1} \frac{\partial^n}{\partial t^n} k(s, t) dt\, du.$$

Now, it remains to show that the kernel of this transformation:

$$\frac{(-1)^n}{\Gamma(n)}\int_u^s (t-u)^{n-1}\frac{\partial^n}{\partial t^n}k(s,t)dt, \tag{4.6}$$

reduces to $k(s, u)$. Using (4.2) and integrating by parts $n-1$ times in (4.6) we get

$$-\int_u^s \frac{\partial}{\partial t}k(s,t)dt = k(s,u).$$

Since the steps in these operations are reversible we are thus able to obtain the transformation (4.5) directly without restricting the class of bounded and integrable functions x which we are studying. This completes the proof of Theorem 3.

5. An illustration. To illustrate the use of Theorem 3 we shall check the known result $(C, \gamma)\supset(C, n)$, $\gamma>n$. We observe first of all that the kernel associated with summability (C, γ) satisfies the requirement (4.2) for $\gamma>n$. It remains to prove that the kernel $K(s, t)$, as defined by (4.1), with $k(s, t)$ replaced by the kernel for summability (C, γ), $\gamma>n$, is regular. We set $\gamma=n+\delta$, $0<\delta<1$, and write

$$\begin{aligned} K(s,t) &= \frac{(-1)^n(n+\delta)}{\Gamma(n+1)}\frac{t^n}{s^{n+\delta}}\frac{\partial^n}{\partial t^n}(s-t)^{n+\delta-1} \\ &= \frac{\Gamma(n+\delta+1)}{\Gamma(n+1)\Gamma(\delta)}\frac{t^n}{s^{n+\delta}}(s-t)^{\delta-1} \\ &= \frac{\Gamma(n+\delta+1)}{\Gamma(n+1)\Gamma(\delta)}\frac{t^n}{s^{n+1}}\left(1-\frac{t}{s}\right)^{\delta-1}. \end{aligned}$$

We see by inspection that condition (ii) of the regularity conditions in Theorem 1 is fulfilled. Moreover, we have at once

$$\int_0^s K(s,t)dt = \frac{\Gamma(n+\delta+1)}{\Gamma(n+1)\Gamma(\delta)s^{n-\delta}}\int_0^s t^n(s-t)^{\delta-1}dt = 1.$$

Thus, conditions (i) and (iii) of Theorem 1 are also satisfied.

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THE EULER-DIDEROT ANECDOTE

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No anecdote with regard to a mathematician is better known than the story of the discomfiture of Diderot by Euler. The story was first told by Thiébault [1]; it was later retold, with additions, by De Morgan [2]. Since then a great many authors, all following the more highly colored version of De Morgan, have repeated the story. It is the purpose of this note to show that one addition by De Morgan—the addition which really gives point to the story—is manifestly absurd, and that the credibility of the original story by Thiébault is open to suspicion.

It will be sufficient to give the De Morgan version, noting what he added:

"The following story is told by Thiébault, in his *Souvenirs de vingt ans de séjour à Berlin*, published in his old age, about 1804. This volume was fully received as trustworthy; and Marshall Mollendorff told the Duc de Bassano in 1807 that it was the most veracious of books written by the most honest of men. Thiébault says that he has no personal knowledge of the truth of the story, but that it was believed throughout the whole of the north of Europe. Diderot paid a visit to the Russian court at the invitation of the Empress. He conversed very freely, and gave the younger members of the court circle a good deal of lively atheism. The Empress was much amused, but some of the councillors suggested that it might be desirable to check these expositions of doctrine. The Empress did not like to put a direct muzzle on her guest's tongue, so the following plot was contrived. Diderot was informed that a learned mathematician was in possession of an algebraical demonstration of the existence of God, and would give it to him before all the court, if he desired to hear it. Diderot gladly consented; though the name of the mathematician is not given, it was Euler. He advanced toward Diderot, and said gravely and in a tone of perfect conviction: *Monsieur*, $(a+b^n)/n=x$, *donc Dieu existe*; *répondez*! Diderot, to whom algebra was Hebrew, was embarrassed and disconcerted; while peals of laughter rose on all sides. He asked permission to return to France at once, which was granted."

This differs from the Thiébault account in three respects:

(1) The formula is slightly different; this affects neither the validity of the proof nor the credibility of the story.

(2) It identifies the mathematician as Euler.

(3) The expression "to whom algebra was Hebrew" is an addition. Thiébault says: "Diderot, voulant prouver la nullité et l'ineptie de cette prétendue preuve, mais ressentant malgré lui, l'embarras où l'on est d'abord lorsqu'on découvre chez les autres, le dessein de nous jour, n'avoit pu échapper aux plaisanteries dont on étoit prêt à l'assaillir."

Since then the story has, as I have said, been repeated many times. To cite only two instances—both of them by authors of popular books—we find Hogben

beginning his *Mathematics for the Million* with this story, but with the substitution of "Arabic" for "Hebrew," and Bell in his *Men of Mathematics* giving a modified version, but with "Chinese" for "Hebrew."

That is the story, and it is a very good story, except that it isn't true. To Diderot algebra was neither Hebrew, nor Arabic, nor even Chinese. Diderot was a very good mathematician, and prior to his Russian trip, was the author of five creditable memoirs on mathematics [3]. To mention only one of these, in the second memoir, *Examen de la développante du cercle*, Diderot shows that if, instead of the Euclidean tools of ruler and compass, we assume a circle and its involute, which last is easily constructed mechanically, then the classical problems, trisection of the angle, duplication of the cube, and quadrature of the circle, may be easily and neatly solved. In proving this, Diderot shows a complete mastery of algebra, geometry, and the calculus.

The anecdote as told by De Morgan and by all who have followed him, is thus seen to be absurd. But it may be noted that Thiébault's story is not so unskillful as to aver that Diderot could not reply; it merely says that he sensed the hostility of the audience. The Thiébault story *may* have been true; and the mathematician *may* have been Euler, who was in Russia at that time. What evidence is there for the original story? Thiébault, writing many years later, says merely that the story was believed throughout the north of Europe. No one else tells the story, in particular there is no known Russian source for the story. On the other hand it is known that Frederick the Great, King of Prussia, was a bitter enemy of Diderot. Several sources indicating that stories about Diderot at Saint Petersburg emanated from Berlin are cited and summarized by Tourneux [4].

The alternatives seem to be, first, a rather pointless incident as told by Thiébault; second, and more probable, a canard, inspired by Frederick the Great or by his courtiers.

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JACOBIAN CIRCLES OF THE BIQUADRATIC

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1. Introduction. The transformations of inversive geometry consist of all homographies $w=(az+b)/(cz+d)$ and all antigraphies $\bar{w}=(az+b)/(cz+d)$, where, in each, $ad-bc\neq 0$, the letters represent complex numbers, and $\bar{w}$ is the conjugate of w. A homography of period 2 is a *polarity*; and an antigraphy of period 2 is an *inversion* [1].

Any inversion may be put in the form

$$(A):\quad a_{11}z\bar{w}+a_{10}z+a_{01}\bar{w}+a_{00}=0, \tag{1.1}$$

where $a_{ik}=\bar{a}_{ki}$, *i.e.*, a_{ii} is real. In the Argand plane an inversion sets up a (1, 1) correspondence between points z and w of the plane; and two corresponding points constitute a couple $[z, w]$ which represents a complex point on the locus defined by (1.1). We shall say that *the couple lies on the locus.* Since the equation (1.1) is self-conjugate, *i.e.*, real, the conjugate couple $[w, z]$ is also on the locus. The locus of a bilinear equation in z and $\bar{w}$ is called a circle, and when the equation is self-conjugate, as above, the locus is a real circle with or without a real trace [1]. If the circle (a) has a real trace, its real points z are represented by self-conjugate couples $[z, z]$ which satisfy the equation

$$a_{11}z\bar{z}+a_{10}z+a_{01}\bar{z}+a_{00}=0.$$

This equation is also the locus of fixed points, if any, of the inversion (1.1).

The circle (a) is a *null* circle if and only if $a_{10}a_{01}-a_{11}a_{00}=0$; and the condition that two circles (a) and (a') be orthogonal [2] is

$$a_{11}a'_{00}+a_{00}a'_{11}-a_{10}a'_{01}-a_{01}a'_{10}=0. \tag{1.2}$$

A *biquadratic* is a curve which determines four couples with any circle. If any one of the couples is self-conjugate, it represents a real point of intersection of the two curves. The general equation of the biquadratic, therefore, is of the second degree in z and $\bar{w}$, and we write it

$$a_{22}z^2\bar{w}^2+2a_{21}z^2\bar{w}+2a_{12}z\bar{w}^2+a_{20}z^2+4a_{11}z\bar{w}+a_{02}\bar{w}^2+2a_{10}z+2a_{01}\bar{w}+a_{00}=0,$$

or, in the so-called matrix form,

$$(B):\quad \begin{array}{c|ccc} & 1 & 2z & z^2 \\ \hline 1 & a_{00} & a_{10} & a_{20} \\ 2\bar{w} & a_{01} & a_{11} & a_{21} \\ \bar{w}^2 & a_{02} & a_{12} & a_{22} \end{array} = 0. \tag{1.3}$$

This equation is real if $a_{ik}=\bar{a}_{ki}$, *i.e.*, a_{ii} is real; and then the determinant $|a_{ik}|$ is Hermitian and is an inversive invariant of (B). We assume in the sequel that the equation of (B) is real.

Interpreted as a (2, 2) correspondence between points z and w of the Argand plane, equation (1.3) associates two image points w with every point z, and reciprocally.

2. Bipolar theory. If ξ is a given point and x its harmonic conjugate with respect to the images of w, as given by equation (1.3), we have a (1, 2) correspondence between points x and w of the plane, viz.,

$$(2.1) \qquad \begin{array}{c|ccc} & 1 & \xi + x & \xi x \\ \hline 1 & a_{00} & a_{10} & a_{20} \\ 2\bar{w} & a_{01} & a_{11} & a_{21} \\ \bar{w}^2 & a_{02} & a_{12} & a_{22} \end{array} = 0,$$

which associates one image point x with every point w; and with every point x two images w. Consider now the harmonic conjugate y of a second given point η (which may coincide with ξ) with respect to the two images of x as given by equation (2.1). This sets up a (1, 1) correspondence between the points x and y of the plane, viz.,

$$(2.2) \qquad (\sigma): \quad \sigma_{11}x\bar{y} + \sigma_{10}x + \sigma_{01}\bar{y} + \sigma_{00} = 0,$$

where,

$$\begin{aligned} \sigma_{11} &= a_{11} + a_{12}\bar{\eta} + a_{21}\xi + a_{22}\xi\bar{\eta} \\ \sigma_{10} &= a_{10} + a_{11}\bar{\eta} + a_{20}\xi + a_{21}\xi\bar{\eta} \\ \sigma_{01} &= a_{01} + a_{02}\bar{\eta} + a_{11}\xi + a_{12}\xi\bar{\eta} \\ \sigma_{00} &= a_{00} + a_{01}\bar{\eta} + a_{10}\xi + a_{11}\xi\bar{\eta}. \end{aligned}$$

The equation (2.2) of the antigraphy between points x and y is also the equation of a complex circle associated with the couple $[\xi, \eta]$, and this circle is real when the couple is self-conjugate, *i.e.*, $\eta = \xi$.

DEFINITION 1. The *bipolar circle* (σ) of a couple $[\xi, \eta]$, with respect to the biquadratic, is the locus defined by equation (2.2).

The bipolar circle (σ) of a given couple $[\xi, \eta]$ is a linear combination of four circles, viz.,

$$(2.3) \qquad s_{11}\xi\bar{\eta} + s_{10}\xi + s_{01}\bar{\eta} + s_{00} = 0,$$

where the s_{ik} are the same functions of x and $\bar{y}$ that the σ_{ik} are of ξ and $\bar{\eta}$. Conversely, since any circle (c) may be written as a linear combination of these same four circles $s_{ik} = 0$, i.e.,

$$(c): \quad s_{11}k_{11} + s_{10}k_{10} + s_{01}k_{01} + s_{00}k_{00} = 0,$$

provided that

$$\Delta_4 = \begin{vmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ a_{10} & a_{11} & a_{20} & a_{21} \\ a_{01} & a_{02} & a_{11} & a_{12} \\ a_{00} & a_{01} & a_{10} & a_{11} \end{vmatrix} \neq 0,$$

such a circle (c) will be the bipolar circle of a couple $[\xi, \eta]$ if and only if

$$k_{11}k_{00} - k_{10}k_{01} = 0,$$

and then

$$\xi = k_{10}/k_{00} \quad \text{and} \quad \eta = \bar{k}_{01}/\bar{k}_{00}.$$

Consequently, every couple has a bipolar circle but, reciprocally, every circle is *not* the bipolar circle of a couple.

If $[x, y]$ is a couple on (σ), the bipolar of $[\xi, \eta]$, then the bipolar of $[x, y]$ may be written

$$s_{11}z\bar{w} + s_{10}z + s_{01}\bar{w} + s_{00} = 0,$$

which, by virtue of equation (2.3) is satisfied by $[\xi, \eta]$. This proves,

THEOREM 1. *The bipolar circles of all couples on the bipolar circle of a couple* $[\xi, \eta]$ *pass through (i.e., are orthogonal to)* $[\xi, \eta]$.

THEOREM 2. *The locus of couples whose bipolar circles are orthogonal to a given circle is a circle.*

This theorem follows from Definition 1 and equation (1.2). If the given circle is

$$(\alpha): \quad \alpha_{11}z\bar{w} + \alpha_{10}z + \alpha_{01}\bar{w} + \alpha_{00} = 0,$$

the locus of couples whose bipolars are orthogonal to (α) is the circle (β), where

$$\beta_{ik} = a_{ik}\alpha_{11} - a_{i,k+1}\alpha_{10} - a_{i+1,k}\alpha_{01} + a_{i+1,k+1}\alpha_{00}, \qquad (i,\ k = 0,\ 1). \tag{2.4}$$

DEFINITION 2. The *bipolar circle* (β) of a circle (α), with respect to the biquadratic, is the locus of couples whose bipolars are orthogonal to (α).

We observe that this definition includes Definition 1, for when (α) is the null circle $[\xi, \eta]$, *i.e.*, (α) has the equation

$$(z - \xi)(\bar{w} - \bar{\eta}) = 0,$$

then

$$\alpha_{11} = 1, \qquad \alpha_{10} = -\bar{\eta}, \qquad \alpha_{01} = -\xi, \qquad \alpha_{00} = \xi\bar{\eta},$$

and $\beta_{ik} = \sigma_{ik}$. Hence, equations (2.4) constitute a (1, 1) transformation of circles (including null circles) in the plane of the biquadratic.

THEOREM 3. *The bipolars of couples on a circle are orthogonal to a circle.*

For, if $[\xi, \eta]$ lies on (α), it may be shown that (σ), the bipolar of $[\xi, \eta]$, is orthogonal to a circle (γ), where (γ) bears to (α) the same relation that (α) bears to its bipolar (β). Thus,

DEFINITION 3. The *antibipolar circle* (γ) of a circle (α), with respect to the biquadratic, is the circle orthogonal to the bipolars of couples on (α).

THEOREM 4. *If (β) is the bipolar of (α), then (α) is the antibipolar of (β).*

3. The Jacobian circles. Since harmonic and orthogonal properties are invariant under transformations of the inversive group, the bipolar and antibipolar relations implicit in equations (2.4) are invariant. That is, if (β) is the bipolar of a circle (α) with respect to a biquadratic (B) and if an inversive transformation sends (β), (α), and (B) into (β'), (α'), and (B') respectively, then (β') is the bipolar of (α') with respect to (B').

Inversions are fundamental in that any inversive transformation is the product of a finite number of inversions. Consequently, we are interested in knowing whether there are any inversions which transform a given biquadratic into itself [3, 4]. If (α) is the circle associated with an inversion I_α which transforms the biquadratic (B) into itself, (B) is said to be *anallagmatic* with respect to the circle (α). If there is such a circle (α), its bipolar (β) with respect to (B) is transformed into itself by I_α. Moreover, by Definition 2, the bipolar of every couple on (β) is orthogonal to (α) and therefore transforms under I_α into itself. In fact, not only is (β) unchanged by I_α but so is every circle orthogonal to (α); hence, every couple of (β) is unchanged by I_α. Since, however, the only couples which are invariant under I_α are the couples on (α), it follows that (β) coincides with (α). Conversely, we can show that if (α) coincides with (β), then (B) is anallagmatic with respect to (α). Thus,

THEOREM 5. *A biquadratic is anallagmatic with respect to a given circle if and only if that circle is identical with its bipolar with respect to the biquadratic.*

From equations (2.4) and the condition that (α) is identical with (β), i.e., $\beta_{ik} = \mu\alpha_{ik}$ where μ is a proportionality factor, we have the coefficients of (α) given by four equations like the following:

$$(a_{11} - \mu)\alpha_{11} - a_{12}\alpha_{10} - a_{21}\alpha_{01} + a_{22}\alpha_{00} = 0. \tag{3.1}$$

These equations may be solved for the α_{ik} when μ is so chosen that D, the determinant of the coefficients a_{ik}, vanishes. Since

$$D \equiv \mu^4 + \Delta_2\mu^2 + \Delta_3\mu + \Delta_4 = 0,$$

there are four such values of μ.* The four values of μ are distinct when the biquadratic has distinct foci; two are coincident when the biquadratic has a node; three are coincident when the biquadratic has a cusp [1].

* The Δ_i are invariants of the biquadratic and bear the following relations to those given in reference [2]:

$$\Delta_2 = -I_2,\ \Delta_3 = -2I_3,\ \Delta_4 = (I_2^2 - I_4)/4.$$

Equations (3.1), with $D=0$, express the condition that (α) is orthogonal to each of four circles like

$$a_{22}z\bar{w} + a_{21}z + a_{12}\bar{w} + (a_{11} - \mu) = 0, \tag{3.2}$$

where μ is a root of $D=0$. Now, three circles (a), (b), and (c) have a common orthogonal circle whose equation is

$$\begin{vmatrix} a_{11} & a_{10} & a_{01} & a_{00} \\ b_{11} & b_{10} & b_{01} & b_{00} \\ c_{11} & c_{10} & c_{01} & c_{00} \\ 1 & -\bar{w} & -z & z\bar{w} \end{vmatrix} = 0$$

if and only if the matrix of the coefficients a_{ik}, b_{ik}, c_{ik} is of rank three. If this matrix is of rank two, the three circles are linearly dependent and have a one-parameter family of orthogonal circles; and if the rank is one, the three circles are identical and have a two-parameter family of orthogonal circles.

Thus, with respect to the circles of (3.2), for each value of μ, we consider the three possibilities:

(1) *D is of rank* 3. The circle (α) is then uniquely determined as the common orthogonal circle of three of the four circles given by (3.2). So determined, (α) is also orthogonal to the fourth circle, since $D=0$.

(2) *D is of rank* 2. The four circles of (3.2) form a coaxal system, and (α) is any circle of the orthogonal coaxal system.

(3) *D is of rank* 1. The four circles of (3.2) are identical, and (α) is any circle of the two-parameter family of orthogonal circles.

There are, therefore, four circles (α) with respect to each of which the general biquadratic is anallagmatic; and certain special biquadratics are anallagmatic with respect to one- and two-parameter families of circles. Moreover, when the rank of D is three, an inversion with respect to any one of the circles transforms each of the others, as well as the biquadratic, into itself. Therefore,

THEOREM 6. *The general biquadratic is anallagmatic with respect to four mutually orthogonal circles.*

The four mutually orthogonal circles of the theorem are called the *Jacobian circles* J_i of the biquadratic. One of them, at least, has no real trace.

4. Inversive transformations which leave a biquadratic invariant. With each Jacobian circle there is associated an inversion I_i which transforms the biquadratic into itself; hence, in the general case, there are four inversions leaving the biquadratic invariant.

Now, it is known that the product of two inversions in orthogonal circles is commutative and is a polarity. The fixed points of this polarity are the common couples of the two circles; that is, the common points of intersection if the circles intersect, or the common inverse points if the circles do not intersect.

Hence, the four inversions I_i, associated with the Jacobian circles, generate six polarities $P_{ik}=I_iI_k$ which leave the biquadratic invariant. Since, however, the Jacobian circles are mutually orthogonal, the common couples of two of them, say J_1 and J_2, are identical with the common couples of the other two, J_3 and J_4. Thus,

$$P_{12} = P_{34}, \qquad P_{13} = P_{24}, \qquad P_{14} = P_{23},$$

and the six polarities are not distinct, but are equivalent in pairs.

The product of two polarities with distinct fixed points is, in general, a homography whose fixed points are the common harmonic conjugates of the two pairs of fixed points of the component polarities. In the case at hand, however, the product of any two of the three distinct polarities P_{12}, P_{13}, P_{14} is the third. For example,

$$P_{12}P_{13} = I_1I_2I_1I_3 = I_2I_1I_1I_3 = I_2I_3 = P_{23} = P_{14},$$

since a product such as I_1I_2 is commutative, and a repeated inversion such as I_1I_1 is the identity transformation. Hence, there are three, and only three, polarities which leave the biquadratic invariant.

The product of an inversion and a polarity is, in general, an antigraphy. Here, however, nothing new is obtained for the antigraphy turns out to be one of the four inversions I_i. For example,

$$I_1P_{23} = I_1P_{14} = I_1I_1I_4 = I_4.$$

To summarize, the Jacobian circles may be found by the method of Section 3, and then the inversive transformations which leave a biquadratic invariant are known: they consist of (1) four inversions in mutually orthogonal circles, (2) three polarities, and (3) the identity transformation. These eight elements constitute the abelian group G_8 of type (1, 1, 1).

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MATRIX METHODS IN THE SOLUTION OF ALGEBRAIC EQUATIONS*

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Although the theory of solution of algebraic equations has received its rightful share of attention on the part of mathematicians in the past, and even perhaps more than its due, we must not forget that this theory gave rise to a large number of elegant developments without which mathematics, and particularly modern algebra, would not be what it is today. For this and other reasons we feel justified in making another contribution to this already highly cultivated field. By means of the method to be given here the solution of a system of algebraic equations which arise in the determination of multiple roots of algebraic equations is reduced to the study of a single algebraic equation which can be more easily manipulated. This algebraic equation is related to the apolar of a binary form so extensively treated in the literature.†

1. Roots. The algebraic equation,

$$f(x) = k_0a_0x^n + k_1a_1x^{n-1} + \cdots + k_{n-1}a_{n-1}x + k_na_n = 0, \qquad a_0 \neq 0, \tag{1}$$

where k_i is the coefficient of x^i in the expansion of $(1+x)^n$, and the a's are real or complex numbers,‡ has a root α repeated at least r times if and only if

$$f(x) = 0, f'(x) = 0, f''(x) = 0, \cdots, f^{[r-1]}(x) = 0. \tag{2}$$

A value of x which satisfies (2) will also satisfy the equations

$$\begin{aligned} t_0a_0x^{n-r+1} &+ t_1a_1x^{n-r} + \cdots + t_{n-r+1}a_{n-r+1} = 0,\\ t_0a_1x^{n-r+1} &+ t_1a_2x^{n-r} + \cdots + t_{n-r+1}a_{n-r+2} = 0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ t_0a_{r-1}x^{n-r+1} &+ t_1a_rx^{n-r} + \cdots + t_{n-r+1}a_n \qquad = 0, \end{aligned} \tag{3}$$

where t_i is the coefficient of x^i in the expansion of $(1+x)^{n-r+1}$. The equivalence of the systems (2) and (3) can be proved by eliminating $x^n, \cdots, x^{n-r+2}$ successively from the equations in (2).

The system (3) is a system of equations linear in $t_0x^{n-r+1}, t_1x^{n-r}, \cdots, 1$ with matrix of coefficients a_{n-r+1} given by

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* An address delivered before the Mathematical Association of America at the invitation of the Program Committee at Chicago, September 2, 1941.

† E. B. Elliott, Algebra of quantics, pp. 260–267. The polynomial $F_i(x)$, which will appear later in this paper, is an apolar.

‡ The a's may belong to any field k with the property that the a's in $f(x)$ are uniquely determined by $f(x)$. If the characteristic of k is greater than n the field k has this property.

$$A_{n-r+1} = \left\| \begin{matrix} a_0 & a_1 \cdots a_{n-r+1} \\ a_1 & a_2 \cdots a^{n-r+2} \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ a_{r-1} & a_r \cdots a_n \end{matrix} \right\|.$$

Instead of the system (3) we consider the system of equations

$$(4) \qquad \begin{aligned} a_0\xi_0 + a_1\xi_1 + \cdots + a_{n-r+1}\xi_{n-r+1} &= 0, \\ a_1\xi_0 + a_2\xi_1 + \cdots + a_{n-r+2}\xi_{n-r+1} &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{r-1}\xi_0 + a_r\xi_1 + \cdots + a_n\xi_{n-r+1} \qquad &= 0, \end{aligned}$$

linear in $\xi_0, \cdots, \xi_{n-r+1}$ with the matrix of coefficients A_{n-r+1}. For the system (3) to have a solution $x=\alpha$, it is necessary that $A_{n-r+1}\xi=0$ have a solution $\xi\neq 0$, where

$$\xi = \left\| \begin{matrix} \xi_0 \\ \xi_1 \\ \cdot \\ \cdot \\ \cdot \\ \xi_{n-r+1} \end{matrix} \right\|.$$

We introduce the polynomial $F_{n-r+1}(x)$ given by

$$F_{n-r+1}(x) = \xi_0 - \xi_1 x + \xi_2 x^2 - \cdots \pm \xi_{n-r+1} x^{n-r+1},$$

where the ξ's are the components of the general solution of (4). Some of the coefficients in the polynomial $F_{n-r+1}(x)$ may be arbitrary in which case $F_{n-r+1}(x)$ is really a linear family of polynomials. In particular, if A_{n-r+1} has rank t, by the theory of linear equations, there are $n-r+1-t$ arbitrary components in the solution ξ of $A_{n-r+1}\xi=0$.

There is a solution $x=\alpha$ of (3) if and only if we can choose the coefficients in $F_{n-r+1}(x)$ so that

$$F_{n-r+1}(x) = t_0\alpha^{n-r+1} - t_1\alpha^{n-r}x + \cdots \pm t_{n-r+1}x^{n-r+1} = (\alpha - x)^{n-r+1}.$$

We have arrived at the following theorem where "r-fold" means "repeated at least r times."

THEOREM 1. *The equation* (1) *has an r-fold root* α *if and only if the coefficients in* $F_{n-r+1}(x)$ *can be chosen so that* α *is an* $(n-r+1)$*-fold root of* $F_{n-r+1}(x)=0$.

The determination of the roots of (1) and the multiplicities of these roots thus reduces to a study of the polynomials F_i for the various values of i.

2. Matrices and polynomials associated with (1). For each i in the range $0, 1, \cdots, n$ there is a matrix A_i of $f(x)$. Since for each pair of values of i and j the matrices A_i and A_j contain the same elements, it is to be hoped that the

ranks of these matrices are related in a simple way. Such a relation does exist and will now be given.

Let m be the smallest value of i for which $F_i(x)$ is not identically zero. We state a theorem whose proof will appear elsewhere.*

THEOREM 2. *The ranks of A_i and A_{n-i} are equal to $i+1$ for $i<m$ and to m for $i \geqq m$, $(i \leqq n/2)$.*

Thus m determines the ranks of the matrices $A_0, A_1, \cdots, A_n$. From the dimensions of the matrices, $A_0, \cdots, A_n$, it follows that

$$m \leqq 1 + n/2. \tag{5}$$

Since A_m has $n-m+1$ rows the rank of A_m is m when $n-m+1 \geqq m$; that is, $m<1+n/2$. If $m=1+n/2$, the rank of A_m is $m-1$. It follows that if m satisfies the condition, $m<1+n/2$, there is one arbitrary coefficient in $F_m(x)$. In the remaining case there are two arbitrary coefficients.

THEOREM 3. *For each $i>m$ we can choose the coefficients in $F_i(x)$ so that*

$$F_i(x) = F_m(x)g(x),$$

where $g(x)$ is the general polynomial of degree $i-m$.

By "general polynomial" we mean a polynomial with arbitrary coefficients.

We write $F_m(x)$ and $g(x)$ as $\eta_0-\eta_1x+\cdots\pm\eta_mx^m$, $\beta_0+\beta_1x+\cdots+\beta_{i-m}x^{i-m}$ respectively. We introduce M_r and L_r, where

$$M_r = \left\|\begin{matrix} a_r & \cdots & a_{r+m} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ a_{r+n-i} & \cdots & a_{r+m+n-i} \end{matrix}\right\|, \qquad L_r = \left\|\begin{matrix} 0 \\ \cdot \\ \cdot \\ 0 \\ \eta_0 \\ \cdot \\ \cdot \\ \eta_m \\ 0 \\ \cdot \\ \cdot \\ 0 \end{matrix}\right\|,$$

r zeros preceding η_0 in L_r. Now $A_iL_r=M_r\eta$. Since M_r is a minor of A_m, and $A_m\eta=0$, it follows that $M_r\eta=0$, whence $A_iL_r=0$. Thus the linear combination

$$\xi = \beta_0L_0 - \beta_1L_1 + \beta_2L_2 - \cdots \pm \beta_{i-m}L_{i-m} \tag{6}$$

of the L's is a solution of $A_i\xi=0$ for each choice of the β's. Since for each j the vector ξ is the column vector whose jth component is $(-1)^j$ times the jth coefficient in the polynomial $F_m(x)g(x)$, it follows that $F_m(x)g(x)\equiv F_i(x)$ or $F_m(x)g(x)$ is a specialization of $F_i(x)$.

* Factorization and representation, Transactions of the American Mathematical Society.

THEOREM 4. *For each i in the range,*

$$n - m + 1 \geqq i \geqq m, \tag{7}$$

the polynomial $F_i(x)$ is identically $F_m(x)g(x)$ where $g(x)$ is the general polynomial of degree $i-m$.

We need but consider the case $i \neq m$ since if $i = m$ we have $F_i(x) \equiv F_m(x)$.

By Theorem 2 the rank of A_i is m, while A_i has $i+1$ columns. It follows from the theory of linear equations that, given A_i, there are $i+1-m$ linearly independent solutions of $A_i\xi = 0$. This means that there are $i+1-m$ arbitrary components in the solution ξ of $A_i\xi = 0$, and thus $i+1-m$ arbitrary coefficients in $F_i(x)$. When $n-m+1 \geqq m$, we have $m < 1+n/2$, whence $F_m(x)$ has one arbitrary coefficient which may be taken equal to 1. Since the general polynomial $g(x)$ of degree $i-m$ has $i-m+1$ arbitrary coefficients, in the product $F_m(x)g(x)$ there are $i-m+1$ arbitrary coefficients. That is, the vector ξ of coefficients in $F_m(x)g(x)$ given in (6) is the general linear combination of $i-m+1$ linearly independent vectors $L_0, \cdots, L_{i-m}$. By the theory of linear equations the general solution of $A_i\xi = 0$ is (6) and

$$F_i(x) \equiv F_m(x)g(x). \tag{8}$$

For i not in the range (7) the identity (8) is no longer valid, although by Theorem 3 the coefficients in $F_i(x)$ can be chosen so that $F_i(x) = F_m(x)g(x)$.

3. The roots of (1), and the number m. The number m yields a bound on the multiplicities of the roots of (1) as is clear from the following theorem.

THEOREM 5. *The equation* (1) *has an r-fold root α, $r \geqq m$, if and only if $F_m(x) = 0$ has α as an m-fold root.*

Since $i = n-r+1$ is in the range (7), by Theorem 4 we cannot have $F_i(x) = (x-\alpha)^i$ unless $F_m(x) = (x-\alpha)^m$. If $F_m(x) = (x-\alpha)^m$, by (8) we have $F_i(x) = (x-\alpha)^i$ when we choose $g(x) = (x-\alpha)^{i-m}$.

COROLLARY 1. *The equation* (1) *has at most one r-fold root where $r \geqq m$.*

If $m < 1+n/2$ Corollary 1 follows from Theorem 5 and the property that $F_m(x) = kH(x)$, where $H(x)$ is free of arbitrary coefficients.

If $m = 1+n/2$, since $2m = n+2$, we cannot have two m-fold roots.

COROLLARY 2. *If* (1) *has α as an r-fold root, $r \geqq m$, the root α is an* $(n-m+1)$-*fold root of* (1).

Corollary 2 is a consequence of Theorems 4 and 5.

COROLLARY 3. *The equation* (1) *has no* $(n-m+1)$-*fold root.*

If (1) has a root α of multiplicity at least $n-m+2$, for some $i < m$ we have $F_i(x) = (x-\alpha)^i \neq 0$, a contradiction.

COROLLARY 4. *If* (1) *has an r-fold root we have* $m \leqq n-r+1$.

If (1) has an r-fold root, we have $F_{n-r+1}(x) = (x-\alpha)^{n-r+1} \not\equiv 0$.
It can be proved that if $r \geqq n/2$ we have $m = n-r+1$.

4. Equations with only one root. In view of Theorem 1 we are concerned with equations possessing only coincident roots. The following theorem is useful in the study of these equations.

THEOREM 6. *The equation* (1) *has coincident roots if and only if* A_1 *has rank* 1.

If A_1 has rank 1, for some value α we have $a_i = -\alpha a_{i-1}$. It follows that $f(x) = a_0(x-\alpha)^n$. The converse follows readily.

In place of (1) we consider $F_i(x) = 0$. For $F_i(x) = 0$ there is the matrix

$$\left\|\begin{array}{cc} \xi_0 & \dfrac{-\xi_1}{t_1} \\ \dfrac{-\xi_1}{t_1} & \dfrac{\xi_2}{t_2} \\ \cdots & \cdots \\ \dfrac{\mp \xi_{i-1}}{t_{i-1}} & \pm \xi_i \end{array}\right\|, \tag{9}$$

corresponding to A_1 for (1), where t_j is the coefficient of x^j in the expansion of $(1+x)^i$. Since $\xi_i \neq 0$ when $F_i(x) = (x-\alpha)^i$, we have $F_i(x) = (x-\alpha)^i$ if and only if all rows of (9) are multiples of the last. This means that

$$\xi_i^{i-j-1}\xi_j = t_j \bar{\xi}_{i-1}^{i-j}, \qquad j = 0, 1, \cdots, i-2, \tag{10}$$

with $\bar{\xi}_{i-1} = \xi_{i-1}/t_{i-1}$. The relations (10) and $A_i\xi = 0$ obviously yield a system of equations equivalent to (3) with $i = n-r+1$.

5. Illustration. We consider (1) where

$$f(x) = x^6 - 2x^5 - x^4 + 4x^3 - x^2 - 2x + 1.$$

To avoid fractions we replace $f(x)$ by $15f(x)$. Much of the development below is given, not because it is needed to determine the roots of (1) and their multiplicities, but rather to illustrate each of the main points in the theory of earlier sections.

The matrix A_3 is given by

$$\left\|\begin{array}{rrrr} 15 & -5 & -1 & 3 \\ -5 & -1 & 3 & -1 \\ -1 & 3 & -1 & -5 \\ 3 & -1 & -5 & 15 \end{array}\right\|.$$

The rank of A_3 is 3, whence $m=3$. By Theorem 5 the equation (1) does not have a 3-fold root unless $F_3(x)=0$ has such a root. For this reason we proceed to compute $F_3(x)$. Solving $A_3\xi=0$ for ξ we obtain

$$\xi = k \begin{Vmatrix} 1 \\ 3 \\ 3 \\ 1 \end{Vmatrix},$$

whence with $k=-1$ we have $F_3(x)=(x-1)^3$. By Theorem 1 the value 1 is a 4-fold root of (1). The matrix A_4, given by

$$\begin{Vmatrix} 15 & -5 & -1 & 3 & -1 \\ -5 & -1 & 3 & -1 & -5 \\ -1 & 3 & -1 & -5 & 15 \end{Vmatrix}$$

has rank 3 in view of Theorem 2. By Theorem 4 we have $F_4(x)\equiv(x-1)^3(\beta_0+\beta_1 x)$, where the β's are arbitrary. Thus the solution ξ of $A_4\xi=0$ is given by

$$\xi = \beta_0 \begin{Vmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 0 \end{Vmatrix} - \beta_1 \begin{Vmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 1 \end{Vmatrix}.$$

Solving $A_5\xi=0$, where

$$A_5 = \begin{Vmatrix} 15 & -5 & -1 & 3 & -1 & -5 \\ -5 & -1 & 3 & -1 & -5 & 15 \end{Vmatrix},$$

we obtain $\xi_1=\xi_2-2\xi_4+5\xi_5$, $5\xi_0=2\xi_2-\xi_3-3\xi_4+10\xi_5$. We take $\xi_5=-1$. Solving the two relations between the ξ's, just obtained, with (10), we derive a quartic equation E in ξ_4. By Theorem 3 we can restrict the ξ's so that $F_5(x)=(x-1)^5$, whence E has the solution $\xi_4=-5$. The other root of E is $\xi_4=5$, in which case $F_5(x)=(x+1)^5$. By Theorem 1 the equation (1) has -1 as a double root. Thus the roots of (1) are 1, 1, 1, 1, -1, -1.

TOPOLOGY OF THE TWO-BODY PROBLEM*

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1. Introduction. The differential equations of the two-body problem of celestial mechanics can be written as four first order equations,

$$dx_i/dt = X_i(x_1, x_2, x_3, x_4), \qquad (i = 1, 2, 3, 4),$$

which represent a flow in 4-space. If the energy constant is fixed, the trajectories fill out a three-dimensional manifold in the 4-space. This manifold we term the *energy phase space.*

It is the purpose of the present paper to answer the following questions: 1. What is the topological† structure of the energy phase space? 2. If the area constant is also fixed, what is the topological structure of the two-dimensional subspace of the energy phase space thereby determined? (This surface will be called the *energy-area phase space.*) 3. As the area constant varies, how do the surfaces of question 2 fill out the energy phase space? 4. How do the individual trajectories fill out each energy-area phase space? 5. What is the topological structure of the whole family of trajectories in the energy phase space?

Throughout the energy constant will be assumed to be negative, so that the paths are elliptical.

Poincaré was the first to consider such questions. Recently Birkhoff has considered similar questions for the three-body problem and for general dynamical systems. Exact references are given in the bibliography.

The answers to the above questions will be given in detail below. In brief they are as follows: 1. The energy phase space has the structure of 3-space minus a line. 2. The energy-area phase space has (in general) the structure of a torus. 3. Imagine the paraboloid of revolution $x^2+y^2=z-1$ to be cut by the family of all planes through the y-axis. This gives a curve-family consisting of ellipses plus one parabola and two points as limiting cases. Imagine the paraboloid mapped homeomorphically on the half-plane $y>0$ of the yz-plane, so that the above curve family becomes one filling the half-plane. (See Fig. 2.) Now imagine the half-plane rotated about the z-axis. The curves of the family sweep out surfaces which are in general of the type of the torus. This family of surfaces is the answer to question 3. 4. When the energy-area space is not degenerate and is hence like a torus, the family of trajectories is the same as a family of "parallel" circles on a torus. 5. Assume one curve of the family in the yz-plane in the answer to question 3 to be the positive y-axis. As the yz-plane is rotated each curve of the family in $z\neq 0$ sweeps out a surface like a torus. Now imagine each point of each such curve to move and make one complete revolution of the curve as the curve makes one complete rotation. The revolution is to follow the posi-

* Presented to the American Mathematical Society in Chicago, September 5, 1941.

† A topological property means one invariant under a homeomorphism. A homeomorphism is a transformation which is one-to-one and continuous in both directions. See, for example, Seifert and Threlfall, Lehrbuch der Topologie, Leipzig, 1934, especially Chapter I.

tive orientation of the curve for $z>0$ and the negative orientation for $z<0$. Thus a family of trajectories is formed on the family of torus-like surfaces. The remainder of the trajectories are two circles, obtained from the two point-curves and the family of rays from the origin of the xy-plane. This gives a rough picture of a model of the family of trajectories in the energy phase space.

2. The differential equations and integrals. If the origin of coordinates is taken at one of the two bodies and units are properly chosen, the differential equations for the motion of the second body can be written:

$$d^2x/dt^2 = -x/r^3, \qquad d^2y/dt^2 = -y/r^3, \qquad r = \sqrt{x^2+y^2}. \tag{1}$$

(See Wintner [3], p. 178.)

These have integrals

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{2}{r} + c_1, \tag{2}$$

$$x\frac{dy}{dt} - y\frac{dx}{dt} = c_2, \tag{3}$$

where c_1 is termed the *energy constant*, c_2 the *area constant*.

We rewrite (1) as four first order equations thus:

$$dx/dt = z, \qquad dy/dt = w, \qquad dz/dt = -x/r^3, \qquad dw/dt = -y/r^3. \tag{4}$$

These represent a vector field in 4-dimensional $xyzw$-space. For $x=y=0$ the differential equations break down. We shall therefore delete the plane $x=y=0$ from the 4-space. In the remaining space (4) then defines a non-singular family of trajectories.

If (4) is used, (2) and (3) become

$$z^2 + w^2 = 2/r + c_1, \tag{5}$$

$$xw - yz = c_2. \tag{6}$$

For fixed c_1, (5) represents a 3-dimensional variety E in $xyzw$-space such that every trajectory of (4) which meets E lies wholly in E. The same holds for the variety A defined by (6), when c_2 is fixed. We term E the *energy phase space* and A the *area phase space*. If both c_1 and c_2 are fixed, the intersection of E and A is a two-dimensional variety EA which is again built up of whole trajectories.

Equations (1) or (4) have a further integral, which is usually simplified by means of (2) and (3). In its general form the third integral can be written

$$\theta - \cos^{-1}\left(\frac{r-(xw-yz)}{r\sqrt{(xw-xz)^2(z^2+w^2-2/r)+1}}\right) = c_3 \tag{7}$$

where θ is the polar coördinate angle in the xy-plane.

If c_1 and c_2 are fixed, (7) reduces to

$$r = \frac{c_2^2}{1 - \sqrt{1 + c_1 c_2^2}\cos(\theta - c_3)} \tag{8}$$

which shows that the solutions project on the xy-plane as conic sections. The eccentricity e satisfies

$$e^2 = 1 + c_1 c_2^2. \tag{9}$$

We shall assume throughout that c_1 is negative, so that $e<1$ and the motion is elliptical. Furthermore we assume that c_2 satisfies

$$c_2^2 < -1/c_1 \tag{10}$$

in order to obtain real solutions.

If $c_2=0$, (8) breaks down, but (7) gives $\theta=c_3$, so that the solutions project on straight lines. Furthermore it follows from (2) that, since c_1 is negative, the path in the xy-plane must lie in the region $0<r\leqq -2/c_1$.

3. Topology of the energy phase-space and of the area phase-space.

THEOREM 1. *E is homeomorphic with 3-space minus a straight line.*

Proof: Since the points of the zw-plane are deleted from the 4-space, the transformation

$$r' = 1/r, \qquad \theta' = \theta, \qquad z' = z, \qquad w' = w, \tag{11}$$

which is simply an inversion in the unit circle in the xy-plane, is a homeomorphism of the deleted 4-space on itself. (5) becomes

$$z'^2 + w'^2 = 2r' + c_1. \tag{5'}$$

Now since θ' does not appear in (5′), the hypersurface which it represents is a hypersurface of revolution. The 2-surface from which it is obtained by revolution can be found by plotting (5′) in $z'w'r'$-space, where r' is now a rectangular coordinate satisfying $r'\geqq 0$. But in that space (5′) represents a paraboloid of revolution not meeting the plane $r'=0$. Thus in 4-space (5′) represents a hypersurface obtained from the paraboloid by revolution about the $z'w'$-plane. Since the paraboloid is homeomorphic with the interior of a half-plane, it follows that the hypersurface is homeomorphic to one obtained by revolving the interior of a half-plane about its boundary line. This gives a 3-space minus a straight line as desired.

In terms of topological products* the hypersurface (5′) is paraboloid$\times$circle

* The topological product of two geometrical objects α and β can here be defined as follows: Suppose α to lie in $(x_1, \cdots, x_n)$-space and β to lie in $(y_1, \cdots, y_m)$-space. The product $\alpha\times\beta$ consists of those points of $(x_1, \cdots, x_n, y_1, \cdots, y_m)$-space whose x-coordinates give a point of α and whose y-coordinates give a point of β. Any object homeomorphic with $\alpha\times\beta$ as thus defined is also termed the topological product $\alpha\times\beta$. Examples: square = line segment$\times$line segment; torus = circle$\times$circle.

or line$\times$line$\times$circle, where line = infinite straight line.

COROLLARY. *If $c_2 \neq 0$, A is homeomorphic with 3-space minus a line.*

Proof: First note that for $c_2 \neq 0$, A contains none of the deleted points of $x=0$, $y=0$.

Suppose $c_2<0$. By rotations in the xw-plane and yz-plane (6) becomes

$$x''^2 - w''^2 - y''^2 + z''^2 = 2c_2. \tag{12}$$

Let ρ, ϕ be polar coordinates in the $y''w''$-plane. (12) becomes

$$x''^2 + z''^2 = \rho^2 + 2c_2. \tag{13}$$

The transformation

$$x''' = x'', \qquad z''' = z'', \qquad \rho''' = \rho^2, \qquad \phi''' = \phi$$

is a homeomorphism of the 4-space. (13) becomes

$$x'''^2 + z'''^2 = \rho''' + 2c_2. \tag{14}$$

Since $2c_2<0$, this is the same as (5′), and the corollary follows as above.

The same method applies if $c_2>0$, but it breaks down if $c_2=0$. If $c_2=0$, it can be shown that A (with the plane $x=0$, $y=0$ excluded) has still the same topology.

4. The energy-area phase space. Suppose now that c_1 and c_2 are fixed ($c_1<0$). Consider then the intersection EA of E and A.

THEOREM 2. *If $0<c_2^2<-1/c_1$, then EA is homeomorphic with the surface of a torus. If $c_2=0$, EA is homeomorphic with the interior of an annulus. If $c_2^2=-1/c_1$, EA is homeomorphic to a circle.*

Proof: Use the coordinates r', θ', z', w' of §3. The equations of EA are

$$\begin{aligned} z'^2 + w'^2 &= 2r' + c_1, \\ w' \cos\theta' - z' \sin\theta' &= c_2 r'. \end{aligned} \tag{15}$$

Suppose now θ' fixed, so that we can plot in $r'z'w'$-space. Then (15) gives the intersection of a paraboloid and a plane. The plane passes through the origin and makes an angle of arc sin $c_2/\sqrt{1+c_2^2}$ with the r'-axis. Its line of intersection with the $w'z'$-plane makes an angle of θ' with the z'-axis. Thus as θ' varies from 0 to 2π the plane starts from the position $w'=c_2r'$ and makes a complete revolution, in the direction from the positive z'-axis to the positive w'-axis, about the r'-axis. Since the paraboloid is one of revolution about the r'-axis, the type of intersection will be independent of θ'. For $\theta'=0$ it reduces to the curve,

$$z'^2 + w'^2 = 2r' + c_1, \qquad w' = c_2 r', \tag{16}$$

whose projection on the $z'w'$-plane is

$$z'^2 + \left(w' - \frac{1}{c_2}\right)^2 = \frac{1 + c_1 c_2^2}{c_2^2}. \tag{17}$$

Thus, if $1+c_1c_2^2>0$ and $c_2\neq 0$, the intersection is an ellipse. As θ' varies from 0 to 2π the ellipse rotates about the $z'w'$-plane and finally returns to its original position. This gives a surface with the structure of a torus. In fact we can immediately introduce coordinates (θ', ϕ) on the torus, where ϕ denotes position on the ellipse (15), for fixed θ'. The coordinate ϕ can be specifically defined thus: Project the ellipse on the $z'w'$-plane; let Q^* be the center of the image and O^* the origin of the $z'w'$-plane. Assume the $z'w'$-plane is oriented so that angles from

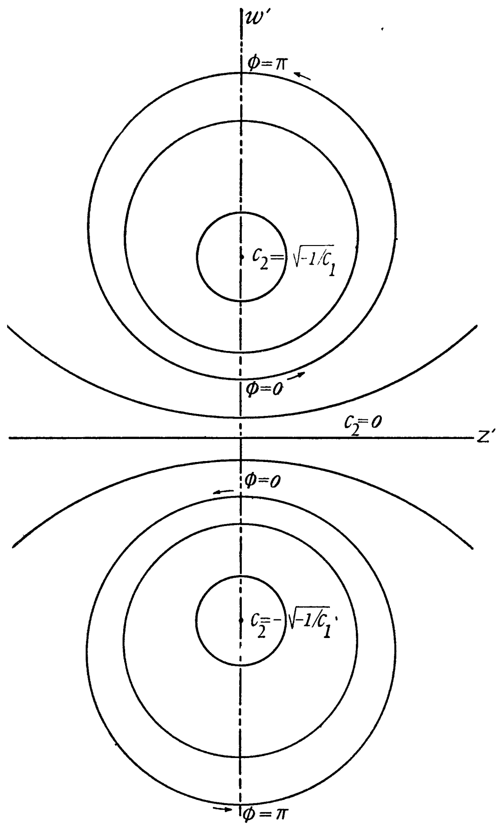

FIG. 1

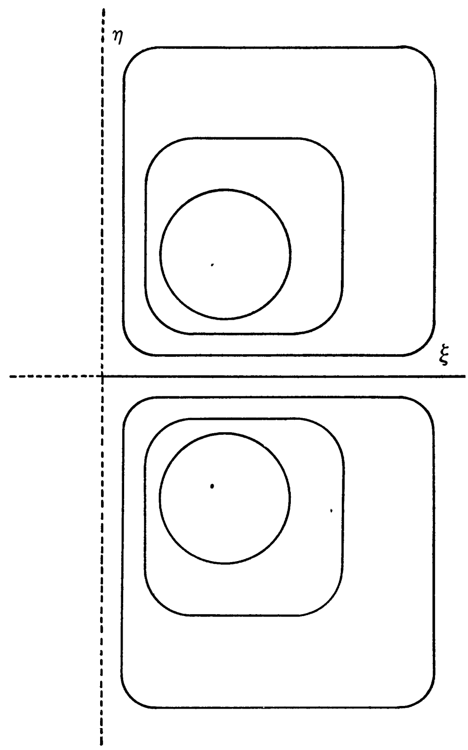

FIG. 2

the positive z'-axis to the positive w'-axis are counted positive. Let ϕ be the angle from Q^*O^* to Q^*P^* for any point P^* on the image. This gives the coordinate ϕ for the point P above P^*.

If $c_2=0$, the ellipse (15) becomes the parabola

$$z'^2 = 2r' + c_1, \quad w' = 0.$$

As θ' varies, this generates the topological product of a parabola with a circle. Since a parabola is homeomorphic with an open interval, the surface here is homeomorphic with the interior of an annulus.

If $1+c_1c_2^2=0$, the ellipse (15) degenerates to the point $(0, 1/c_2, 1/c_2^2)$. As θ' varies this generates a simple closed curve. (Note that this occurs for $c_2=\sqrt{-1/c_1}$ or $c_2=-\sqrt{-1/c_1}$.) Thus Theorem 2 is established.

5. Decomposition of the energy phase space. The trajectories (4), for fixed c_1, form a curve-family filling E. For c_2 fixed, the trajectories form a curve-family filling EA. As c_2 varies, the surfaces EA sweep out E. Thus they decompose E into surfaces, each of which is a collection of trajectories.

The range of c_2 is from $-\sqrt{-1/c_1}$ to $\sqrt{-1/c_1}$. At the two extremes EA reduces to a closed curve. In between EA is topologically a torus, except for $c_2=0$, when EA is homeomorphic with the interior of an annulus. Now E itself is homeomorphic with 3-space minus a line. The question to be considered is how the family of surfaces EA fills out E.

This is immediately answered by considering the picture of §4. There we held θ' fixed and considered the locus in $r'w'z'$-space. If c_2 varies over its interval $[-\sqrt{-1/c_1}, \sqrt{-1/c_1}]$, the paraboloid is decomposed as indicated in Figure 1. In that figure $\theta'=0$ and the paraboloid is viewed from along the r'-axis. The picture changes with varying θ' only by a rotation about the origin. In the figure the ellipses are drawn as circles, but from a topological point of view this is no specialization.

As θ' varies from 0 to 2π the configuration pictured is multiplied topologically by a circle. The resulting family of surfaces can be pictured topologically in three-space if we first map the $w'z'$-plane of Figure 1 on a half-plane $\xi>0$ of the $\xi\eta$-plane. This gives the picture of Figure 2. Now rotate the family of curves pictured about the η-axis. This gives a family of tori (with degenerate cases) filling three-space minus a line as desired.

6. Trajectories on each torus EA. It now remains to consider how the trajectories are distributed on each torus. The trajectories have projections on the xy-plane given by (8). For fixed c_1, c_2, c_3 with $c_1<0$, $c_2\neq 0$ (8) represents an ellipse. If c_3 is allowed to vary, the ellipse is rotated about the origin.

If we make the transformation (11), so that the ellipse is inverted in a circle, (8) becomes

$$r' = \frac{1-\sqrt{1+c_1c_2^2}\cos(\theta'-c_3)}{c_2^2} \tag{8'}$$

which is the equation of a limaçon. The important properties of this curve for the following are that r' is a single-valued function of θ', and that as θ' goes from 0 to 2π, r' has exactly two extrema, one maximum and one minimum.

Now (8′) is the projection of a trajectory which is on a torus EA, given by (15). The part of EA in a 3-space $\theta'=$constant is an ellipse, as in Figure 1. For each θ', (8′) gives just one r'; hence there is exactly one point $P_{\theta'}$ of the trajec-

tory on each ellipse $\theta' =$constant of (15). As θ' varies, $P_{\theta'}$ must then vary on the torus. The trajectory can thus be written as

(19) $$\phi = f(\theta') \qquad (f(\theta' + 2\pi) \equiv f(\theta') \bmod 2\pi).$$

Furthermore all the trajectories on the same torus will have the same equation as (19), except for a translation of θ' ,as follows from (8'); that is, the family of trajectories is given by

(20) $$\phi = f(\theta' + \alpha) \qquad 0 \leqq \theta' \leqq 2\pi, \qquad 0 \leqq \alpha \leqq 2\pi.$$

A direct study of the curve (8') on the torus shows that the function f is monotone strictly decreasing if $c_2 > 0$ and monotone strictly increasing if $c_2 < 0$. Hence

$$f(\theta' + 2\pi) - f(\theta') = 2k\pi, \qquad |k| \geqq 1,$$

where k is an integer and $kc_2 < 0$ ($c_2 \neq 0$). If $|k|$ were greater than 1, r' would have more than one maximum, since r' has a maximum every time ϕ crosses a value $(2n+1)\pi$ ($n = 0, \pm 1, \pm 2, \cdots$). But r' has just one maximum, and hence $|k| = 1$. Thus each curve (20) has an increase (if $c_2 < 0$) or a decrease (if $c_2 > 0$) in ϕ of 2π for every circuit.

From the monotone character of f we conclude that the family (20) can also be written

(21) $$\theta = g(\phi) - \alpha, \qquad 0 \leqq \phi \leqq 2\pi, \qquad 0 \leqq \alpha \leqq 2\pi. \qquad (\theta' = \theta).$$

Such a family of curves on the torus is homeomorphic with a family obtained from the family of generators of a finite cylinder $x^2 + y^2 = 1$, $0 \leqq z \leqq 1$ by identifying boundary points with the same xy-coordinates. The appearance of a twist of 2π has to do solely with the way the torus is imbedded in E.

7. The degenerate cases of EA. If $c_2 = 0$, EA is homeomorphic to the interior of an annulus. In this case the trajectories have projections $\theta' =$constant on the $x'y'$-plane. The part of EA above a line $\theta' =$constant is a parabola such as (18). It follows that the parabola itself is the trajectory. The family of trajectories is thus homeomorphic with the family of lines $\theta' =$constant of the annulus $1 < r < 2$.

If $c_2 = \pm\sqrt{-1/c_1}$, EA reduces to a closed curve. In each of these two cases (8') reduces to a circle. Thus the two closed curves EA are trajectories, each of which projects on a circle.

8. The family of trajectories in the energy phase space. We have seen how E is decomposed into tori and how the trajectories are placed on each torus. It remains to give a picture of the whole collection of trajectories in E. This can be seen from Fig. 2. Imagine that figure rotated around the η-axis in a $\xi\eta\zeta$-space. As each ellipse traces out its torus, the trajectories through it trace out the paths given as above by $\theta = g(\phi) - \alpha$, where θ denotes the angle through which the figure has been rotated. The function g will depend on the ellipse chosen, which in turn depends on c_2. Thus we write $\theta = g(\phi; c_2) - \alpha$ as the equation of the family.

For $c_2=0$, the trajectories are given by the rays from the origin in the $\xi\zeta$-plane. For $c_2=\pm\sqrt{-1/c_1}$ the trajectories are two circles, one above and one below the $\xi\zeta$-plane. Since the trajectories must fit together as do the solution of a differential equation, we must have

$$\lim_{c_2\to 0,\phi\to 0}\frac{\partial\theta}{\partial\phi}=0, \qquad \lim_{c_2\to\pm\sqrt{-1/c_1}}\frac{\partial\theta}{\partial\phi}=\pm\infty.$$

It can be proved that this description completely characterizes the family from a topological point of view; *i.e.*, if two such families, F_1 and F_2, are given, both of which satisfy the above description, then there is a homeomorphism of $\xi\eta\zeta$-space (minus the η-axis) on itself which transforms F_1 onto F_2.

9. Introduction of time parameter. Thus far in considering the trajectories we have neglected the time parameter. But work of Whitney* has shown that from the topological point of view the parameter is almost completely determined by the family of trajectories.

One point of interest however is how the directions of motion orient the family of trajectories. For $c_2>0$, the motions are known to be direct $(d\theta/dt>0)$ and for $c_2<0$ the motions are retrograde $(d\theta/dt<0)$ This implies that in the above picture of the family, the motions for $c_2>0$ are in the direction of increasing θ, those for $c_2<0$ are in the direction of decreasing θ, and those for $c_2=0$ are away from the origin.

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3. A. Wintner, The Analytical Foundations of Celestial Mechanics, Princeton, 1941.

* See Hassler Whitney, Regular families of curves, Annals of Mathematics, Vol. 34, 1933, pp. 244–270, esp. pp. 269–270.

DISCUSSIONS AND NOTES

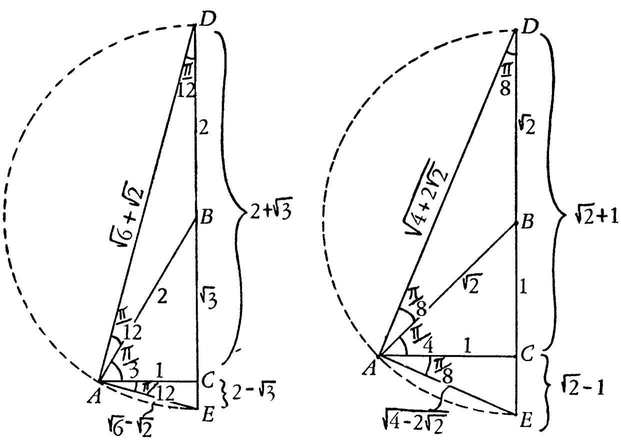
EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

VALUES OF THE TRIGONOMETRIC RATIOS OF $\pi/8$ AND $\pi/12$

H. L. DORWART, Washington and Jefferson College

Although every textbook in trigonometry begins with the determination of the exact values of the trigonometric ratios of $\pi/3$, $\pi/4$ and $\pi/6$ from appropriate triangles, all texts that the writer has seen reserve the exact values of $\pi/8$ and $\pi/12$ until after the functions of sum and difference and half angles have been derived. Since the exact values of the ratios of these latter angles can be



just as useful at the beginning of a trigonometry course as those of the former, the following simple construction* may be of some interest.

In each figure, starting with the basic triangle ABC, a semicircle of radius AB is described on BC (extended) with B as a center. ABD is thus an isosceles triangle and angle DAE is a right angle.

From the right triangle DAE, we have

$$\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}, \qquad \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2},$$

$$\cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}, \qquad \cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2},$$

* The idea of making such a construction was suggested by one of my students, Mr. R. L. Mills.

from the right triangle ACE

$$\tan\frac{\pi}{12} = 2 - \sqrt{3}, \qquad \tan\frac{\pi}{8} = \sqrt{2} - 1,$$

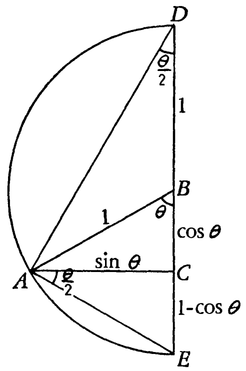
$$\sec\frac{\pi}{12} = \sqrt{6} - \sqrt{2}, \qquad \sec\frac{\pi}{8} = \sqrt{4 - 2\sqrt{2}},$$

and from the right triangle ACD

$$\cot\frac{\pi}{12} = 2 + \sqrt{3}, \qquad \cot\frac{\pi}{8} = \sqrt{2} + 1,$$

$$\operatorname{cosec}\frac{\pi}{12} = \sqrt{6} + \sqrt{2}, \qquad \operatorname{cosec}\frac{\pi}{8} = \sqrt{4 + 2\sqrt{2}}.$$

Note by the Editor. Professor Dorwart's construction suggests a simple derivation of the half-angle formulas for angles less than 180°. It is used thus occa-



sionally in texts in trigonometry. The accompanying figure is self-explanatory, and we see that

$$\begin{aligned}\tan \theta/2 &= AC/CD = \sin\theta/(1 + \cos\theta)\\ &= EC/AC = (1 - \cos\theta)/\sin\theta.\end{aligned}$$

The other half-angle formulas are easily obtained from these. R. J. W.

A SIMPLE GEOMETRICAL PARADOX

J. L. Coolidge, Harvard University

1. The general form for the equation of a quadric surface is

$$\Sigma a_{ij}x^i x^j = 0, \qquad i, j = 1, \cdots, 4, \qquad a_{ij} = a_{ji}.$$

Here we have ten independent coefficients a_{ij}. If we require the surface to pass through a given point, we impose a linear homogeneous condition on these

coefficients. Hence we may pass a single surface through nine arbitrary points, or, what amounts to the same thing,

Ten general points in space do not lie on a quadric surface.

2. A quadric surface may be generated, after the procedure of Seydewitz, by the intersection of the lines of a bundle with the planes of a second bundle projectively related to it by a correlation. The surface will pass through the centers of these two bundles. A correlation between two bundles is essentially the same thing as a correlation between two planes. This may be expressed

$$\Sigma b_{ij}x^iy^j = 0, \qquad i, j = 1, 2, 3, \qquad |b_{ij}| \neq 0.$$

This tells us that the point (y) correspond to a line through a point (x). There are nine independent coefficients b_{ij}, hence we may set up a correlation in which eight general points correspond to lines through eight other general points. Or we may set up a correlation between two bundles so that eight general lines of one correspond to planes through eight general lines of the other.

Now let A, B, $C_1, \cdots, C_8$ be ten general points of space. We may set up such a correlation that the eight lines $AC_1, \cdots, AC_8$ correspond to planes through the eight lines $BC_1, \cdots, BC_8$.

Ten general points in space lie on a quadric surface.

Which is right?

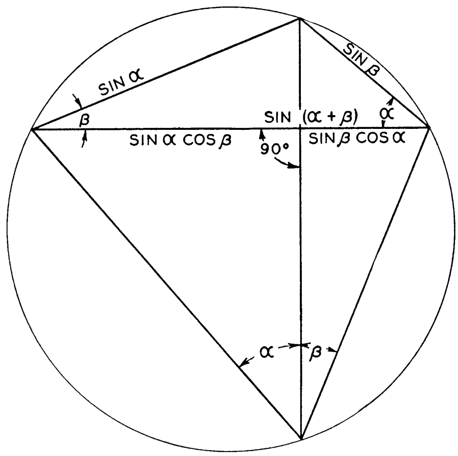
THE ADDITION FORMULAS IN TRIGONOMETRY

A. S. Householder, University of Chicago

Cauchy's derivation of the addition formulas in trigonometry, given by Hobson in his *Treatise* and presented recently by McShane, is general and shorter than the standard proof given by most texts. The proof suggested by the accompanying figure (the circle has unit diameter but the figure is otherwise self-explanatory) is admittedly not general and moreover requires a preliminary lemma to the effect that the length of a chord in a circle of unit diameter is equal to the sine of the subtended inscribed angle. Nevertheless, this lemma follows almost immediately from standard high school geometry theorems relating to inscribed angles, and is otherwise useful in providing a simple proof for the law of sines. Once the lemma is established, the addition formula for the sine is immediately evident from this figure, for the case α, β, and $\alpha+\beta$ numerically less than 180°, and no algebraic manipulation is required. The same figure is easily adapted to the cosine formula.

This proof can hardly be new. Its relation to Ptolemy's theorem is too close, and that these formulas are special cases of Ptolemy's theorem is well known. But it does not seem to have found its way into the standard texts, though even if a single general proof is to be demanded it should be pedagogically a worthy adjunct to this.

Note by the Editor. Professor E. F. Beckenbach also has pointed out that McShane's method of deriving the addition formulas is essentially due to



Cauchy, and that this derivation, along with a history of these formulas, appears in Enciclopedia delle Matematiche Elementari, Milano, 1937, vol. 2, part 1, pp. 551–552.

R. J. W.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

College Algebra. Second Edition. By H. P. Pettit and P. Luteyn. New York, John Wiley and Sons, Inc. 1941. 14+247 pages. $1.90.

The first edition of this fine text was reviewed in the MONTHLY, vol. 40, 1933, pp. 288–289, with but few criticisms.

The present edition still embodies the noteworthy feature of introducing the student to new materials immediately, such as functions, graphs, and the summation notation, thereby reviewing indirectly a considerable amount of elementary algebra. This is followed by the usual topics of a college algebra text, but arranged so that the ideas and notations are readily accessible for later use and developing in the student "a greater capacity for independent thought." A chapter on probability, omitted in the first edition, has been included and makes for a well-rounded text.

Especially good are the many problems which are of a practical value and related to the various sciences. These in themselves provide the student with

much worth-while information and integrate his experiences in mathematics with those of the natural sciences. Whole chapters are presented on problem solving, and answers are given for the odd-numbered examples.

Few errors were noted. Minor mistakes were found on pages 27, 57, 119, 129, and 246, the most confusing being the writing of triangle OMA on page 149, instead of triangle ONA.

The book is well written, and the typography is extremely pleasing to the eye, beginning with an attractive bookplate and concluding with exponential and logarithmic tables.

R. A. HARRISON

Intermediate Algebra. By Neil McArthur and Alexander Keith. London, Methuen and Company, 1942. 10+356 pages. Price 8/6.

As stated by the authors "This book is intended for use in school and first-year university classes composed of students who are not (or not yet) specializing in mathematics."

The introduction, which is a brief review of the number system, is well done.

As compared with United States texts on college algebra, this work assumes a somewhat better preparation than is given in most of the secondary schools in the United States. Most of our college algebras include a chapter on Interest and Annuities which is not included in this text. They also usually give solution of determinants of any order while only second and third order determinants are considered here. Also the approximation of irrational roots of equations of higher degree are usually considered while this is not included. As a whole, however, this treatment of the subject is more complete than is usual in similar texts in the United States. Their treatment of inequalities and surds is also more exhaustive.

In addition to the usual topics discussed in our texts the following are included in this book: work on symmetrical functions and reciprocal equations; the treatment of sequences and series includes Arithmetico-Geometrical series and series in which the nth term is a polynomial in n; an excellent discussion of graphs is given: symmetry and continuity are discussed, also two-point and three-point contact of curves, and maxima and minima; the idea of derivative is introduced in connection with the slope of the tangent to the curve at a point, and the equation of this tangent; the slope is also determined by the method of limits. Also, the multinomial theorem, that is $(a+b+c+\cdots)^n=\sum_{r=0}^{n} {}_nC_r a^r(b+c+\cdots)^{n-r}$ is given in connection with the binomial theorem; arithmetical series of higher order, and expansion in power series are considered.

The book seems to me to be well written. The explanations are clear and adequate; illustrative examples are sufficient in number; and there is a wealth of examples and problems for class use.

SARA L. NELSON

The Mathematics of Finance. By Llewellyn Rood Perkins and Ruth Marion Perkins. New York, John Wiley and Sons, Inc., 1941. 20+321 pp. $3.25.

This is a well written textbook, covering in a precise and lucid manner the conventional fields of the so-called mathematics of finance. According to the preface, it is directed to the student who already has a good grounding in algebra. Hence, with the exception of a discussion of rounded multiplication and division, no explanations of algebraic and other techniques are given. Some instructors, therefore, may miss the usual explanation of such elementary questions as logarithms or an appendix of log-tables. It should be stressed, however, that a decided asset of the book is a very complete set of tables which—especially in the field of life insurance—go far beyond similar textbooks.

The authors duly emphasize points of connection with actual financial practice and it seems to the reviewer that they go successfully farther in this direction than many others (*e.g.*, in chapter VII on building and loan associations).

As far as the method is concerned, it is noteworthy for its rather unusual employment of graphical representation, somewhat similar to the manner of exposition more commonly found in economic textbooks. This should prove helpful, especially to students not primarily trained in mathematics and who, otherwise, might have difficulties to grasp the analytical pattern of thought directly.

For a revised edition, the following minor suggestions might be made. The increasing mechanization of business analysis might justify the addition, in an appendix, of a short paragraph on the appropriate arrangement of data for and the use of calculating machines (as in most textbooks of economic statistics). Perhaps also, a hint at the technique of handling special slide rules, and their adaptation to some of the standard problems in the mathematics of finance would be useful. Many students are surprisingly unfamiliar with such manipulations. While admiring the technical excellence of this as well as of most American textbooks as compared with much less elaborately equipped European books, the present reviewer cannot suppress the suggestion to print tables on an extended hinge. This would enable the student to have the table in sight while reading the text and would also make it possible to print some of the larger tables in a more comprehensive way.

The book should certainly be a valuable help to the teacher as well as to the student.

J. E. Morton

The Trisection Problem. By R. C. Yates. Baton Rouge, La., The Franklin Press, Inc., 1942. 68 pages. $1.00.

This little book has for its purpose the explanation of the trisection problem, and to show why it is impossible to solve it by straight edge and compasses. No knowledge of mathematics beyond plane geometry, elementary algebra, and the rudiments of trigonometry and of plane analytic geometry is presupposed.

It is first shown that the solution of the problem depends on that of a reduced cubic equation $x^3-3x-2a=0$ and that a construction based on these instruments leads only to certain values of a. This part is well done. It is then shown that an angle of the form $2\pi/n$, n an integer, can or cannot be trisected by straight edge and compasses according as n is not or is a multiple of 3. It is not shown under what restrictions on n such angles can be constructed.

The second chapter discusses a number of curves which intersect a given circle in the required points and adds the remark that infinitely many such curves exist, none of which can be constructed by the instruments allowed. This chapter is followed by one on mechanical trisectors, based on linkages, which is rather extensive. This is followed, in turn, by one on approximations. A brief historical note is added to many sections and a bibliography is given of works referred to in them. It makes no claim at being complete.

The back fly-leaf is a museum of newspaper clippings on "solutions." May this book do its part to quell the flood of trisectors.

VIRGIL SNYDER

NEW BOOKS RECEIVED

Elementary Mathematics in Artillery Fire. By J. M. Thomas. (With Tables prepared by Vincent H. Haag.) New York and London, McGraw-Hill Book Company, Inc., 1942. 11+256 pages. $2.50.

The Trisection Problem. By R. C. Yates. Baton Rouge, La., The Franklin Press, Inc., 1942. 68 pages.

Plane and Spherical Trigonometry. By P. R. Rider. New York, The Macmillan Company, 1942. 7+180 pages. $1.75.

An Introduction to Analytic Geometry and Calculus. By T. K. Raghavachari. Madras, Humphrey Milford and Oxford University Press, 1941. 8+192 pages. Rs 2.

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT AND J. S. FRAME

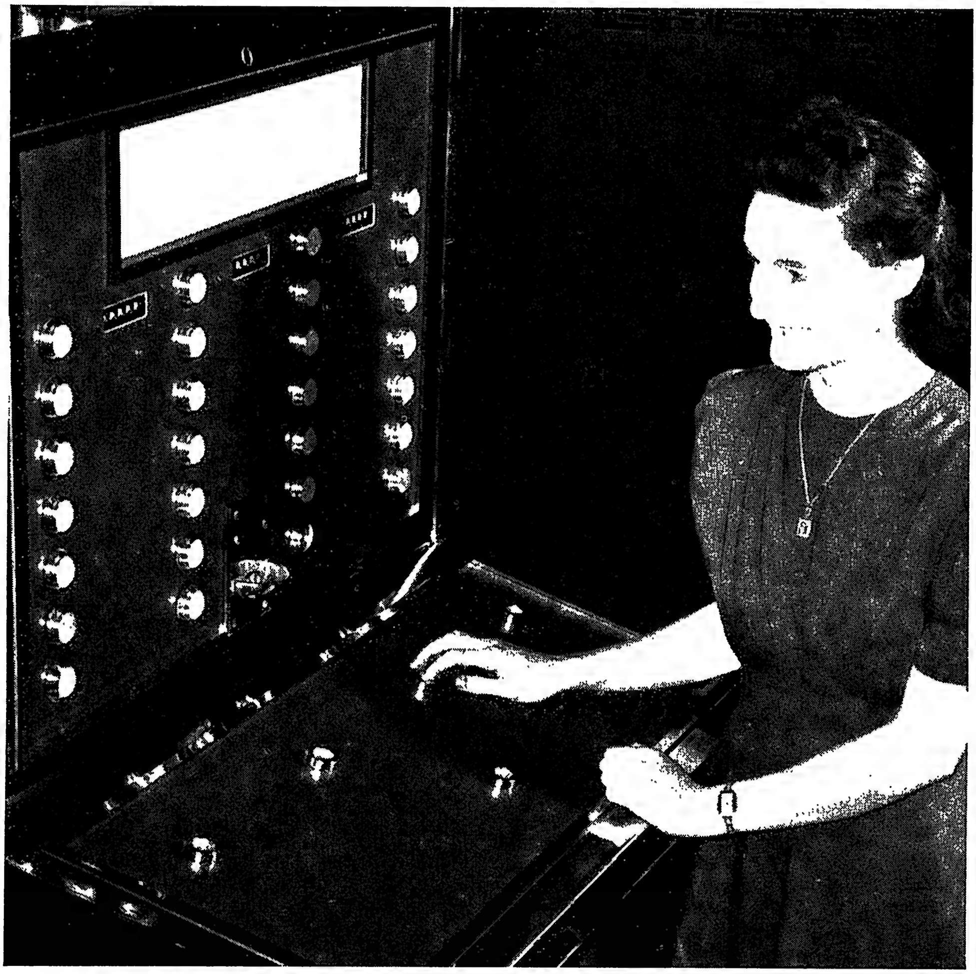
Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Brown University, Providence, R. I.

THE NIMATRON

E. U. CONDON, Westinghouse Electric and Manufacturing Co.

The Nimatron is a machine which is very skillful at playing the game of Nim. Unlike other mathematical machines, the Nimatron serves no other useful purpose than to entertain, unless it be to illustrate how a set of electrical relays can be made to make a "decision" in accordance with a fairly simple mathematical procedure.

The machine was built in the spring of 1940 and was exhibited at the Westinghouse Building of the New York World's Fair, where it played more than 100,000 games and won 90,000 of them. Most of its defeats were at the hands of the exhibit attendants as demonstrations to folks who, after numerous trials, became convinced that the machine couldn't be beaten. Now it belongs to the scientific collections of the Buhl Planetarium in Pittsburgh. It was invented by



two members of the staff of the Westinghouse Research Laboratories during a long lunch hour, and considerably improved by one of the engineers of the switchgear department of the East Pittsburgh Works of the Westinghouse Electric and Manufacturing Company, where it was designed and built. The Nimatron made its last "personal appearance" at the convention of the Allied Social Science Associations in New York City under the sponsorship of the American Statistical Association and the Institute of Mathematical Statistics.

Full details of the circuit diagrams together with a detailed description are given in U.S. Patent Number 2,215,544, obtainable from the U.S. Office in Washington, D.C.

Editorial Note. The theory of the game of Nim is due to C. L. Bouton, *Annals of Mathematics*, ser. II, vol. 3, 1901, p. 35. It was recently discussed by D. P. McIntyre, this MONTHLY, vol. 49, 1942, p. 44.

CLUB REPORTS 1940–41

PI MU EPSILON NATIONAL CONVENTION

The triennial convention of the chapters of *Pi Mu Epsilon* was held in Lamberton Hall of Lehigh University on Thursday, January 1, 1942 at 12:30 P.M. Twenty-nine members and friends representing ten chapters were present. Following the luncheon a short business meeting was held with Professor Shook of Lehigh presiding. Professor Shook called the roll by chapters and members present were introduced. A partial report for the nominating committee was made by Professor Owens. Dr. Moses Richardson of the Department of Mathematics of Brooklyn College addressed the convention on *Some aspects of freshman mathematics*, listing some of the difficulties which freshmen experience and making suggestions as to their solution. The convention closed at 2:00 P.M. with a vote of thanks to the Department of Mathematics of Lehigh University and Professor Cutler, chairman of the committee on arrangements.

Mathematics Club, Massachusetts Institute of Technology

Five meetings were held during the year at which the following topics were discussed by members from the faculty of the departments at the Institute: *The applications of the theory of waves* by Professor Morse of the physics department, *Nomographic charts* by Mr. Adams of the graphics department, *The differential analyser* by Professor Taylor of the electrical engineering department, *Industrial statistics* by Mr. Hermistone, and *Foundations of statistics* by Professor Wadsworth of the mathematics department. Officers were: President, Charles Papas; Vice-President, O. K. Smith; Secretary-Treasurer, Earl Singleton; Program Manager, Marvin Epstein.

Mathematics Club, New Jersey State Teachers College at Montclair

Semi-monthly meetings were held throughout the year and the following topics were presented: *Books, old and new* by Professor V. S. Mallory, *Complex numbers* by Henry Hausdorff, *Fun in mathematics* by Shirley Stamer, *Mathematical puzzles* by Barbara Stauffer, *Paper folding* by Robert Maurer, *Topology* by Jean Monsees, *Navigation* by Philip Stanger, *Plane linkages* by Philip Egeth, *Tricks with numbers* by Lillian Sprung and Virginia Florin, and *The duodecimal system* by Carlton Michelson. Each year the club invites one of the alumni to discuss experiences in the teaching of mathematics. This year the guest speaker was Mrs. Edna H. Young of East Rutherford High School who spoke on *Teaching locus problems with the aid of models.* A joint meeting of mathematics clubs in the state was held in November with representatives from the clubs at New Jersey College for Women, Rutgers University and Upsala College in attendance. Professor Richard Courant of New York University spoke on *Problems of Maxima and Minima.* At a joint meeting with the Science Club, Mr. E. C. Molina of the Bell Telephone Company Laboratories spoke on *Mathematics in the Bell Telephone Industry.* Officers were: President, John Macchi; Vice-President, Jean Monsees; Secretary, Anne Beaumont; Treasurer, Virginia Florin; Librarian, Audrey Vincentz.

Kappa Mu Epsilon, Albion College

Seven meetings were held during the year. *Mathematics in radio* and *Mathematics in aviation* were topics discussed by Ernest Longman and John Telander. Other subjects were presented as follows: *Addition and subtraction of logarithms* by Gerald Allen, *Summation of series* by Mark

Putham, *Theoretical mathematics is practical* by David Lawler, *Pike's early American arithmetics* by Margaret Ingram, *Introduction of zero into the number system* by Webster Sawyer, and *Archimedes* by Helen Shepard.

Pi Mu Epsilon, University of Illinois

Entertaining programs were the aim of the administrative committee during the year. At the opening meeting members heard a humorous paper on the troubles of a newlywed written by Professor A. R. Crathorne entitled *Statistics in the kitchen.* On the evening of election day members took part in a mock election and revised predictions as returns of the voting were received. At a Get-Acquainted Party Dr. Pepper entertained guests with her collection of puzzles and spoke on the subject *A Yank at Oxford.* In February a talk entitled *A night with probability* by Dr. E. R. Blanche was followed by opportunities for all members to compete against gambling devices. Methods of higher algebra were used at another meeting by Professor Harry Levy in solving mathematical puzzles. Professor H. F. Moore of the College of Engineering was guest speaker at the initiation banquet and used as his topic *Wishful thinking and wishful observation.* Officers were: Director, Dr. E. R. Blanche; Adviser, Dr. Echo Pepper; Secretary, DeLos De Tar; Treasurer, Eleanor Ewing.

Mathematics Club, Tennessee Polytechnic Institute

Members of the club participated in a radio program entitled, *Battle between the departments,* consisting of a quiz program in which teams representing the mathematics and engineering departments competed. The mathematics group consisted of Robert Tate, Joseph Lane, Margaret Plumlee and Charles Tabor. Topics discussed at club meetings were: *History of mathematics* by Margaret Plumlee, *Recent trends in arithmetic* by Thurman Webb, *Relation of science to mathematics* by Dr. Moorman, *The number system if we had six fingers* by Dr. Hutchinson, *Simple mathematics problems in electrical work* by Mr. Duncan, *Mathematics in war* by Professor Mattson, *Trisection of angles* by Dr. Hutchinson. At the close of the year the club was accepted as the *Tennessee Alpha Chapter* of *Kappa Mu Epsilon* by the national organization. Officers were: President, K. Walthall; Vice-President, J. Lane; Corresponding Secretary, Dr. R. A. Moorman; Recording Secretary, Margaret Plumlee; Treasurer, W. Fitzgerald.

Pi Mu Epsilon, St. Lawrence University

This chapter held regular meetings throughout the year in conjunction with the local mathematics club, *Alpha Mu Gamma.* Included among the talks were *Actuarial mathematics and the use of statistics in industry* given by Nathan Niles, *Your chance to win* by Roy Jefferey, and *Dimensional analysis* by Walter Boris. Two of the members, Stuart Wadsworth and John Boudiette, demonstrated a mechanical differentiator which they built.* The third annual Pi Mu Epsilon Interscholastic Mathematics Contest was held on May 3, 1941, following the pattern of previous years. Schools competing are members of the Northern New York Interscholastic League which compete annually in football, basketball, and baseball. A cup was awarded to the winning high school team from Potsdam, New York, and medals were given to the three students with the highest scores: John Dooley of Ogdensburg Free Academy, John Turner of Malone Franklin Academy and Ronald Greene of Potsdam High School. Certificates of merit were also awarded to the highest ranking individual of each competing high school. Officers for the year were: President, Constance Weeks; Secretary, Cameron Geraghty; Treasurer, Gerald Bradshaw; Director, Dr. O. K. Bates.

Mathematics Club, Chicago Teachers College

Six meetings were held during the year and the following topics were presented: *The story of the calculus* by George Benyek, *Higher plane curves* by John Conway, *The teaching of mathematics*

* For references used, see this department of the MONTHLY, October 1941, p. 553. Also *Scientific American Supplement* No. 2093, Feb. 12, 1916; *Proc. Royal Society of Edinburgh,* May 1904.

by Dr. Bartky, *Planetary motions* by R. R. Reynolds. The film, *Einstein's Theory of Relativity*, was presented at one program and another meeting was devoted to a description and discussion of the Isograph, slides and a motion picture film being supplied by the Bell Telephone Laboratories and the discussion led by Dr. Mansfield. Officers were: Chairman, H. J. Williams; Vice-Chairman, R. R. Reynolds; Secretary, Asta Einarson; Adviser, Dr. Ralph Mansfield.

Kappa Mu Epsilon, Nebraska State Teachers College at Wayne

Mathematics and amateur radio work was the topic used by Gerald Wright at a fall meeting of the chapter. He based his discussion on his experiences in amateur radio work in which he holds a number of national prizes and he demonstrated his talk with some of his equipment. At another meeting, Mr. Van Bearinger told of his use of the planimeter in measuring aerial photographs while working in a soil conservation office during the summer and this work led to further study of the calculus involved. Officers were: President, J. Ahern; Vice-President, E. Klein; Treasurer, C. Winter; Secretary, Van Bearinger; Faculty Sponsor, Miss E. Marie Hove.

Mathematics Club, Mount Mary College

In addition to attending four meetings of the Intercollegiate Mathematics Association of Milwaukee the club members held two discussion meetings, a joint meeting with the Science Club, a Christmas party and a steak fry. Topics presented were: *Diophantine analysis* by Virginia Altenhofen and *Brocard points* by Marie Hiegel. Officers were: President, Marianne Schueler; Vice-President, Marie Hiegel; Secretary-Treasurer, Margaret Weeks; Adviser, Sister Mary Felice.

Mathematics Club, Butler University

What is mathematics and why study it? was the topic discussed by the club adviser, Mrs. Juna L. Beal at the opening meeting of the year. Later meetings were devoted to talks on *Slide rules and their uses* by Maribelle Foster, *Theory of relativity* by Robert Stump, *History of the calendar* by Jane Gibson, *Some recent discoveries pertaining to the mathematics of the ancient Babylonians* by Helen Caster, *Concepts of the calculus and their development* by Blanchalice Barrett. The final meeting was held at the Goethe Link Observatory at Brooklyn, Indiana, where Dr. Getchell gave a lecture on Astronomy. Officers were: President, Blanchalice Barrett; Vice-President, Helen Caster; Secretary, Maribelle Foster; Treasurer, Robert Stump.

Mathematics-Physics Club, College of Saint Teresa

This organization consisted of 33 members who met bi-monthly to discuss topics in mathematics and physics. The various reports presented by both the faculty members and the students during the year were: *The duo-decimal system, Spectra of molybdenum, Life of Newton, Mathematical magic, The spider lady, Culture in mathematics, Applications of the parallelogram, Telling direction by a watch.* The following films were also presented: *Precisely So, Elgin Tells Time,* and *Geometry in Action.* Officers were: President, Marian Heinen; Vice-President, Mildred Bertrand; Secretary-Treasurer, Margaret Reckers; Faculty Adviser, Sister M. Thomas á Kempis.

Mathematics Club, Boston University

At the first meeting of the year, Professor Bruce gave an illustrated lecture on his travels in India and used for his subject *Algebra's Land of the Dawn.* Mr. C. H. Mergendahl, head of the mathematics department of the Newton Massachusetts high school, was guest speaker at another meeting and illustrated his topic *So what?* with student's reactions to problems. Other subjects presented were: *Natural logarithms* by Mr. Gould, *Magic squares* by Julia Lowe, *In defense of the fourth dimension* by Joseph Rizzo, *Algebraic series* by Joseph Lahage, *History of pi* by Elizabeth Campbell, *Life of Rene Descartes* by Agnes Caneiro, *Brain Teasers* by Francis Scheid. The final program was an Information Please program conducted by Dr. Frye. The members also attended two meetings of the Boston Intercollegiate Mathematics Club Association held at Boston College

and Regis College. Officers were: President, William Gould; Vice-President, Julia Lowe; Secretary, Elizabeth Campbell; Treasurer, Philip Nassisse; Faculty Adviser, Professor Bruce.

Kappa Mu Epsilon, Texas Technological College

In its first year as a chapter of *Kappa Mu Epsilon*, ten program meetings were held. Topics included: Biographies of the mathematicians for whom the chapter officers were named, *Non-Euclidean geometry* by R. K. Wakerling, *The coconut problem* by R. S. Underwood, *Division without a divisor*, by E. R. Heineman, *Number numerology and number theory* by F. W. Sparks, *Projective measurements* by William Wallis, *Curve tracing using Newton's diagram* by Lester LaGrange, *Energy transformations as a source of wealth* by E. A. Hazelwood. Joe R. Foots, first president of the chapter, was appointed part-time instructor in the department of Pure Mathematics of the University of Texas. Lee Michie, another charter member, received his wings and commission as second lieutenant in the Army Air Corps at Stockton, California, on April 25,1941, and sailed from San Francisco on June 3 for the Philippine Islands. Officers were: President Lobatchewsky, William Wallis; Vice-President Agnesi, Aliene May; Secretary Noether, Marie McCrummen; Treasurer Cayley, Rance Jones; Corresponding Secretary Descartes, Mrs. Opal L. Miller; Faculty Sponsor, Dr R. K. Wakerling.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 521. *Proposed by J. R. Musselman, Western Reserve University*

(a) On the sides BC and CA of a triangle ABC, construct externally any two directly similar triangles, CBA_1 and ACB_1. Show that the midpoints of the three segments BC, A_1B_1, CA form a triangle directly similar to the two given triangles.

(b) On BC externally, and on CA internally, construct any two directly similar triangles CBA_1 and CAB_1. Show that the midpoints of AB and A_1B_1 form with C a triangle directly similar to the two given triangles.

E 522. *Proposed by V. Thêbault, San Sebastián, Spain*

Find the smallest prime radix for which there exists a perfect cube of the form *abcabc*.

E 523. *Proposed by N. A. Court, University of Oklahoma*

With the vertices of a given orthocentric tetrahedron (T) as centers, spheres are drawn orthogonal to a given sphere (M) concentric with the polar sphere of

(T). Show that the radical planes of (M) with the four spheres considered form a tetrahedron which is orthocentric, and that its orthocenter coincides with that of (T).

E 524. *Proposed by R. V. Heath, Wall St., New York, N.Y.*

Write the numbers 9, 10, 11, 12, 13, 14, 15, 16 in one line. Underneath, place the numbers 1, 2, 3, 4, 5, 6, 7, 8 in such an order that the eight sums and eight differences are sixteen different numbers. In how many ways can this be done?

E 525. *Proposed by Maurice Kraïtchik, New School for Social Research, New York City*

Find parallelepipeds with commensurable edges and diagonals.

SOLUTIONS

An Imperfect Square

E 486 [1941, 555]. *Proposed by J. M. Andreas, Pasadena, California*

A quadrilateral $ABCD$ has a right angle at A. The angles at B and C are bisected by the diagonals BD and CA. Is the quadrilateral necessarily a square?

I. *Solution by Howard Eves, Chattanooga, Tenn.*

We shall answer the question in the negative as follows. Let M be a variable point on a fixed line parallel to a horizontal segment CB. In the triangle MCB, bisect the angles C and B, and let the bisectors meet the respectively opposite sides in A and D. When M is equidistant from C and B, $\angle DAB$ is obtuse; but when M is sufficiently far to the right, this angle is acute. By considerations of continuity it follows that there must be some position of M for which $\angle DAB$ is a right angle. Then $ABCD$ is a non-square quadrilateral satisfying the conditions of the problem.

II. *Solution by E. P. Starke, Rutgers University*

No. If $\angle ABC$ is any obtuse angle, there are two values of the ratio $AB:AC$ such that a quadrilateral may be completed to satisfy the given conditions. To show this, write $\theta = \angle ABD = \frac{1}{2}\angle ABC$, $a = BC$, $c = AB$. By considering projections parallel and perpendicular to AB, one computes without difficulty:

$$AD = c \tan\theta, \quad CD = \sqrt{(a^2 - 2ac + c^2 \sec^2\theta)}, \quad AC = \sqrt{(a^2 + c^2 - 2ac \cos 2\theta)}.$$

Equating expressions for the cosines of $\angle ACD$ and $\angle ACB$, one obtains a relation which may be reduced to

$$(a^2 + c^2)(1 - 4\cos^2\theta) + 2ac(1 - 2\cos^2\theta + 4\cos^4\theta) = 0$$

by suppressing a factor $1 - \cos^2\theta$. The discriminant of this quadratic in a/c reduces to

$$16\cos^2\theta(1 - \cos^2\theta)(1 - 4\cos^4\theta),$$

which is positive for all θ between 45° and 90°. The statement made above is now obvious.

For a numerical example take $AB=3$, $AD=3\sqrt{2}$, $BC=7+2\sqrt{10}$, $CD=8+\sqrt{10}$, determining $BD=3\sqrt{3}$ and $AC=4\sqrt{2}+4\sqrt{5}$.

Also solved by D. H. Browne, W. B. Clarke, C. W. Emmons, A. K. Waltz and the Clarkson College Mathematical Club. The proposer remarks that the negative answer invalidates a traditional method for obtaining an allegedly square piece of paper from one that is approximately rectangular, by folding and tearing.

The Orthic Triangle

E 487 [1941, 555]. *Proposed by V. Thébault, San Sebastián, Spain*

Prove that if the orthocenter of a triangle is conjugate to the three vertices with regard to the incircle and two of the excircles, respectively, then these three circles touch the respective sides of the orthic triangle, and conversely.

Solution by Howard Eves, Chattanooga, Tenn.

(1) *Lemma*: Given a triangle ABC circumscribed to a conic K, consider points M on AC, N on AB. Then MN touches K if and only if the point of intersection (BM, CN) is conjugate to A.

Let the points of contact of K with AC and AB be S and T, respectively. Suppose MN touches K. Then the simple quadrilateral $BCMN$ is circumscribed to K. Therefore BM and CN are concurrent with ST, the polar of A. Conversely, if BM, CN, ST are concurrent, let the second tangent to K from N cut AC in M'. Then, by the same argument, BM', CN, ST are concurrent; therefore M' coincides with M, and MN is a tangent.

(2) *Theorem*: Given a triangle ABC circumscribed to three conics K_a, K_b, K_c, and given a point H, construct the intersections $L=(AH, BC)$, $M=(BH, CA)$, $N=(CH, AB)$. Then the sides of the triangle LMN touch the respective conics K_a, K_b, K_c if and only if H is conjugate to A, B, C with regard to these respective conics.

This follows at once from the lemma.

(3) The given problem is a highly specialized case of (2).

The proposer remarks that, for such a triangle, the diameter of the circumcircle is equal to the radius of the third excircle.

Differences of Factorials

E 488 [1941, 556]. *Proposed by D. H. Browne, Buffalo, N.Y.*

The factorial $u_k=k!$, "sub-factorial" $v_k=k!\sum_0^k(-1)^r/r!$, and "super-factorial" $w_k=k!\sum_0^k 1/r!$, may be defined by the recurrence formulas

$$u_0 = v_0 = w_0 = 1, \qquad u_k = ku_{k-1}, \qquad v_k = kv_{k-1} + (-1)^k, \qquad w_k = kw_{k-1} + 1.$$

Show that $\Delta^n w_0 = u_n$, $\Delta^n u_0 = v_n$.

Solution by R. D. Specht, University of Florida

We have

$$\Delta^n w_0 = (E-1)^n w_0 = \sum_{r=0}^{n}(-1)^r\binom{n}{r}w_{n-r}$$
$$= \sum_{r=0}^{n}(-1)^r\binom{n}{r}(n-r)!\sum_{s=0}^{n-r}\frac{1}{s!} = n!F_n$$

where

$$F_n = \sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-1)^r}{r!s!}.$$

But

$$\Delta F_n = \sum_{r=0}^{n+1}\frac{(-1)^r}{r!(n+1-r)!} = \frac{(1-1)^{n+1}}{(n+1)!} = 0$$

and $F_0=1$. Therefore $F_n=1$, and $\Delta^n w_0 = n! = u_n$.

Also

$$\Delta^n u_0 = \sum_{r=0}^{n}(-1)^r\binom{n}{r}u_{n-r} = n!\sum_{r=0}^{n}\frac{(-1)^r}{r!} = v_n.$$

Editorial Note. The first part may be proved more simply by observing that

$$\Delta^n w_0 - n\Delta^{n-1}w_0 = \sum_{r=0}^{n}(-1)^r\binom{n}{r}w_{n-r} - n\sum_{r=0}^{n-1}(-1)^r\binom{n-1}{r}w_{n-r-1}$$
$$= \sum_{r=0}^{n}(-1)^r\binom{n}{r}\{w_{n-r}-(n-r)w_{n-r-1}\}$$
$$= \sum_{r=0}^{n}(-1)^r\binom{n}{r} = 0.$$

Sections of a Prismatoid

E 489 [1941, 556]. *Proposed by Howard Eves, Chattanooga, Tenn.*

Let A_0, A_m, A_h be the areas of the lower base, midsection, and upper base of a pristmatoid. If $A_h=A_0$, prove that

(1) sections equidistant from the midsection are equal in area;
(2) the midsection bisects the volume of the prismatoid;
(3) if $A_m=A_0$, all sections have the same area;
(4) if $A_m\neq A_0$, A_m is the maximum or minimum section.

Solution by the Proposer

Let S be a solid figure extending between two parallel planes. Let h be the distance between the planes, and let A_x be the area of the section at distance x

from one of the planes (the "lower base"). Suppose S is such that A_x is a quadratic function of x. (The prismatoid, along with many other figures, is such a solid.) Now we have

$$\text{(i)} \qquad A_x = rx^2 + sx + t,$$

where r, s, t are constants to be determined. Putting $x=0$, h, $h/2$ in turn, and solving for r, s, t, we find

$$r = (2A_0 - 4A_m + 2A_h)/h^2, \qquad s = (4A_m - A_h - 3A_0)/h, \qquad t = A_0.$$

Substituting in (i) we get

$$h^2A_x = (2A_0 - 4A_m + 2A_h)x^2 + (4A_m - A_h - 3A_0)hx + h^2A_0.$$

When $A_h = A_0$, this reduces to

$$\text{(ii)} \qquad h^2A_x = (2x - h)^2A_0 - 4x(x - h)A_m.$$

From (ii) it is clear that $A_x = A_{h-x}$, which is part (1) of the problem. Cavalieri's Theorem makes part (2) an immediate corollary of this. Putting $A_m = A_0$ in (ii), we see that $A_x = A_0$, which is part (3). Differentiating the right member of (ii) with respect to x, we see that A_x is a maximum or minimum when $x = h/2$; this establishes part (4).

Derangements

E 490 [1941, 556]. *Proposed by J. F. Kenney, University of Wisconsin at Milwaukee*

In a gambling game, a player is permitted to deal ten cards from a bridge deck (which has been thoroughly shuffled) and wins if, at any stage of the dealing, the number on a card is the same as the number of cards dealt. (Face cards are assigned the number 0.) Find the probability that the dealer will win.

I. *Solution by R. W. Wagner, Oberlin College*

Let N_i denote the number of ways of arranging the deck so that the first i cards are respectively an ace, a two, and so on. Then $N_i = 4^i(52-i)!$. By Proposition XIV of Whitworth, *Choice and Chance*, p. 68, we deduce that the number of ways of arranging the deck so that no one of the first ten cards has a number equal to its position is

$$\sum_{i=0}^{10} (-1)^i \binom{10}{i} N_i = \sum_{i=0}^{10} (-4)^i \binom{10}{i} (52 - i)!.$$

The total number of ways of arranging the deck is 52!. Thus the probability of winning is

$$-\sum_{i=1}^{10} (-4)^i \binom{10}{i} (52 - i)!/52! = .54815 \cdots .$$

II. *Solution by Irving Kaplansky, Harvard University*

This is a special case of the problem treated in my note *On a generalization of the "Problème des Rencontres"* [1939, 159]. Putting $n=13$, $a_i=4$, $p_1=\cdots=p_{10}=1$, $p_{11}=p_{12}=p_{13}=0$ in the formula obtained there, we get

$$E^{12}(E^4-4E^3)^{10}0! = E^{42}(E-4)^{10}0! = \sum_{i=0}^{10}(-4)^i\binom{10}{i}(52-i)!$$

for the number of arrangements.

III. *Experimental check by D. H. Browne, Buffalo, N. Y.*

By a trial of several hundred deals, using a single close ruff and a lift cut between each deal and the next, I get an average of 54.5%.

Also solved by E. P. Starke and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4036. *Proposed by L. A. Santaló, Rosario, Argentina*

Let C_1 be an oval with a continuously varying radius of curvature R; at each point of C_1 a normal of length R is drawn exteriorly giving points of a second curve C_2 (which may not be convex); and let A be the area enclosed between the two curves. From a chosen fixed point a vector is drawn parallel to the normal at a point of C_1 and of length R for that point, thus giving as the point varies on C_1 a curve C_3 having the area A_3 and length L_3. If L_2 is the length of C_2 and A_1 is the area of C_1, show that

$$\text{(a)}\quad A = 3A_3; \qquad \text{(b)}\quad L_2L_3 \geqq 8\pi A_1;$$

the equality in (b) is true only when C_1 is a circle.

4037. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

Integrate

$$(x^{n+1}+y^n)y' - x^n y = 0;$$

calculate and examine the radius of curvature of the integral curves at the origin.

4038. *Proposed by V. Thébault, San Sebastián, Spain*

The point M is chosen arbitrarily on a bisector of angle A of the triangle ABC, and let M' be its isogonal conjugate with respect to ABC. Show that the two circles each through M and M' and tangent to the side BC are tangent also to the circumcircle of ABC.

4039. *Proposed by N. A. Court, University of Oklahoma*

The circumcenter of a tetrahedron (T) and any point M are isogonal conjugates with respect to the tetrahedron formed by the centers of the four spheres passing through M and the circumcircles of the faces of (T).

Correction of editorial errors in 3994 [1941, 273]. *Proposed by C. E. Springer, University of Oklahoma*

If

$$a_{11} = a_{22} = a_{33} = \sum_j \binom{K}{j}(n-1)^{K-j}, \qquad j \equiv 0 \pmod 3;$$

$$a_{12} = a_{23} = a_{31} = \sum_j \binom{K}{j}(n-1)^{K-j}, \qquad j \equiv 1 \pmod 3;$$

$$a_{13} = a_{32} = a_{21} = \sum_j \binom{K}{j}(n-1)^{K-j}, \qquad j \equiv 2 \pmod 3;$$

show that the determinant

$$| a_{ij} | = [(n-1)^3 + 1]^K.$$

SOLUTIONS

Symmetric Functions

3980 [1941, 69]. *Proposed by Esther Szekeres, Budapest, Hungary*

The symmetric polynomials $y_1, y_2, \cdots, y_n$ in the variables $x_1, x_2, \cdots, x_n$ are of the degrees indicated by the subscripts, and are algebraically independent. If $f(x_1, x_2, \cdots, x_n)$ is any given polynomial symmetric in the x's, show that it can be expressed as a polynomial in the y's.

Solution by the Proposer

Since the symmetric polynomial $f(x_1, x_2, \cdots, x_n)$ can be expressed as a polynomial in terms of the elementary symmetric functions $\sigma_1, \sigma_2, \cdots, \sigma_n$, it suffices to show that each σ_i can be expressed as a polynomial in the y_i's.

We may write

$$y_k = c_k\sigma_k + g_k(\sigma_1, \sigma_2, \cdots, \sigma_{k-1}),$$

where the second term in the right member is a polynomial in the indicated arguments whose terms are of weights not exceeding k, or in other words it is the sum of terms such as

$$\sigma_1^{t_1}\sigma_2^{t_2}\cdots\sigma_{k-1}^{t_{k-1}}, \qquad t_1 + 2t_2 + \cdots + (k-1)t_{k-1} \leqq k,$$

multiplied by a constant coefficient. We show first that no c_k is zero, If any one is zero let c_k be the first one to be zero. Then we have

$$\begin{aligned} y_1 &= c_1\sigma_1 + c_0 \\ y_2 &= c_2\sigma_2 + g_2(\sigma_1) \\ &\;\;\vdots \\ y_{k-1} &= c_{k-1}\sigma_{k-1} + g_{k-1}(\sigma_1, \sigma_2, \cdots, \sigma_{k-2}) \\ y_k &= \qquad\qquad g_k(\sigma_1, \sigma_2, \cdots, \sigma_{k-1}). \end{aligned}$$

These equations may be solved in turn for $\sigma_1, \sigma_2, \cdots, \sigma_{k-1}$ so that $\sigma_i (i \leqq k-1)$ is a polynomial in $y_1, y_2, \cdots, y_i$; and then inserting these values in the last equation, we shall have

$$y_k = g(y_1, y_2, \cdots, y_{k-1}),$$

where the right member is a polynomial in the indicated arguments, But this is contrary to the hypothesis. Hence no c_k is zero, and each σ_i can be expressed as a polynomial in $y_1, y_2, \cdots, y_i$. This concludes the proof.

Solved also by C. M. Stein. A solution similar to the one by the proposer was received from Li Ming-hsien after the preparation of the above for printing.

Spheres Associated with a Tetrahedron

3982 [1941, 70]. *Proposed by V. Thébault, San Sebastián, Spain*

The vertices of the tetrahedron $ABCD$ are centers of spheres the squares of whose radii are equal respectively to one-third of the sum of the squares of the edges through the considered vertex. Show that the sphere orthogonal to the four spheres is concentric with the twelve point sphere of $ABCD$.

Note. See N. A. Court, *Modern Pure Solid Geometry*, p. 250, for the twelve point sphere.

3983 [1941, 70]. *Proposed by V. Thébault, San Sebastián, Spain*

The vertices of the tetrahedron $ABCD$ are centers of spheres the squares of whose radii are equal respectively to k times the sum of the squares of the edges of the face opposite to the vertex considered, and they are also centers of spheres the squares of whose radii are equal respectively to k times the sum of the squares of the edges through the considered vertex. Let ω_1 and ω_2 be the centers of the spheres, radii R_1 and R_2, orthogonal respectively to the two sets of four spheres, G the centroid, and O the circumcenter of $ABCD$. (1) Show that the points O, G, ω_1, ω_2 are collinear and determine their relative positions. (2) Show that $(R_1^2 - R_2^2)/k$ remains constant when k varies.

Solution by J. W. Peters, University of Illinois

Since the first problem is a special case of part of the second, let us consider the second problem first.

Let $\overline{A}_i$, $(i=1, 2, 3, 4)$, be the position vectors of the vertices of the tetrahedron $ABCD$. There will be no loss in generality if the origin is taken as the circumcenter, O, and if the unit of measurement is so chosen that the radius of the circumsphere is 1. The vectors $\overline{A}_i$ are thus unit vectors. Let $\overline{S}=\sum_{i=1}^4\overline{A}_i$. Then from the reference to Court we learn that the centroid is $\overline{S}/4$, the Monge point is $\overline{S}/2$, and the center of the twelve point sphere is $\overline{S}/3$.

Let the first set of spheres mentioned in problem 3983 be given by

$$(1)\qquad (\overline{X}-\overline{A}_i, \overline{X}-\overline{A}_i) = r_i^2,$$

where

$$r_i^2 = k[(\overline{A}_j-\overline{A}_m, \overline{A}_j-\overline{A}_m) + (\overline{A}_m-\overline{A}_n, \overline{A}_m-\overline{A}_n) + (\overline{A}_n-\overline{A}_j, \overline{A}_n-\overline{A}_j)],$$
$$r_i^2 = 2k[3-(\overline{A}_j, \overline{A}_m)-(\overline{A}_m, \overline{A}_n)-(\overline{A}_n, \overline{A}_j)],$$

where k is a real scalar and $i, j, m, n, =1, 2, 3, 4; i\neq j\neq m\neq n$. The parentheses represent scalar products of the enclosed vectors.

Since $(\overline{X}-\bar{\omega}_1, \overline{X}-\bar{\omega}_1)=R_1^2$ is orthogonal to the spheres (1), we have $(\overline{A}_i-\bar{\omega}_1, \overline{A}_i-\bar{\omega}_1)-R_1^2-r_1^2=0$, which reduces to

$$(2)\qquad 1-2(\overline{A}_i, \bar{\omega}_1)+(\bar{\omega}_1, \bar{\omega}_1)-R_1^2-r_i^2=0.$$

Eliminating R_1^2 from this set of four equations, we obtain three independent equations of the form

$$(3)\qquad 2(\overline{A}_i-\overline{A}_j, \bar{\omega}_1)+r_i^2-r_j^2=0.$$

On replacing the r_i^2 by their values in terms of k and A_i, this set of equations reduces to a set of three equations of the form

$$(4)\qquad (\overline{A}_i-\overline{A}_j, \bar{\omega}_1)+k(\overline{A}_i-\overline{A}_j, \overline{S})=0.$$

If the four points $ABCD$ do not lie in a plane, the equations (4) will have the solution $\bar{\omega}_1=-k\overline{S}$. With this value for $\bar{\omega}_1$ and any one of equation (2), we find

$$(5)\qquad R_1^2=(1-2k)^2+2k(1+k)\cdot\sum^6(\overline{A}_i, \overline{A}_j), \qquad i, j=1, 2, 3, 4; i\neq j.$$

The second set of four spheres with centers at the vertices $ABCD$ are given by the equations

$$(6)\qquad (\overline{X}-\overline{A}_i, \overline{X}-\overline{A}_i)=s_i^2,$$

where

$$s_i^2=k[(\overline{A}_i-\overline{A}_j, \overline{A}_i-\overline{A}_j)+(\overline{A}_i-\overline{A}_m, \overline{A}_i-\overline{A}_m)+(\overline{A}_i-\overline{A}_n, \overline{A}_i-\overline{A}_n)]$$

or

$$s_i^2=2k[4-(\overline{A}_i, \overline{S})].$$

Since $(\overline{X}-\bar{\omega}_2, \overline{X}-\bar{\omega}_2)=R_2^2$ is orthogonal to the spheres (6), it follows that

$$1 - 2(\overline{A}_i, \bar{\omega}_2) + (\bar{\omega}_2, \bar{\omega}_2) - R_2^2 - s_i^2 = 0. \tag{7}$$

Eliminating R_2^2 from these four equations, one obtains a set of three independent equations of the form

$$2(\overline{A}_i - \overline{A}_j, \bar{\omega}_2) + s_i^2 - s_j^2 = 0. \tag{8}$$

On replacing the s_i^2 by their values in terms of k and A_i, this set of equations reduces to

$$(\overline{A}_i - \overline{A}_j, \bar{\omega}_2) - k(\overline{A}_i - \overline{A}_j, \overline{S}) = 0. \tag{9}$$

If the four points are not in a plane, the three equations (9) will have the solution $\bar{\omega}_2=k\overline{S}$. With this value for $\bar{\omega}_2$ and any one of the equations (7), it is found that

$$R_2^2 = 1 - 8k + 4k^2 + 2k^2\cdot \sum^{6} (\overline{A}_i, \overline{A}_j). \tag{10}$$

It is evident that $\bar{\omega}_1$ and $\bar{\omega}_2$ lie on the line OG and that O is the bisector of the segment $\bar{\omega}_1\bar{\omega}_2$. Furthermore it is easily seen that $(R_1^2-R_2^2)/k=4+2\sum(\overline{A}_i, \overline{A}_j)$ and this ratio, being independent of k, remains constant when k varies.

For problem 3982, set $k=1/3$ in equations (6), (7), (8), and (9). Then $\bar{\omega}_2=\overline{S}/3$ and this is the center of the twelve point sphere.

Solved also by the proposer.

Isogonal Conjugates

3984 [1941, 152]. *Proposed by R. Goormaghtigh, Bruges, Belgium*

The two points P, Q are symmetric as to the circumcenter of a triangle, the isogonal conjugates of P, Q are P', Q', and M is the midpoint of $P'Q'$; prove that $PQ\cdot P'Q'=4R\cdot HM$, where H is the orthocenter and R is the circumradius of the triangle.

Solution by Li Ou, Yenching University, Peiping, China

Let the vertices of the triangle $A_1A_2A_3$ be the turns t_i satisfying $t^3-\sigma_1t^2+\sigma_2t-\sigma_3=0$; and let the point P be the complex number z. Since P and Q are symmetric as to the circumcenter of the triangle, the complex number of Q will be $-z$ Then it may be shown* that the complex number of P', the isogonal conjugate of P, is

$$P' \equiv \frac{\bar{z}^2\sigma_3 - \bar{z}\sigma_2 + \sigma_1 - z}{1 - z\bar{z}},$$

where $\bar{z}$ is the conjugate of z. Similarly the isogonal conjugate of Q will be

* J. H. Weaver, On Isogonal Points, this MONTHLY, vol. 42, 1935, p. 496.

$$Q' \equiv \frac{\bar{z}^2\sigma_3 + \bar{z}\sigma_2 + \sigma_1 + z}{1 - z\bar{z}}.$$

Hence we have the lengths

$$\overline{PQ} = 2|z|, \qquad \overline{P'Q'} = \frac{2|\bar{z}\sigma_2 + z|}{|1 - z\bar{z}|}.$$

Thus, their product

$$\overline{PQ}\cdot\overline{P'Q'} = \frac{4|z\bar{z}\sigma_2 + z^2|}{|1 - z\bar{z}|}.$$

Now the complex numbers of M and H are easily shown to be

$$M \equiv \frac{\bar{z}^2\sigma_3 + \sigma_1}{1 - z\bar{z}}, \qquad H \equiv \sigma_1$$

respectively; and the length

$$\overline{HM} = \frac{|\sigma_3|\,|z\bar{z}\sigma_2 + \bar{z}^2|}{|1 - z\bar{z}|} = \frac{|z\bar{z}\sigma_2 + z^2|}{|1 - z\bar{z}|},$$

since $|\sigma_3| = 1$. Hence $\overline{PQ}\cdot\overline{P'Q'} = 4\cdot\overline{HM}$. We note that here we take $R=1$.

Solved also by J. W. Clawson, J. H. Weaver, and the proposer, each using complex coordinates.

An Operational Identity

3985 [1941, 152]. *Proposed by Charles M. Stein, New York, N. Y.*

Prove that if $f(x)$ is analytic within and on a circle C having x as center and containing y, then

$$\cdots\left(1 + \frac{y-x}{n}D\right)\cdots\left(1 + \frac{y-x}{2}D\right)\left(1 + \frac{y-x}{1}D\right)f(x) = f(y),$$

where $D = d/dx$.

Solution by Kwan Chao-Chih, Yenching University, Peiping, China

Since $f(x)$ is analytic in C, all its derivatives exist. We observe

$$\left(1 + \frac{y-x}{1}D\right)f(x) = f(x) + \frac{y-x}{1}f'(x) \equiv \sum_{\gamma=0}^{1}\frac{(y-x)^\gamma}{\gamma!}f^{(\gamma)}(x).$$

We shall write

$$(1)\qquad \left(1 + \frac{y-x}{n-1}D\right)\cdots\left(1 + \frac{y-x}{2}D\right)\left(1 + \frac{y-x}{1}D\right)f(x) = \sum_{\gamma=0}^{n-1}\frac{(y-x)^\gamma}{\gamma!}f^{(\gamma)}(x),$$

and prove it by induction for any n. For,

$$\left(1+\frac{y-x}{n}D\right)\cdots\left(1+\frac{y-x}{2}D\right)\left(1+\frac{y-x}{1}D\right)f(x)$$
$$=\sum_{\gamma=0}^{n-1}\frac{(y-x)^\gamma}{\gamma!}f^{(\gamma)}(x)+\frac{(y-x)}{n}\left[\sum_{\gamma=0}^{n-1}\frac{(y-x)^\gamma}{\gamma!}f^{(\gamma+1)}(x)\right.$$
$$\left.-\sum_{\gamma=1}^{n-1}\frac{(y-x)^{\gamma-1}}{(\gamma-1)!}f^{(\gamma)}(x)\right]$$
$$=\sum_{\gamma=0}^{n}\frac{(y-x)^\gamma}{\gamma!}f^{(\gamma)}(x).$$

When $n\to\infty$, the right member of (1) is Taylor's expansion of $f(x)$ at y, and thus represents $f(y)$ since the circle C contains y.

Solved also by Huang K'un and the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to C. O. Oakley, Haverford College, Haverford, Pennsylvania.

The National Council of Teachers of Mathematics will hold its next annual summer meeting jointly with the Department of Secondary Education of the National Education Association on June 29 and 30, 1942, at Denver, Colorado.

The twenty-fifth anniversary of Dr. H. E. Hawkes, professor of mathematics, as dean of Columbia College was celebrated at a dinner given in his honor on April 16, 1942.

Dr. Archie Blake, associate mathematician of the U. S. Coast and Geodetic Survey, has been transferred to the National Inventors Council at Washington.

C. A. Bridger, statistician for the Idaho Department of Public Health, has been given a civil service appointment as junior metallurgist with the Bureau of Mines at Salt Lake City.

Assistant Professor Marguerite D. Darkow of Hunter College has been promoted to an associate professorship.

Assistant Professor B. E. Gatewood of Louisiana Polytechnic Institute is on leave of absence to take a position at the MacDonnell Aircraft Corporation at St. Louis.

Assistant Professor C. W. Hook of Georgia School of Technology, a member of the Naval Reserve, has reported to the Naval Academy for duty as an instructor in mathematics.

At Smith College, Associate Professors N. H. McCoy and Deane Montgomery have been promoted to professorships.

R. M. Pinkerton, associate physicist at Langley Field, has been appointed associate professor in the Aeronautical Engineering Department at Agricultural and Mechanical College of Texas.

Professor D. W. Pugsley of Berea College has succeeded Professor W. R. Hutcherson as head of the department of mathematics, Professor Hutcherson having been granted leave of absence on a defense mechanics fellowship at Brown University.

Assistant Professor E. J. Purcell has been promoted to an associate professorship at the University of Arizona.

Professor Suzan R. Benedict of Smith College died of a heart attack on April 8, 1942. She was a charter member of the Mathematical Association and had retired from teaching last January after thirty-five years teaching at Smith College.

Dr. Robert Henderson, vice president and actuary (retired) of the Equitable Life Assurance Society, died February 16, 1942. He was a charter member of the Mathematical Association of America and was for a long term of years a trustee of the American Mathematical Society.

Assistant Professor Elizabeth E. Knight of State Teachers College of Milwaukee died on March 26, 1942. She had been a member of the Mathematical Association since 1925.

Dr. W. W. Landis, for forty-seven years professor of mathematics at Dickinson College, died on April 8, 1942, at the age of seventy-three years. He was a charter member of the Mathematical Association.

Professor J. H. Weaver of Ohio State University was instantly killed by a train near his home on April 7, 1942. He was a charter member of the Mathematical Association and had assisted on the staff of the MONTHLY for a number of years.

SUMMER COURSES

The following courses in mathematics are announced for the summer of 1942:

University of Chicago. In view of the national emergency the Summer Quarter has been lengthened to twelve weeks, extending from *June 22 to September 12, 1942*. However, most of the following courses, with the exception of trigonometry, algebra and possibly calculus, will be completed by *August 29*, for the benefit of those teachers who must return to their positions. In addition to courses in trigonometry, algebra, analytic geometry, calculus and differential equations, the following will be offered: By Professor Bliss (with the cooperation of Professor Graves, Hestenes, Reid, Radó and Smiley): Seminar on the calculus of variations. By Professor Barnard: Lattice theory. By Professor Graves: Func-

tions of real variables. By Professor Hartung: Advanced problems in teaching mathematics in the secondary school and the junior college. By Professor Hestenes: Calculus of variations. By Professor Lane: Metric differential geometry, Projective differential geometry of hyperspace. By Professor Radó: Subharmonic functions. By Professor Reid: Fourier series and Bessel functions, Exterior ballistics. By Dr. Schilling: Introduction to algebraic theories, Galois theory.

Iowa State College. Statistics will be emphasized in the first summer session, *June 8 to July 15.* Courses in the mathematical theory of statistics and its applications will be offered; the design of experiments and of other sampling investigations important in the present emergency will receive special consideration.

University of North Carolina. In addition to the regular courses through the calculus, the following will be offered: *First Term, June 11 to July 21.* By Professor Henderson: Advanced algebra, Theory of equations. By Professor Browne: Theory of numbers. By Professor Hoyle: Advanced calculus. *Second Term, July 22 to August 28.* By Professor Lasley: Projective geometry. By Professor Mackie: Differential equations. By Professor Hill: History of mathematics.

University of Pennsylvania. In addition to the usual elementary courses, the following advanced work will be offered: By Professor Rademacher: Partial differential equations of mathematical physics, Elliptic functions. By Professor Shohat: Theory and practice of approximations. By Professor Clarkson: Navigation.

THE WILLIAM LOWELL PUTNAM COMPETITION

The following are the results of the fifth annual William Lowell Putnam Mathematical Competition held March 7, 1942.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of the University of Toronto. The members of the team were K. S. Hoyle, H. V. Lyons, M. A. Preston; to each of these is awarded a prize of forty dollars.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Yale University. The members of the team were F. H. Brownell, 3rd, A. M. Gleason, A. E. Roberts, Jr.; to each of these is awarded a prize of thirty dollars.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Massachusetts Institute of Technology. The members of the team were E. D. Calabi, W. S. Loud, G. P. Wachtell; to each of these is awarded a prize of twenty dollars.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the College of the City of New York. The members of the team were Herman Chernoff, Harvey Cohn, Edward Gordon; to each of these is awarded a prize of ten dollars.

The five persons ranking highest in the examination, names in alphabetical order, were Harvey Cohn, College of the City of New York; A. M. Gleason, Yale University; W. S. Loud, Massachusetts Institute of Technology; H. V. Lyons, University of Toronto; M. A. Preston, University of Toronto. Each of these will receive a prize of fifty dollars.

The following teams won honorable mention: Department of Mathematics, Cooper Union Institute of Technology, the members of the team being Harold Grad, M. S. Klamkin, Kenneth Robinson; Department of Mathematics, Harvard University, the members of the team being R. M. Bloch, L. S. Shapley, J. A. Zilber; Department of Mathematics, New York University, the members of the team being Melvin Lax, Harold Lewis, Henry Shenker; Department of Mathematics, Swarthmore College, the members of the team being N. V. Hannay, W. H. Mills, M. S. Raff. This list is alphabetical.

Five individuals are given honorable mention, the names listed in alphabetical order being: E. D. Calabi, Massachusetts Institute of Technology; C. P. Gadsden, Tulane University; K. S. Hoyle, University of Toronto; Melvin Lax, New York University; W. H. Mills, Swarthmore College.

A total of one hundred fourteen undergraduate students representing thirty-one institutions took part in the competition.

W. D. CAIRNS, *Secretary-Treasurer*

EXAMINATION QUESTIONS FOR THE FIFTH WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION, MARCH 7, 1942

MORNING SESSION: 9:00 A.M. to 12:00 NOON. (*Answer the questions in any order and by any method. Show all your work in logical sequence, and indicate your answers clearly. No tables or other books may be used.*)

1. A square of side $2a$, lying always in the first quadrant of the XY plane, moves so that two consecutive vertices are always on the X- and Y-axes respectively. Find the locus of the mid-point of the square.

2. The polynomial $f(x)$ is divided by $(x-a)^2\,(x-b)$, where $a\neq b$, derive a formula for the remainder.

3. Is the following series convergent or divergent

$$1+\frac{1}{2}\cdot\frac{19}{7}+\frac{2!}{3^2}\left(\frac{19}{7}\right)^2+\frac{3!}{4^3}\left(\frac{19}{7}\right)^3+\frac{4!}{5^4}\left(\frac{19}{7}\right)^4+\cdots?$$

4. Find the orthogonal trajectories of the family of conics $(x+2y)^2=a(x+y)$. At what angle do the curves of one family cut the curves of the other family at the origin?

5. A circle of radius a is revolved through 180° about a line in its plane, distant b from the center of the circle, where $b>a$. For what value of the ratio b/a does the center of gravity of the solid thus generated lie on the surface of the solid?

6. Any circle in the XY (horizontal) plane is "represented" by a point on the vertical line through the center of the circle, and at a distance "above" the plane of the circle equal to the radius of the circle.

Show that the locus of the representations of all the circles which cut a fixed circle at a constant angle is a (portion of a) one-sheeted hyperboloid.

By consideration of suitable families of circles in the plane, demonstrate the existence of two families of rulings on the hyperboloid.

AFTERNOON SESSION: 2:00 P.M. TO 5:00 P.M. (*Answer the questions in any order and by any method. Show all your work in logical sequence, and indicate your answers clearly. No tables or other books may be used.*)

7. A square of side $2a$, lying always in the first quadrant of the XY plane, moves so that two consecutive vertices are always on the X- and Y-axes respectively. Prove that a point within or on the boundary of the square will in general describe a (portion of a) conic. For what points of the square does this locus degenerate?

8. For the family of parabolas

$$y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a,$$

(a) Find the locus of vertices.
(b) Find the envelope.
(c) Sketch the envelope and two typical curves of the family.

9. Given

$$x = \phi(u, v)$$
$$y = \psi(u, v)$$

where ϕ and ψ are solutions of the partial differential equation

$$\frac{\partial\phi}{\partial u}\frac{\partial\psi}{\partial v} - \frac{\partial\phi}{\partial v}\frac{\partial\psi}{\partial u} = 1. \tag{1}$$

By assuming that x and v are the independent variables, show that (1) may be transformed to

$$\frac{\partial y}{\partial v} = \frac{\partial u}{\partial x}. \tag{2}$$

Integrate (2), and show how this effects in general the solution of (1). What other solutions does (1) possess?

10. A particle moves under a central force inversely proportional to the kth power of the distance. If the particle describes a circle (the central force proceeding from a point *on the circumference of the circle*), find k.

11. Sketch the curve

$$y = \frac{x}{1 + x^6 \sin^2 x},$$

and show that

$$\int_0^\infty \frac{x\,dx}{1 + x^6 \sin^2 x}$$

exists.

The examination lists were formulated by Professors B. H. Brown and Marie Litzinger.

Note. Chairmen of mathematics departments may obtain copies of the examination questions for the Putnam Competition for 1938, for 1939, for 1940, for 1941, and for 1942 by a post card request to Professor W. D. Cairns, 97 Elm Street, Oberlin, Ohio.

THE NEW SECRETARY-TREASURER

At the meeting of the Board of Governors held at Chicago on September 1, 1941, Professor William D. Cairns, Secretary-Treasurer of the Association since its organization in 1915, announced his wish and purpose to retire from that position at the end of his present term of office, January 1, 1943.

On behalf of the officers and Board of Governors of the Mathematical Association of America, I am very happy to announce the election of Professor Walter B. Carver of Cornell University as Secretary-Treasurer for a five-year term beginning in January, 1943. As a result of this appointment Professor Carver will continue his long service to the Association, which includes five years as Editor-in-Chief, two years as President, and further service on the Finance Committee. The Association is most fortunate to have its affairs in such competent hands during the important years just ahead.

R. W. BRINK, *President*

PRE-INDUCTION TRAINING

Bulletin 23 on *Higher Education and National Defense*, published by the American Council on Education, is devoted to the subject "Pre-induction Training Needs on the College Level for Enlisted Men in the Armed Forces." Since any pre-training needed by enlisted men obviously is at least equally important for officers, we may look upon the following quotations from this bulletin as applying to officers as well as to enlisted men. In the preparation of the bulletin, an analysis of scholastic needs was made through the medium of consultations with the officers in charge of training in the various branches of the armed forces. The following quotations from Bulletin 23 are of interest as relating to mathematics or to subjects demanding a mathematical basis.

"Their [the officers'] statements of suggested educational background were carefully digested for this report, and their opinions should be given serious thought even though there seems to be only a new emphasis and not new subject-matter material in this list of studies. Two things should be kept in

mind, however, to interpret properly the suggested studies by the officers. First, they had been asked to consider students who would be in college, short of graduation. Second, they naturally were thinking, in the main, of background material which would shorten, for the armed forces, the training period for specific jobs, They, therefore, strongly recommend the following studies:

Mathematics: arithmetic, algebra, geometry, trigonometry, and some calculus, functions, graphs, and some surveying.

Physics: standard course with special emphasis on the following: electricity and magnetism, hydraulics, mechanics, heat, light, sound, force and motion, optics, principles of internal combustion engine, telephone, telegraph and radio communication, Morse code and International code.

Chemistry: general course to include principles of explosives.

Physical Geography: map interpretation, especially topographic maps and aerial photography, winds and weather.

Astronomy: as it relates to air and marine navigation, descriptive.

W. L. HART

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fifth Summer Meeting, Poughkeepsie, N. Y., September 7–9, 1942.

Twenty-seventh Annual Meeting, New York, N. Y., December 30–31, 1942.

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Decatur, May 8–9, 1942
INDIANA, Notre Dame, April 9–10, 1943
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Ashland, Va., May 2, 1942
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Northfield, May 9, 1942
MISSOURI, fall, 1942
NEBRASKA, Omaha, May 9, 1942
NORTHERN CALIFORNIA, San Francisco, Jan. 30, 1943
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 28, 1942
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, fall, 1942
WISCONSIN, Milwaukee, May 7, 1943

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BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 450 Ahnaip Street, Menasha, Wisconsin, or 97 Elm Street, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, 97 Elm Street, Oberlin, Ohio, to reach him five weeks before the change becomes effective.

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MATHEMATICAL ASSOCIATION OF AMERICA

The following thirty-nine persons have been elected to membership on applications duly certified.

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P. H. ANDERSON, Ph.D.(Illinois) Instr., Louisiana State Univ., Baton Rouge, La.

I. L. BATTIN, A.M.(Swarthmore) Instr., Drew Univ., Madison, N. J.

A. V. BAUSER, A.M.(New York University) Teacher, Cooper High School, Shenandoah, Pa.

A. C. BERRY, Ph.D.(Harvard) Asso. Prof., Lawrence Coll., Appleton, Wis.

A. L. BILLIG, Ed.D.(Temple) Teacher, High School, Allentown, Pa.

B. H. BISSINGER, A.M.(Syracuse) Instr., Cornell Univ., Ithaca, N. Y.

PAUL BROCK. Student(class of '42), Brooklyn Coll., Brooklyn, N. Y.

R. E. BROWN, M.S.(Mass. Inst. of Tech.), A.M.(Boston Univ.) Asso. Prof., Mech. Engineering, Rhode Island State Coll., Kingston, R.I.

R. L. CALVERT, A.M.(Illinois) Instr., Utah State Agric. Coll., Logan, Utah

MARY D. CLEMENT, Ph.D.(Chicago) Instr., Wells Coll., Aurora, N. Y.

CLARA M. COOPER, A.M.(Pennsylvania State) Head of Dept., St. Vincent's Coll., Shreveport, La.

PLUMMER CRAFT, B.S.(Mississippi State) Stattistician, Research Dept., Masonite Corp., Laurel, Miss.

R. P. DILWORTH, Ph.D.(Calif. Inst. of Tech.) Instr., Yale Univ., New Haven, Conn.

R. T. DUFFNER. Student(class of '42), Colorado School of Mines, Golden, Colo.

J. D. ELDER, A.M.(Princeton) Asso. Prof., Math. and Physics, Lynchburg Coll., Lynchburg, Va.

N. J. FINE, A.M.(Pennsylvania) Instr., Cornell Univ., Ithaca, N. Y.

DOUGLAS FIRST. Apprentice Shipwright, New York Navy Yard, Brooklyn, N. Y.

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W. D. CAIRNS, *Secretary-Treasurer*

THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Georgetown University, Washington, D. C., on Saturday, December 6, 1941, with a morning session, luncheon, and an afternoon session. Professor E. J. McShane, chairman of the Section, presided at the sessions.

The attendance was seventy-two including the following forty-nine members of the Association: O. S. Adams, M. W. Aylor, N. H. Ball, J. D. Bankier, H. J. Barten, C. V. Bertsch, Archie Blake, C. C. Bramble, Lillian O. Brown, F. L. Celauro, C. R. Clark, Abraham Cohen, H. E. Crull, Alexander Dillingham, J. A. Duerksen, P. J. Federico, E. J. Finan, W. C. Flaherty, Michael Goldberg, T. N. E. Greville, G. A. Hedlund, L. C. Hutchinson, L. M. Kells, Evelyn M. Kennedy, Solomon Kullback, W. D. Lambert, O. E. Lancaster, A. E. Landry, Florence P. Lewis, E. J. McShane, Sister Thomas Marie Maloney, W. K. Morrill, G. W. Patterson, E. C. Phillips, O. J. Ramler, C. H. Rawlins, J. N. Rice, Irwin Roman, R. E. Root, J. B. Scarborough, Arthur Schach, E. D. Schell, W. F. Shenton, F. W. Sohon, Mary C. Varnhorn, G. C. Vedova, C. H. Wheeler III, G. T. Whyburn, R. H. Wilson.

At the invitation of the Section, Dr. L. P. Harrison, associate meteorologist of the Weather Bureau, U. S. Department of Commerce, gave an address on certain phases of meteorology.

A motion was passed expressing the appreciation of the Section to the authorities of Georgetown University for their generous hospitality.

After an address of welcome by Rev. Arthur A. O'Leary, president of Georgetown University, the following papers were read:

1. "On minimizing certain functions of triangular numbers" by Rev. E. C. Phillips, Georgetown University.
2. "Notes on the numerical evaluation of elliptic integrals" by W. D. Lambert, U. S. Coast and Geodetic Survey.
3. "Conditioned maxima and minima of functions" by Professor E. J. McShane, University of Virginia.
4. "Geometric stereograms and how to make them" by Professor W. F. Shenton, American University.
5. "On the foundations of the calculus of Eudoxus and Archimedes" by Professor G. C. Vedova, University of Maryland.
6. "Recent investigations concerning the nature of thunderstorms and their electrical manifestations" by Dr. L. P. Harrison, U. S. Weather Bureau.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Dr. Phillips discussed the functions arising from the solution of the following mathematical recreation: Given a triangle composed of $n(n+1)/2$ coins to invert it, so that its apex which was directed upward now points downward,

by moving the least possible number of coins. He endeavored to find a quadratic expression for the number, N, of coins which have to be moved in order that the new base will lie along an arbitrarily chosen line x spaces from the apex in the form $N=ax^2+bx+c$. It was found that the general value of N cannot be expressed by a single quadratic function of x but requires two distinct quadratic functions. The minimum integral values of these were determined.

2. Mr. Lambert gave two similar but not identical formulas, derived by quite different procedures, for the numerical computation of elliptic integrals of the first and second kinds for the extreme case where the modulus, k, is very nearly unity and the amplitude ϕ nearly 90°, so nearly that $k' \tan \phi > 1$. These formulas apply when standard procedures are inapplicable or inconvenient. He also discussed the computation of elliptic integrals of the third kind and found as a result of his experience with many special methods that the formulas involving Jacobi's theta functions and Jacobi's quantity q are in general most satisfactory. He also discussed tables and collections of formulas from the point of view of the computer.

4. Geometric stereograms or anaglyphes, as Vuibert called them in 1910, are geometric figures prepared in two complementary colors and viewed through glasses of the same colors. They will produce sharp black and white figures in three dimensions. Professor Shenton explained a simple method whereby an elementary school student can make successful drawings in three dimensions. He displayed drawings made by various processes.

5. Professor Vedova discussed how the attitude of the Greeks toward the infinite had influenced their conception of number and how this in turn forced the development of their integral calculus (method of exhaustions).

6. Dr. Harrison gave the results of recent investigations regarding the nature of thunderstorms. Slides were presented showing the formation of thunderheads and their electrical manifestations.

C. H. WHEELER III, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-second regular meeting of the Southern California Section of the Mathematical Association of America was held at Occidental College, Los Angeles, California, on Saturday, March 14, 1942. Professor L. J. Adams, chairman of the Section, presided.

The attendance was sixty-five, including the following thirty-eight members of the Association: C. M. Ablow, L. J. Adams, O. W. Albert, C. K. Alexander, L. D. Ames, Harry Bateman, Clifford Bell, E. T. Bell, L. T. Black, F. A. Butter, Jr., P. H. Daus, D. C. Duncan, W. H. Glenn, Jr., H. J. Hamilton, P. G. Hoel, C. G. Jaeger, G. R. Kaelin, Ada A. McClellan, G. F. McEwen, W. E. Mason, B. C. Moore, P. M. Niersbach, W. T. Puckett, Jr., H. R. Pyle, V. V. Quilliam, E. C. Rex, J. M. Robb, G. E. F. Sherwood, D. V. Steed, A. E. Taylor, C. W. Trigg, S. E. Urner, Morgan Ward, R. L. White, W. M. Whyburn, Clyde Wolfe, Euphemia R. Worthington, M. A. Zorn.

The following officers were elected for the coming year: Chairman, Morgan Ward, California Institute of Technology; Vice-Chairman, D. C. Duncan, Los Angeles City College; Program Committee, F. A. Butter, Jr., Chairman, L. T. Black, and the Secretary. The next meeting was tentatively scheduled to be held March 13, 1943, at the University of Southern California.

The following nine papers were read. The paper by Professor M. A. Zorn was an invited hour expository lecture.

1. "A boundary-value problem with conditions at k points" by Dr. C. P. Brady, Los Angeles City College, introduced by Professor Duncan.

2. "Statistics of nebulae" by Dr. G. F. N. Mulders, University of Redlands, introduced by Professor Albert.

3. "Square inscribed in a simple, closed, convex curve" by Professor H. J. Hamilton, Pomona College.

4. "Algebra—retrospect and prospect" by Professor M. A. Zorn, University of California at Los Angeles.

5. "Airplane lofting" by Professor L. J. Adams, Santa Monica Junior College.

6. "Analytic functions and conformal mapping on surfaces" by Professor H. R. Pyle, Whittier College.

7. "On certain polynomial differential operators" by E. C. Rex, University of Southern California.

8. "The part played by mathematical tables in the development of applied mathematics" by Professor Harry Bateman, California Institute of Technology.

9. "A property of Bernoulli numbers" by Carl Savit, California Institute of Technology, introduced by Professor Ward.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Dr. Brady used the method of successive approximations to establish the existence and uniqueness of a solution of a system of ordinary differential equations, with associated conditions at k points.

2. Dr. Mulders discussed the statistical treatment of nebular counts by comparing the dispersion of the nebulae with that of a random distribution, and showed that the probability of a random distribution of the external galaxies brighter than magnitude 12.7 is 1:420,000,000.

3. Professor Hamilton showed how the existence of a square inscribed in a simple, closed, convex curve followed from the consideration of a continuously turning inscribed rhombus and a limiting process.

4. Professor Zorn illustrated the principles of modern algebra by several examples from algebra and analysis. He stressed the value of the concept "indeterminate" (as opposed to "variable"), and predicted that its introduction into other fields will produce significant results.

5. Professor Adams gave a brief explanation of airplane lofting, with some of the current geometric and analytical practices used, and an indication of some of the mathematical problems which arise.

6. If we are given two surfaces S_1 and S_2 defined by their metrics and so related that the point (x, y) of S_1 corresponds to the point $(\bar{x}, \bar{y})$ of S_2 where $\bar{x}=\phi\ (x, y)$, $\bar{y}=\psi(x, y)$, we can find the functions Z and W, such that $Z=c_1$ is a minimal curve on S_1 and $W=c_2$ is a minimal curve on S_2. Professor Pyle showed that the conditions that W be an analytic function of Z are the same as those for the conformal mapping of S_2 on S_1. The conditions were found in terms of the differential parameter of the first order of differential geometry. A generalization of the Laplace equation was found in terms of the differential parameter of the second order.

7. Mr. Rex factorized the operator $f(D)=D^3+a_1D^2+a_2D+a_3$ for certain a's (which are rational integral polynomials in x). Then a particular solution was obtained for $[f(D)]y=0$.

8. As indicated by Professor Bateman, great progress was made when men interested in the stars thought about the navigation of the primitive sailing ship which was probably driven before a wind making the "right angle" with the sail. Records began to be kept of the direction of the wind and a voyage along the "right line" joining two places was probably found to be much more desirable than one along two legs of a triangle when oars had to be used part of the time. Eratosthenes of Cyrene recommended the measurement of the zenith distance of a star as an aid to the location of position. The idea was developed by Hipparchus, trigonometry was invented, and a table of chords or sines published in the Almagest of Ptolemy became well known, particularly when this book was prescribed as part of a pilot's education in the fifteenth century. Until the error in the compass was established there must have been doubts about the accuracy of the tables and, indeed, an error was located by Johannes Müller (Regiomontanus) who constructed his own tables of sines and tangents. Ten place tables with first differences were soon afterwards constructed by Georg Joachim (Rheticus) and these became of additional interest when it was claimed that a table of sines was also useful in predicting the range of a cannon ball. Tartaglia wisely dedicated his book of questions and inventions to King Henry the Eighth of England who, perhaps envious of the fame of Prince Henry the Navigator, preferred to be remembered as an authority on ballistics rather than matrimony. It soon became the fashion for a monarch to provide funds for work on tables and great mathematical progress was made when Kepler worked at the construction of new tables for Rudolph of Bohemia, when Flamsteed was appointed by Charles II of England to provide new tables for the use of English seamen and when Euler was appointed by Frederick the Great. A. Inglis in the Mathematical Gazette of 1936 has indeed surmised that Napier had the needs of navigation in mind when he invented logarithms; he may at any rate have been familiar with the work of Wright on Mercator's Projection.

9. Mr. Savit gave a proof by an elementary construction that the denominators of Bernoulli numbers recur infinitely many times.

P. H. DAUS, *Secretary*

WHAT IS ANALYSIS IN THE LARGE?

MARSTON MORSE, Institute for Advanced Study

1. Introduction. All mathematics is more or less "in the large" or "in the small." It is highly improbable that any definition of these terms could be given that would be satisfactory to all mathematicians. Nor does it seem necessary or even desirable that hard and fast definitions be given. The German terms "im Grossen" and "im Kleinen" have been used for some time with varying meanings. It will perhaps be interesting and useful to the reader to approach the subject historically by way of examples.

No proofs are given. In attempting to give the reader a conception of analysis in the large two ways are open. The first is to attempt an elementary exposition of the fundamental techniques. Unfortunately, this method of exposition is attempted much too often. The explanations given are fragmentary and give an exaggerated notion of the importance of some special technique, and no adequate notion of the subject as a whole. In a new and comprehensive field possibly the only way to give the beginner a stimulating and adequate notion of what the subject is about is to give examples and results which are themselves relatively complete. The cooperative reader can readily imagine the variety of techniques that might be used to obtain the stated results, and may himself invent new techniques, but in the presence of significant results he is less apt to be concerned with trivialities and subjective bypaths.

2. An example from differential geometry. Most of classical differential geometry is "in the small," that is, most theorems are proved merely in the a point. It is proved, for example, that in the neighborhood of a point P of a surface Σ, Σ can be referred to isothermic parameters so that neighboring P,

$$ds^2 = \lambda(u, v)\left[du^2 + dv^2\right] \tag{1.1}$$

with $\lambda(u, v) \neq 0$.* The question in the large as to what sort of closed surfaces can be represented as a whole with parameters (u, v) and ds^2 of the form (1.1) has been asked and answered in general only in recent years. It is required that there be just one curve $u = \text{const.}$ and just one curve $v = \text{const.}$ through each point. Among two-sided or orientable surfaces which admit such parameters, those of the topological type† of the torus are the only possibilities.

One could continue by asking a more general question. What sort of closed surfaces S admit a representation in terms of parameters (u, v) in such a manner that there is one and only one curve $u = \text{const.}$ and one and only one curve $v = \text{const.}$ through each point? Such a representation of S would in particular imply the existence at each point P of S of a vector tangent to the curve $u = \text{const.}$ through P. There would thus exist a field of vectors, one for each point

* Appropriate hypotheses as to the regularity of the representation of the surface must be made.

† A surface is of the topological type of the torus if it is the (1, 1) continuous image of the torus.

of P, tangent to S and P and varying continuously with P. For such a field to exist S must be the topological type of the torus.

Thus in questions as to the existence of parameter nets without singularities the controlling factors are those of topology. One can see why analysis or geometry in the large depends so heavily on topology.

3. An example from the theory of functions of a complex variable. The theorem that a function $f(z)$ of a complex variable z which has no singularities in the extended plane other than poles is a rational function of z, is a theorem in the large whose proof illustrates some of the salient characteristics of analysis in the large. One begins by representing $f(z)$ neighboring $z=z_0$ as the sum of the "principal part" of $f(z)$ at z_0 and a function analytic at z_0. This is the preliminary analysis in the small.

Upon subtracting the principal parts of $f(z)$ at each pole from $f(z)$ one obtains a function $\phi(z)$ bounded in absolute value and with at most removable singularities. According to Liouville, $\phi(z)$ is a constant. The theorem follows.

The analysis in the large comes in the proper definition of the extended plane and the proof of the Liouville theorem. Details will not be given but it will be of interest to state that the theorem of Liouville can be reduced to a theorem of topological character on the nature of vector fields.

4. Differential equations in the large. An example from the works of Henri Poincaré. It is no mere coincidence that Poincaré was the first to comprehend fully the possibilities of analysis in the large, and at the same time was the father of modern topology. Poincaré was not satisfied with the classical theory of differential equations. He wished to know something concerning the system of trajectories as a whole. He was greatly interested in the movements of the planets but found insufficient generality and completeness in the classical theory. His interest in Celestial Mechanics is in the background of all of his papers on differential equations.

Poincaré's first papers on differential equations are not pretentious in their generality, but in method they are most novel. Poincaré is concerned with an ordinary first order differential equation* defined at each point of a 2-sphere. In terms of any system of local coordinates (u, v) representing the neighborhood of a point (u_0, v_0) on the sphere the differential conditions have the form

$$\frac{du}{U(u, v)} = \frac{dv}{V(u, v)} \cdot$$

The functions U and V are supposed real and analytic in (u, v) neighboring (u_0, v_0). Points (u_0, v_0) at which both U and V vanish are termed "singular points." These points are supposed finite in number on the sphere.

Poincaré makes certain assumptions concerning the singular points (u_0, v_0). To state these conditions we shall take (u_0, v_0) as the origin. Then U and V have

* Poincaré. Sur les courbes définies par les équations différentielles, Jour. de Liouville, 1881, 1882.

developments of the form

$$U = au + bv + \cdots,$$
$$V = cu + dv + \cdots$$

neighboring the origin. Poincaré assumes in most of his work that the roots λ_1 and λ_2 of the equation

$$\begin{vmatrix} a-\lambda, & b \\ c & d-\lambda \end{vmatrix} = 0$$

are distinct, different from 0, never pure imaginary and that neither $\lambda_1|\lambda_2$ nor $\lambda_2|\lambda_1$ is a positive integer. These conditions will be satisfied by most analytic examples.

Curves on the sphere which satisfy the differential equation are termed *characteristics*. In general characteristics are without singularity except at most when they pass through a singular point of the differential equation. Typical of analysis in the large, Poincaré's work permits a subdivision into three parts as follows:

(a). *A study of characteristics neighboring a singular point.*

(b). *The assignment of an index* ± 1 *to each singular point and the establishment of a relation between these indices. This part of the analysis would now be regarded as an essay in combinatorial topology.*

(c). *A description of the characteristics in the large with particular reference to recurrence and limiting trajectories; Results* (a) *and* (b) *are preliminary to* (c).

PART (a). In his study of characteristics neighboring a singular point, Poincaré shows that there are three principal kinds of singular points as follows:

"*Noeud.*" Neighboring a noeud (u_0, v_0) each characteristic tends to (u_0, v_0) with a definite limiting direction. For example the differential equation

$$\frac{du}{u} = \frac{dv}{2v}$$

has a noeud at the origin. In this example the characteristics have the form $kv = hu^2$ where h and k are constants.

"*Foyer.*" The characteristics approach such a singular point in the form of spirals, with the arc length becoming infinite. For example, the differential equation

$$\frac{du}{u-v} = \frac{dv}{u+v}$$

has a foyer at the origin with logarithmic spirals as characteristics.

"*Col.*" There are just two characteristics which tend to a col as a limiting point. For example, the equation

$$\frac{du}{u} = \frac{dv}{-v}$$

has a col at the origin. The characteristics $uv=\text{const.}$ include the two characteristics $u=0$ and $v=0$ passing through the origin.

PART (b). In the development (b), Poincaré assigns an index 1 to each noeud and to each foyer, and an index -1 to each col. Poincaré shows that *the sum of the indices of the of the singular points on the sphere equals* 2. Thus there must exist at least two singular points.

A closed characteristic without a multiple point is called a *cycle.* If a characteristic tends to a noeud or a foyer as a limit point there is in general no natural way to continue the characteristic, and it is agreed that in such cases the characteristic shall end at the noeud or foyer. If a characteristic g tends to a col the convention is made that g may be continued turning either to the right or left and departing from the col on a characteristic. By virtue of this convention the notion of a cycle is enlarged. With this understood we see that a cycle can have no singularity other than those occurring at a col.

PART (c). Poincaré ends with a relatively complete description of the characteristics. He shows that *a characteristic continued without limit in a given sense either terminates at a noeud, or is a cycle, or is asymptotic to a cycle.* A foyer is to be regarded as a degenerate cycle to which the neighboring spirals are asymptotic.

The reader is asked to observe the fundamental difference between the modes of analysis required in Parts (a), (b), and (c), and then to note how (a) and (b) are preliminary to (c) and make (c) possible. The index theorem of Poincaré has its topological generalization in the fixed point theorems of Brouwer, Alexander, Lefschetz, and H. Hopf. The analysis of characteristics in (c) is the predecessor of the modern study of recurrence and transitivity which G. D. Birkhoff has developed so fully and to which Hedlund, Morse, von Neumann, Koopman, E. Hopf and others have made significant contributions.

5. Elementary examples in equilibrium theory in the large. Equilibrium theory in the large makes an extensive use of topology. The principles of analysis brought out in the previous examples appear here again. Briefly summed we have seen in these examples that analysis in the large has involved (a) a preliminary analysis in the small, (b) a local determination of indices, (c) an integration of this local analysis by various means (including topology) into the final theorems in the large. The examples which we shall now present will show how various problems which from a local point of view appear most diverse, from a topological point of view are essentially the same.

We begin with certain results concerning a function f of a point on a closed bounded n-manifold Σ lying in an euclidean space of sufficiently high dimension. We suppose throughout that Σ is locally represented in terms of n parameters (u) with convenient conditions of differentiability and regularity. In terms of the local parameters (u) f shall be a function $F(u)$ at least three times continu-

ously differentiable. A *critical* or *equilibrium point* of f is a point at which each partial derivative of F is null.

For the purposes of this exposition we shall make an assumption which is in general fulfilled. We shall suppose that each critical point is *non-degenerate* in the sense that the terms F_2 of the second order in the Taylor's formula for F about the critical point is a non-degenerate quadratic form. Then, as in the elementary theory of conic sections it is possible to make a real non-singular linear transformation from the variables (u) to the variables (v) such that F_2 takes the form

$$F_2 = -v_1^2 - \cdots - v_k^2 + v_{k+1}^2 + \cdots + v_n^2$$

The number k is called the *index* of the critical point.

A manifold such as Σ possesses an ith Betti number R_i $(i=1, \cdots, n)$. This is the maximum number of independent non-bounding i-cycles* on Σ. For example, if Σ is a torus then $R_0=1$, $R_1=2$, $R_2=1$. We shall be concerned with a 3-dimensional torus T_3. Such a manifold can be obtained by starting with a 2-dimensional torus T_2 and a 2-plane π_2 lying in a euclidean 3-plane, with π_2 not intersecting T_2. To obtain T_3 we revolve T_2 about π_2 in a 4-plane containing our 3-plane. Such a T_3 is sometimes called a product of three circles. For T_3 one has $R_0=1$, $R_1=3$, $R_2=3$, $R_3=1$. These numbers are the binomial coefficients when $n=3$. We can obtain a 1–1 continuous image of an ordinary torus by identifying opposite sides of a square. Similarly one can obtain a 1–1 continuous image of T_3 by identifying opposite faces of a cube. With this identification three mutually perpendicular edges of the cube represent three independent non-bounding 1-cycles, as can be shown. Similarly three mutually perpendicular faces of the cube represent three independent non-bounding 2-cycles. A point is a 0-cycle and T_3 itself is a 3-cycle. In this way one intuitively accounts for the fact that $R_0=1$, $R_1=3$, $R_2=3$, $R_3=1$. A 3-dimensional manifold which is a 1–1 continuous image of T_3 will be called a *topological* 3-torus.

The theorem which will be used in what follows is that on Σ the number M_i of critical points of f of index i satisfies the fundamental relation†

$$M_i \geqq R_i. \tag{5.1}$$

Thus on a topological 3-torus one can infer the existence of at least $1+3+3+1=8$ critical points.

Example 5.1. *Triangles of light.* Let there be given three non-intersecting, simple, closed, non-singular, analytic curves C_1, C_2, C_3 all lying in a 2-plane. We shall be concerned with triangles with vertices p_1, p_2, p_3 on C_1, C_2, C_3 respectively. Such a triangle will be called a *triangle of light* if a ray of light following

* For details see Seifert-Threlfall, Lehrbuch der Topologie, Leipzig, 1934. Chap. III.

† See Morse, Calculus of variations in the large. Colloquium Lectures. Amer. Math. Soc. (1934) Chap. VI.

Also, Seifert-Threlfall. Variationsrechnung im Grossen, Leipzig, 1938.

this triangle is reflected at p_i as if C_i were a mirror, or if the angle in the triangle at p_i is π. How many triangles of light can we affirm to exist?

Let f be the sum of the lengths of the sides of the triangle $p_1p_2p_3$. We can refer C_i to a parameter u_i which is proportional to the arc length and varies from 0 to 2π. Then f becomes a function $f(u_1, u_2, u_3)$. The domain of definition of f is clearly a topological 3-torus. As a matter of analysis in the small one proves by elementary methods that f has a critical point if and only if the corresponding triangle is a triangle of light.

These triangles of light can then be classified according to the index of the corresponding critical point. According to relation (5.1) in the general theory of critical points there are at least $8=1+3+3+1$ of these triangles of light.

Example 5.2. *Normals from a point to a topological* 3-*torus*. Let Σ_3 be a topological 3-torus in a euclidean 4-space. Let p be a fixed point not on Σ_3. We seek normals from p to Σ_3. To obtain these we let f be the distance from p to Σ_3 regarding f as a function of the point (u) of Σ_3. It can be shown that except for a subset of special points p the critical points of f are non-degenerate. Moreover one then shows by a local analysis that f has a critical point (u) if and only if the line segment from p to (u) is normal to Σ_3 at (u). According to our general theorem there are then at least eight normals from p to Σ_3. These normals can be classified and it can be shown that the index of a non-degenerate critical point is the number of centers of principal curvature of Σ_3 between p and (u) on the given normal. Similar theorems hold for a topological 2-torus. Here the number of normals is at least 4.

Example 5.3. 3-*planes passing through a fixed* 2-*plane and tangent to the preceding topological* 3-*torus* Σ_3. We suppose that the fixed 2-plane π_2 does not pass through a hole in Σ_3, that is, we suppose that π_2 can be moved indefinitely away from Σ_3 without intersecting Σ_3. We can then show that if π_2 is non-specialized there are at least eight 3-planes through π_2 tangent to Σ_3.

Example 5.4. *Heavy chain in equilibrium, with ends free to move on a topological* 2-*torus and on a closed curve C respectively*. We suppose that the curve C and the topological 2-torus Σ_2 lie in euclidean three space but that no point of C and Σ_2 lie on the same plumb line. We suppose that a chain is provided which is larger than the maximum distance from a point of C to a point of Σ_2. The end points of the chain are supposed free to move on Σ_2 and C respectively and the chain is permitted to pass through Σ_2 or C. If the position of C is non-specialized relative to Σ_2, then there are at least eight positions of equilibrium of the chain, seven of which are unstable. The function whose critical points are sought gives the height of the center of gravity of the chain as a function of the end points of the chain.

The examples of this section are unified by the fact that the function involved is defined in each case on a topological 3-torus. The examples belong equally well to mechanics, geometry, or the calculus of variations. The restrictions as to non-specialized positions of the configurations involved can all be removed by replacing the definition of a critical point in terms of derivatives by a topological

definition of a critical point, and by replacing the classification of critical points according to their indices by a topological classification of a group theoretic character. This type of generalization both in its form and genesis is characteristic of analysis in the large.

It is possible that analysis in the large may eventually reduce to topology, but not until topology has been greatly broadened. It is equally conceivable that the apparently less general situations which arise with such frequency in problems in analysis in the large may form the canonical cases about which the topology of the future can be built.

Analysis is full of difficult but significant unsolved problems in the large. We mention only one example. How does the topological structure of the contour manifolds of the Jacobi least action integral J in the problem of three or more celestial bodies vary with the value of J? The independent variable in J is a closed path. The solution of this problem may disclose that the planetary orbits exist for essentially topological reasons. On the purely topological side the number of problems whose solution is necessary for a rapid advance of analysis in the large is very great, presenting a field that is virtually untouched.

ON PORISTIC QUADRILATERALS

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1. Introduction. In 1828 Jacobi [1] found the condition that a closed polygon of n sides can be inscribed in one circle and circumscribed about another. Special cases of this problem had been solved prior to 1828 by Fuss [2] and Poncelet [3]. Later Cayley [4] generalized the problem by considering polygons inscribed in one ellipse and circumscribed about another. W. E. Byerly [5] gave some interesting constructions and his paper has further references.

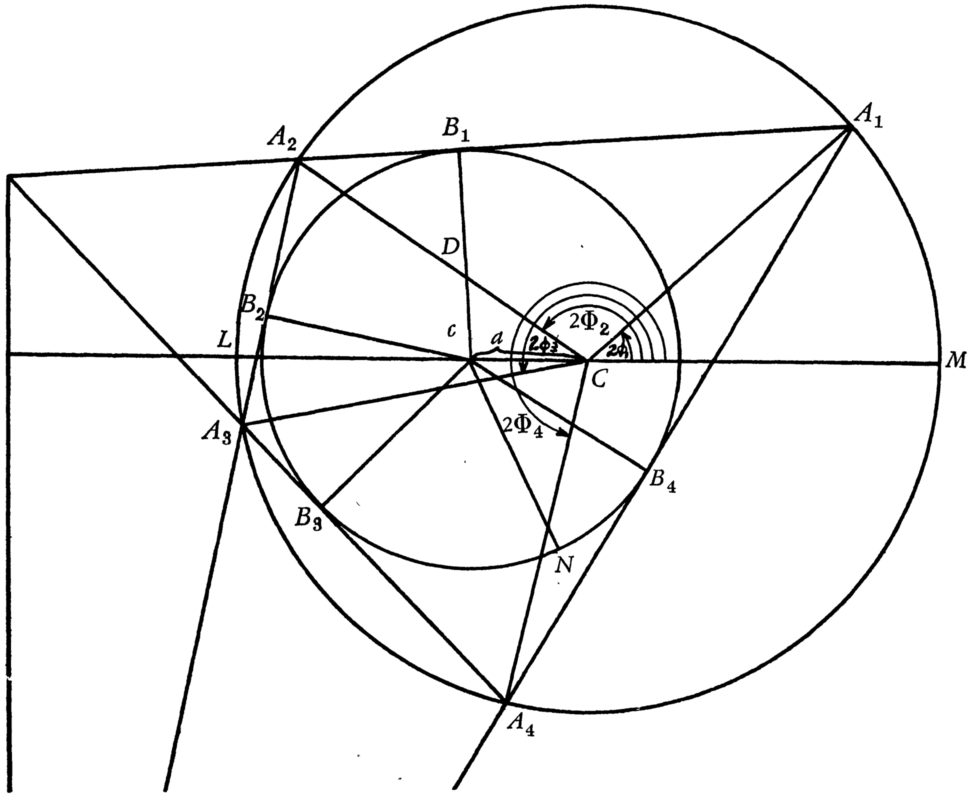
Jacobi used the theory of elliptic functions to obtain his general solution of the problem. Since the elementary theory of elliptic functions is well within the range of information of students of advanced calculus, it seems that a review of Jacobi's results with a few new theorems may be of interest.

2. Definition and relations. A poristic polygon is one which is inscribed in one circle (ellipse) and circumscribed about another. In the adjoining figure $A_1A_2A_3A_4$ is such a polygon. We denote the radii of the two circles by R and r and the distance between their centers by a. Let us choose an arbitrary point, A_1, on the larger circle as a starting point and draw the line A_1A_2 tangent to the smaller circle. Let the centers of the two circles be C and c. The diameter LM passes through c and C. Let CA_1 make an angle $2\phi_1$ with LM. The lines CA_2, CA_3, CA_4 make the angles $2\phi_2$, $2\phi_3$, and $2\phi_4$ respectively with LM. We shall use C as the origin and LM as the x-axis of a rectangular system of coördinates whenever we need to write the equations of any line or the coordinates of a point.

From the figure and from the above definitions, the following simple relations are easily obtained:

The acute angle at D is $\phi_2-\phi_1$, the angle B_1cC is $\phi_2+\phi_1$,

$$cB_1 = R\cos(\phi_2-\phi_1) = a\cos(\phi_2+\phi_1).$$



The last relation can be written in the form

$$\frac{r}{R+a} = \cos\phi_2\cos\phi_1 + \frac{R-a}{R+a}\sin\phi_2\sin\phi_1.$$

Let

$$\cos\alpha = \frac{r}{R+a}, \quad \text{then} \quad 1-k^2\sin^2\alpha = \frac{(R-a)^2}{(R+a)^2} \quad \text{where} \quad k^2 = \frac{4aR}{(R+a)^2-r^2}.$$

One can easily prove that $k^2<1$ if $r<R-a$. Then we have

$$\cos\alpha = \cos\phi_2\cos\phi_1 + \sqrt{1-k^2\sin^2\alpha}\,\sin\phi_2\sin\phi_1. \tag{1}$$

3. Lagrange's formula. At this point Jacobi must have noticed the similarity between equation (1) and Lagrange's formula. In fact, Lagrange's formula is

$$\operatorname{cn}(u-v) = \operatorname{cn} u \operatorname{cn} v + \operatorname{sn} u \operatorname{sn} v \operatorname{dn}(u-v),$$

and if we substitute $v=u+T$, we have

$$\operatorname{cn} T = \operatorname{cn} u \operatorname{cn}(u+T) + \operatorname{sn} u \operatorname{sn}(u+T) \operatorname{dn} T. \tag{2}$$

Equations (1) and (2) are identical if

$$\alpha = \operatorname{am} T \qquad \text{or} \qquad T = \int_0^{\alpha} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}},$$

$$\phi_1 = \operatorname{am} u \qquad \text{or} \qquad u = \int_0^{\phi_1} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}},$$

$$\phi_2 = \operatorname{am}(u+T) \text{ or } u+T = \int_0^{\phi_2} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}.$$

4. The closed polygon. When the second tangent is drawn with one end at A_2 and the other at A_3 it follows that ϕ_3 is related to ϕ_2 just as ϕ_2 is related to ϕ_1. Thus $\phi_3=\operatorname{am}(u+2T)$. And in general $\phi_{n+1}=\operatorname{am}(u+nT)$. Now if the $n+1$ vertex is made to coincide with A_1 we shall have a closed polygon and in order that this be the case we must have $\phi_{n+1}=\phi_1+h\pi$ where h is a positive integer. Hence,

$$u+nT = \int_0^{\phi_1+h\pi} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = \int_0^{\phi_1} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} + \int_0^{h\pi} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}.$$

From well known theorems on elliptic integrals it follows that $u+nT=u+2hK$ and $nT=2hK$ where K is the complete elliptic integral of the first kind. This is Jacobi's criterion for a closed polygon of n sides. If the polynomial is a quadrilateral $n=4$ and $h=1$ so that

$$T = \frac{K}{2}. \tag{3}$$

Jacobi remarked that, since this condition is independent of ϕ_1, it follows that that starting point is quite arbitrary, and we may get as many closed poristic polygons as we please by varying the starting point.

The condition (3) may be expressed in terms of an elliptic integral in the form

$$\frac{K}{2} = \int_0^{\alpha} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}.$$

5. A relation between R, r, and a. Equation (3) together with $r<R<a$ are necessary conditions for a closed poristic quadrilateral. Further, equation

(2) is an identity in u. Hence, if we can draw one quadrilateral, we can draw as many as we please. For convenience we choose $u=K/2$. Noting that sn $K=1$ and cn $K=0$ equation (2) reduces to

$$\text{cn } K/2 = \text{sn } K/2 \text{ dn } K/2.$$

Since

$$\text{cn } K/2 = \text{cn } T = \cos\alpha = \frac{r}{R+a},$$

we have

$$\frac{r}{R+a} = \sqrt{1 - \frac{r^2}{(R+a)^2}}\,\frac{R-a}{R+a}.$$

which easily reduces to

$$\frac{r^2}{(R+a)^2} + \frac{r^2}{(R-a)^2} = 1. \tag{4}$$

It follows that

$$\sin\alpha = \frac{r}{R-a}.$$

6. Sufficient conditions. Equation (3) and $r<R-a$ are necessary for a closed poristic quadrilateral. We shall also prove that they are sufficient. For suppose that they are satisfied. Then (4) is true. Draw two tangent lines, below the x-axis, to the smaller circle, one passing through M and the other through L. Draw the radii to the point of contact. Then the angle McN is the angle α. The acute angle at L is also equal to α since $\sin\alpha=r/(R-a)$. It follows that the tangent line passing through L is perpendicular to the tangent line through M. Hence they must meet on the circumference of the larger circle. Therefore, if (4) is satisfied we can draw at least one poristic quadrilateral. Hence if (4) and (3) are satisfied we can draw as many as we please.

If the poristic polygon is a triangle the relation corresponding to (4) is

$$\frac{r}{R+a} + \frac{r}{R-a} = 1.$$

These two relations and corresponding equations for poristic pentagons and hexagons were known before Jacobi's paper of 1828, but the general conditions for a closed polygon were first given by Jacobi.

7. The quadrilateral $B_1B_2B_3B_4$. Let B_1, B_2, B_3, B_4 be the points at which the sides of the polygon $A_1A_2A_3A_4$ are tangent to the smaller circle. We shall now prove a theorem about the diagonals of the quadrilateral $B_1B_2B_3B_4$. The coordinates of B_1, B_2, B_3 and B_4 are,

$$[r \cos(\phi_1+\phi_2) - a,\ r \sin(\phi_1+\phi_2)],$$
$$[r \cos(\phi_2+\phi_3) - a,\ r \sin(\phi_2+\phi_3)],$$
$$[r \cos(\phi_3+\phi_4) - a,\ r \sin(\phi_3-\phi_4)],$$
$$[-r \cos(\phi_4-\phi_1) - a,\ -r \sin(\phi_4+\phi_1)].$$

The slopes of B_1B_3 and of B_2B_4 are easily found to be $-\cot S/2$ and $\tan S/2$ where $S=\phi_1+\phi_2+\phi_3+\phi_4$. Hence the equations of the internal diagonals of $B_1B_2B_3B_4$ are

$$y - r\sin(\phi_1+\phi_2) = -[x+a-r\cos(\phi_1+\phi_2)]\cot S/2,$$
$$y - r\sin(\phi_2+\phi_3) = [x+a-r\cos(\phi_2+\phi_3)]\tan S/2$$

We are interested in the point of intersection of B_1B_3 and B_2B_4. Solving the equations we find the coördinates to be given by

$$2(x+a)\csc S = r\sin(\phi_1+\phi_2) - r\sin(\phi_2+\phi_3) + r\cos(\phi_1+\phi_2)\cot S/2 + r\cos(\phi_2+\phi_3)\tan S/2,$$

and

$$2y\csc S = r[\cos(\phi_1+\phi_2) - \cos(\phi_2+\phi_3) + \sin(\phi_2+\phi_3)\cot S/2 + \sin(\phi_1+\phi_2)\tan S/2].$$

In order to simplify these expressions we make use of some fundamental relations given by Jacobi on page 285 of his collected works. The relations we need are

$$\tan\frac{\phi_1+\phi_3}{2} = \sqrt{k_1}\tan\phi_2, \qquad \tan\frac{\phi_2+\phi_4}{2} = \sqrt{k_1}\tan\phi_3,$$
$$\tan\frac{\phi_3+\phi_5}{2} = \sqrt{k_1}\tan\phi_4, \qquad k_1^2 = 1 - k^2 = \left(\frac{R-a}{R+a}\right)^2.$$

In the closed quadrilateral $\phi_5=\pi+\phi_1$. Hence the third relation above becomes, by using the first,

$$-\cot\phi_2 = k_1\tan\phi_4.$$

The above formulas enable us to write $\tan S/2$ in the form

$$\tan S/2 = \tan\left\{\frac{\phi_1+\phi_3}{2}+\frac{\phi_2+\phi_4}{2}\right\} = \frac{\sqrt{k_1}\sin(\phi_2+\phi_3)}{\cos\phi_2\cos\phi_3 - k_1\sin\phi_2\sin\phi_3}.$$

But from the addition formulas for the elliptic functions we have

$$\sin\phi_3 = \operatorname{sn}(u+K) = \frac{\operatorname{cn} u}{\operatorname{dn} u} = \frac{\cos\phi_1}{\operatorname{dn} u},$$

and

$$\cos\phi_3 = -\frac{k_1\sin\phi_1}{\operatorname{dn} u}.$$

Hence

$$-\cot \phi_1 = k_1 \tan \phi_3.$$

This enables us to write

$$\tan S/2 = \frac{k_1 \sin \phi_1 \sin \phi_2 - \cos \phi_1 \cos \phi_2}{\sqrt{k_1} \sin (\phi_1 + \phi_2)}.$$

In the equation which gives y above we replace $\cot S/2$ by the reciprocal of the first form for $\tan S/2$ and replace $\tan S/2$ by its second form, then immediately all the terms in the right member cancel and therefore $y=0$.

To find a simpler form for x we multiply by $\sin S$, combine the proper terms, and remove the cosines by the formula

$$\cos \phi_i \cos \phi_{i+1} = -\sqrt{k_1} \sin \phi_i \sin \phi_{i+1} + \frac{r}{R+a}, \qquad i = 1, 2, 3.$$

There results

$$\left\{\frac{2(x+a)}{r} - \frac{4r}{R+a}\right\} \frac{1}{1+\sqrt{k_1}} = -\sin \phi_2(\sin \phi_1 + \sin \phi_3) + \sin \phi_4(\sin \phi_1 - \sin \phi_3),$$

or

$$\frac{2(x+a)(R+a) - 4r^2}{2Rr} = -\operatorname{sn}(u+T)\left(\operatorname{sn} u + \frac{\operatorname{cn} u}{\operatorname{dn} u}\right) + \frac{\operatorname{cn}(u+T)}{\operatorname{dn}(u+T)}\left(\operatorname{sn} u - \frac{\operatorname{cn} u}{\operatorname{dn} u}\right).$$

If we expand $\operatorname{sn}(u+T)$, $\operatorname{cn}(u+T)$, and $\operatorname{dn}(u+T)$ and simplify, all of the terms containing elliptic functions of u cancel and the right member reduces to $-2 \operatorname{sn} K/2$. Hence

$$x = -\frac{2ar^2}{R^2 - a^2} - a.$$

THEOREM. *The internal diagonals of the quadrilateral $B_1B_2B_3B_4$ are mutually perpendicular and intersect in a fixed point on the line of centers of the two circles.*

8. The external diagonal of $A_1A_2A_3A_4$. The external diagonal of the quadrilateral $A_1A_2A_3A_4$ is the polar of the point of intersection of the internal diagonals of the quadrilateral $B_1B_2B_3B_4$. In the notation which we have employed the equation of the external diagonal is

$$x = -\frac{R^2 + a^2}{2a}.$$

Since this is independent of ϕ_1 it follows that the points of intersection of the opposite sides of the poristic quadrilateral $A_1A_2A_3A_4$ move along the line as ϕ_1 varies.

THEOREM. *The external diagonal of all the poristic quadrilaterals that can be drawn to two fixed circles is a fixed straight line perpendicular to the line of centers of the two circles.*

This theorem enables us to draw as many different poristic quadrilaterals as we please with very little labor. We first choose R and a then compute the right member of the equation of the external diagonal and draw the line. After the two circles have been drawn we choose any point on the external diagonal and draw two tangent lines to the smaller circle. These tangent lines cut the larger circle in the vertices of a poristic quadrilateral.

9. Symmetric quadrilaterals. If we start from the point M and draw a poristic quadrilateral then $\phi_1=0$ and am $u=0$. Since $\phi_3=\text{am }(u+2T)$ it follows that $\phi_3=\pi/2$, that is, the end point of the second tangent line is at the end of the diameter opposite to M. Since we can measure ϕ_2 and ϕ_3 clockwise as well as counterclockwise it follows that every poristic quadrilateral which has one vertex at M is symmetrical with respect to the line of centers of the two circles.

10. Poristic quadrilaterals for two ellipses. Let us draw a plane making the angle θ with the plane of the two circles of §2. This plane may pass through the external diagonal of the quadrilateral $A_1A_2A_3A_4$. Project the figure on this plane by drawing perpendiculars from all of its points to the new plane. Then it is obvious that the two circles project into ellipses; that the projections of the diagonals of $B_1B_2B_3B_4$ will still intersect on the line of centers of the two ellipses and that the theorem of §8 remains true in the projected figure.

11. The quadrilateral $B_1B_2B_3B_4$. We have already proved that the internal diagonals of the quadrilateral $B_1B_2B_3B_4$ intersect on the x-axis and at a fixed point. They are also mutually perpendicular. Now we rotate the quadrilateral $A_1A_2A_3A_4$, by varying ϕ_1, so that the point B_2 is on the x-axis. When this is done the diagonal B_2B_4 must coincide with the x-axis, so that B_2B_4 is a diameter of the smaller circle. The diagonal B_1B_3 is perpendicular to B_2B_4 and is bisected by it. Hence $B_1B_2B_3B_4$ is symmetrical to the x-axis and it is obvious that a third circle can be inscribed in it. Hence the quadrilateral $B_1B_2B_3B_4$ is a poristic quadrilateral. The center of the third circle may be determined by bisecting the interior angle at B_1. We may determine the coordinates of the center by computing the lengths of B_1B_2 and B_1B_4 and using the theorem that the bisector of any angle of a triangle divides the opposite side into segments which are in the same ratio as the adjacent sides. If the segments of B_2B_4 are $2r-x$ and x we find

$$\frac{2r-x}{x}=\tan(\phi_1+\phi_2)/2 \quad \text{or} \quad x=\frac{2r}{[1+\tan(\phi_1+\phi_2)/2]}.$$

The coördinates of the center of the third circle turn out to be

$$\left\{ - r\,\frac{[1 - \sin(\phi_1 + \phi_2)]}{\cos(\phi_1 + \phi_2)} - a,\ 0 \right\}.$$

We can simplify still further the expressions $\sin(\phi_1+\phi_2)$ and $\cos(\phi_1+\phi_2)$. We notice that B_2 and B_4 are both on the x-axis and $\cos 2\phi_1 = (r-a)/R$, or $\sin \phi_1 = \sqrt{(R-r+a)/2R}$ and $\cos \phi_1 = \sqrt{(R+r-a)/2R}$, $\cos 2\phi_2 = -(r+a)/R$, or $\sin \phi_2 = \sqrt{(R+r+a)/2R}$, and $\cos \phi_2 = \sqrt{(R-r-a)/2R}$. Then, in terms of R, r, and a, the coördinates of the center of the third circle are

$$\left[\left\{ \frac{\sqrt{(R-r)^2 - a^2} + \sqrt{(R+r)^2 - a^2}}{4aR} \right\} (R^2 - a^2) - a,\ 0 \right].$$

This last expression is not very simple. It is given for the purpose of showing that the coördinates of the third circle may be calculated in terms of R, r, and a.

We can now state the theorem: *Every poristic quadrilateral, $A_1A_2A_3A_4$ determines another poristic quadrilateral $B_1B_2B_3B_4$.*

It is, of course, obvious that we may continue this process indefinitely.

Let us change the notation so that the radius of the first circle is R_1, of the second R_2 and so on. Let the distance between the centers of the first and second circles be a_1, between the centers of the second and third be a_2, and so on. Then we always have the relation

$$\left(\frac{R_{i+1}}{R_i + a_i}\right)^2 + \left(\frac{R_{i+1}}{R_i - a_i}\right)^2 = 1.$$

One cannot help but wonder what happens to this relation as i increases indefinitely. It seems probable that a_i approaches 0 and that R_{i+1} approaches $R_i/\sqrt{2}$.

We have made no attempt to apply these results to the case, which Byerly discusses, in which the second circle is entirely outside the first. It would be interesting to follow the discussion through in that case.

References

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2. Nicolaus Fuss, Nova Acta. Petersburg, Band 13, pp. 166–189.
3. Poncelet, Traite des Propriétés Projectives des Figures. S. 361.
4. Cayley, Collected Works. Vol. IV. p. 292.
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REMARKS ON TERNARY DIOPHANTINE EQUATIONS

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Let $F(x, y)$ be a binary form of order $n \geqq 3$ with integral coefficients, $k \neq 0$ an integer, $P_1, \cdots, P_t$ a finite set of different prime numbers. The following results have been proved:*

THEOREM 1. *If the equation*

$$F(x, y) = k$$

has an infinity of integral solutions x, y, then $F(x, y)$ is a power of a linear form or of an indefinite quadratic form.

THEOREM 2. *If the equation*

$$F(x, y) = \mp P_1^{z_1} \cdots P_t^{z_t}$$

has an infinity of integral solutions $x, y, z_1, \cdots, z_t$, where x and y are relatively prime, and $z_1 \geqq 0, \cdots, z_t \geqq 0$, then $F(x, y)$ is a power of a linear or quadratic form.

Now consider a ternary form $F(x, y, z)$ of order $n \geqq 3$ with integral coefficients, and the representations of integers $k \neq 0$ by this form. If $F(x, y, z)$ is decomposable into a product of linear forms with algebraic coefficients and if n is sufficiently large, then results analogous to Theorem 1 and 2 are true.† On the other hand, very little is known about the more general case when $F(x, y, z)$ is irreducible in the field of all constants. In this note, I construct examples of ternary forms of every order with the property of representing at least one integer $k \neq 0$, or even every integer, in an infinity of different ways. I further show how to form positive definite ternary forms of every even order with the property that for an infinity of different sets of relatively prime integers x, y, z the greatest prime factor of $F(x, y, z)$ is bounded. Special cases of forms with these properties are well known; *e.g.*, the equation

$$x^3 + y^3 + z^3 = 1$$

has an infinity of integral solutions, since identically in t,

$$(9t^4)^3 + (3t - 9t^4)^3 + (1 - 9t^3)^3 = 1;$$

and the equation

$$x^4 + y^4 + z^4 = 2 \cdot 7^{z_1}$$

has an infinity of integral solutions with relatively prime x, y, z, since identically in x and y,‡

* A. Thue, Norske Vid. Selsk. Skr. 1908, Nr. 7. K. Mahler, Math. Ann. 107, 1933, pp. 691–730.
† C. L. Siegel, Math. Z. 10, 1921, pp. 173–213.
E. T. Parry, Journal of the London Mathematical Society, 1940, vol. 15, pp. 293–305.
‡ The equation $x^2+xy+y^2=7^z$ has an infinity of integral solutions with relatively prime x, y.

$$x^4 + y^4 + (x+y)^4 = 2(x^2 + xy + y^2)^2.$$

The stated results are obtained by the construction of simple identities. In a similar way, it is possible to show the following theorem: "*If $F(x, y, z)$ is an irreducible cubic form with integral coefficients, such that the equation $F(x, y, z) = 0$ has at least one solution in integers not all zero, then $F(x, y, z)$ either represents all integers in a suitable linear progression $at+b$ $(t = 0, \mp 1, \mp 2, \cdots)$ or it represents a certain integer $k \neq 0$ in an infinity of different ways.*" Let g, h, k be three integers not all zero such that

$$F(g, h, k) = 0,$$

and denote with F_g, F_h, F_k the values of the three first partial derivatives $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$ for $x = g$, $y = h$, $z = k$. There are three integers not all zero such that

$$GF_g + HF_h + KF_k = 0.$$

Let now t be a parameter; then

$$F(gt + G, ht + H, kt + K) = At^3 + Bt^2 + Ct + D$$

is a cubic polynomial in t. This polynomial cannot vanish identically, since $F(x, y, z)$, by hypothesis, is irreducible. It is however at most of the first degree. For the assumptions about g, h, k; G, H, K are equivalent to $A = B = 0$. According as to whether $C \neq 0$ or $C = 0$, F represents all integers of the progression $Ct + D$, or is equal to D for all values of t. In the second case $D \neq 0$, since $F \not\equiv 0$.

1. Ternary equations with an infinity of solutions. The general ternary form $F(x, y, z)$ of order n has

$$N = \frac{(n+1)(n+2)}{2}$$

coefficients. If

$$p_1(t), \quad p_2(t), \quad p_3(t) \qquad \left(\sum_{h=1}^{3} |\, p_h(t) \,| > 0 \text{ for all } t\right)$$

are three polynomials in a parameter t with integral coefficients and of degree less than or equal to ν, then

$$F(p_1(t), p_2(t), p_3(t)) = \phi(t)$$

is a polynomial in t of degree not greater than $n\nu$; its coefficients are linear forms in the coefficients of $F(x, y, z)$ with integral numerical coefficients; say

$$\phi(t) = \sum_{h=0}^{n\nu} L_h(F) t^h.$$

If $\phi(t)$ is to be a constant, then the coefficients of $F(x, y, z)$ must satisfy the $n\nu$ linear equations

$$L_h(F) = 0 \qquad (h = 1, 2, \cdots, n\nu).$$

For $N > n\nu$, this system has always a non-trivial solution in integers; there is therefore then a ternary form of order n with integral coefficients not all zero such that

$$F(p_1(t), p_2(t), p_3(t)) = L_0(F)$$

is independent of t. The so constructed form $F(x, y, z)$ may, however, be reducible, and the constant $L_0(F)$ on the right-hand side may vanish. In the special case $\nu = 1$ of linear polynomials $p_h(t)$ both complications can always be avoided, and it is possible to determine irreducible forms $F(x, y, z)$ of every order n such that the constant $\phi(t) = L_0(F) \neq 0$.*

The form $F(x, y, z)$ constructed in this manner is *not* definite; it assumes the values $k = L_0(F)$ for every integral value $t = 0, \mp 1, \mp 2, \cdots$. I give here a few examples of this kind:

$x^3 + y^3 + z^3 + \lambda xyz = \lambda^3 + 27$ identically in t for $x = t + \lambda$, $y = -t$, $z = 3$;

$y^2z - 12x^3 + 3z^3 = 12$ identically in t for $x = t + 1$, $y = 3t$, $z = t + 2$;

$2x^4 - y^4 - z^4 + 2x^2y^2 + 2x^2z^2 - 4y^2z^2 = -6$ identically in t for $x = t$, $y = t + 1$, $z = t - 1$;

$2(x^4 + y^4) - (x - y)^2 z(3x - 3y + z) = 4$ identically in t for $x = t + 1$, $y = t - 1$, $z = t^2$.

2. Forms which represent every integer. Let α, β, γ be three integers and assume that the form $F(x, y, z)$ of order n has the following property:

"On replacing z by $\alpha x + \beta y + \gamma$, we get identically in x and y,

$$F(x, y, \alpha x + \beta y + \gamma) = p(x) + ay, \tag{1}$$

where a is a constant and $p(x)$ a polynomial in x both depending on F."

Assume the form has this property, and let

$$\xi_1, \xi_2, \cdots, \xi_s$$

be the different residues of $p(x)$ mod a; since

$$p(x) \equiv p(x') \bmod a \quad \text{for} \quad x \equiv x', \bmod a,$$

each congruence

$$p(x) \equiv \xi_\sigma \bmod a \qquad (\sigma = 1, 2, \cdots, s)$$

has an infinity of integral solutions x. Hence if k is any integer in one of the residue classes

$$k \equiv \xi_\sigma \bmod a \qquad (\sigma = 1, 2, \cdots, s),$$

then

$$p(x) + ay = k$$

* This corresponds to the fact that there are irreducible algebraic curves of every order n which intersect a given straight line only in n coinciding points.

has an infinity of integral solutions x, y, and therefore the equation $F(x, y, z)=k$ has an infinity of integral solutions, x y, z.

The following forms satisfy the identity (1):

$$F_l(x, y, z) = x^{n-l}(z-\beta y)^l \qquad (l = 0, 1, \cdots, n-1); \tag{2}$$
$$F_n(x, y, z) = y(z-\alpha x-\beta y)^{n-1},$$

as is evident on putting $z=\alpha x+\beta y+\gamma$. A further and less obvious solution is obtained in the following way:

We assume that this solution can be written as

$$\begin{aligned} F_{n+1}(x, y, z) = z^n + yf_1(x, y)z^{n-2} + yf_2(x, y)z^{n-3} + \cdots \\ + yf_{n-2}(x, y)z + yf_{n-1}(x, y), \end{aligned} \tag{3}$$

where $f_k(x, y)$, for $k=1, 2, \cdots, n-1$, is a binary form in x, y of order k with integral coefficients. Evidently none of the forms (2) is of this type.

From (1) and (3), we get

$$\begin{aligned} F_{n+1}(x, y, \alpha x+\beta y+\gamma) &= (\alpha x+\beta y+\gamma)^n + yf_1(x, y)(\alpha x+\beta y+\gamma)^{n-2} + \cdots \\ &\quad + yf_{n-1}(x, y) \\ &= p(x) + ay. \end{aligned}$$

Hence, on developing both sides into sums of binary forms in x, y and comparing forms of equal order,

$$(\alpha x+\beta y)^n+yf_1(x, y)(\alpha x+\beta y)^{n-2}+\cdots+yf_{n-2}(x, y)(\alpha x+\beta y)+yf_{n-1}(x, y) = c_nx^n,$$
$$\binom{n}{1}(\alpha x+\beta y)^{n-1}+\binom{n-2}{1}yf_1(x, y)(\alpha x+\beta y)^{n-3}+\cdots+\binom{1}{1}yf_{n-2}(x, y)=c_{n-1}x^{n-1},$$
$$\binom{n}{2}(\alpha x+\beta y)^{n-2}+\binom{n-2}{2}yf_1(x, y)(\alpha x+\beta y)^{n-4}+\cdots+\binom{2}{2}yf_{n-3}(x, y)=c_{n-2}x^{n-2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\binom{n}{n-2}(\alpha x+\beta y)^2+\binom{n-2}{n-2}yf_1(x, y) = c_2x^2,$$
$$\binom{n}{n-1}(\alpha x+\beta y) = c_1x+\binom{n}{n-1}ay.$$

Here the c's are suitable constants; their values are found by putting $y=0$, namely:

$$c_n = \alpha^n,\ c_{n-1} = \binom{n}{1}\alpha^{n-1},\ c_{n-2} = \binom{n}{2}\alpha^{n-2}, \cdots, c_1 = \binom{n}{n-1}\alpha.$$

Let

$$d_k(x, y) = d_k = (\alpha x)^k - (\alpha x+\beta y)^k \quad (k = 2, 3, \cdots, n).$$

Then the last equations take the form,

$$yf_1(x, y)(\alpha x+\beta y)^{n-2}+yf_2(x, y)(\alpha x+\beta y)^{n-3}+\cdots+yf_{n-1}(x, y)=d_n,$$
$$\binom{n-2}{1}yf_1(x, y)(\alpha x+\beta y)^{n-3}+\binom{n-3}{1}yf_2(x, y)(\alpha x+\beta y)^{n-4}+\cdots+\binom{1}{1}yf_{n-2}(x, y)=\binom{n}{1}d_{n-1},$$

$$\binom{n-2}{2} yf_1(x, y)(\alpha x+\beta y)^{n-4}+\binom{n-3}{2} yf_2(x, y)(\alpha x+\beta y)^{n-5}+\cdots+\binom{2}{2} yf_{n-3}(x, y)=\binom{n}{2} d_{n-2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\binom{n-2}{n-2} yf_1(x, y) = \binom{n}{n-2} d_2.$$

This system of linear equations for

$$yf_1(x, y),\ yf_2(x, y), \cdots,\ yf_{n-1}(x, y)$$

is easily solved just in the required form; for the expressions on the right-hand side are all divisible by y. We find for small values of n:

$$n = 1:\ F_2(x, y, z) = z;$$

$$n = 2:\ F_3(x, y, z) = z^2 + d_2;$$

$$n = 3:\ F_4(x, y, z) = z^3 + 3d_2z + (d_3 - 3(\alpha x + \beta y)d_2);$$

$$n = 4:\ F_5(x, y, z) = z^4 + 6d_2z^2 + 4(d_3 - 3(\alpha x + \beta y)d_2)z + (d_4 - 4(\alpha x + \beta y)d_3 + 6(\alpha x + \beta y)^2d_2);$$

$$n = 5:\ F_6(x, y, z) = z^5 + 10d_2z^3 + 10(d_3 - 3(\alpha x + \beta y)d_2)z^2 + 5(d_4 - 4(\alpha x + \beta y)d_3 + 6(\alpha x + \beta y)^2d_2) + (d_5 - 5(\alpha x + \beta y)d_4 + 10(\alpha x + \beta y)^2d_3 - 10(\alpha x + \beta y)^3d_2).$$

It is also clear that the following formulae hold:

$$\begin{aligned} &F_l(x, y, \alpha x + \beta y + \gamma) = x^{n-l}(\alpha x + \gamma)^l \qquad (l = 0, 1, \cdots, n-1);\\ (4)\qquad &F_n(x, y, \alpha x + \beta y + \gamma) = \gamma^{n-1}y;\\ &F_{n+1}(x, y, \alpha x + \beta y + \gamma) = (\alpha x + \gamma)^n + n\beta\gamma^{n-1}y. \end{aligned}$$

Assume in particular that

$$\alpha \neq 0, \qquad \beta \neq 0, \qquad \gamma \neq 0.$$

Then the $n+2$ forms (2) and (3) are linearly independent; for the polynomials in x

$$x^{n-l}(\alpha x + \gamma)^l \qquad (l = 0, 1, \cdots, n-1)$$

are clearly linearly independent, since no two of them vanish to the same order at $x=0$ or at $x=-\gamma/\alpha$. Therefore the form of order n,

$$(5)\qquad F(x, y, z) = \sum_{l=0}^{n+1} C_lF_l(x, y, z),$$

where $C_0, C_1, \cdots, C_{n+1}$ are arbitrary integers not all zero, does not vanish identically; it is again a solution of (1).

Take

$$\gamma = 1, \qquad C_n + C_{n+1}n\beta = 1$$

and choose F such that $F(x, 0, \alpha x+\gamma)$ is a polynomial in x exactly of degree n. Then by the remark at the beginning of this paragraph, $F(x, y, z)$ represents every integer in an infinity of different ways. We furthermore have

$$F(x, y, \alpha x + \beta y + \gamma) = F(x, 0, \alpha x + \gamma) + y,$$

and conclude that $F(x, y, z)$ is an irreducible form in x, y, z, since the expression on the right-hand side is irreducible in x and y and of exact degree n in x. It is again clear, as in the preceding paragraph, that the so constructed form, $F(x, y, z)$ is indefinite.

3. Forms with bounded greatest prime factor. A positive definite ternary form of order n, $F(x, y, z)$, represents every integer k in at most a finite number of ways. Since $F(x, y, z)$ is positive definite, its order is even, say $n=2m$. The values of F on the sphere $x^2+y^2+z^2=1$ are always positive; since F is continuous, they have a minimum value $V>0$ on this sphere. Hence

$$f(x, y, z) = \frac{F(x, y, z)}{(x^2 + y^2 + z^2)^m} \geqq V$$

for all points of this sphere, and therefore for all points of space, since $f(x, y, z)$ is homogeneous of order zero. Hence, if $k>0$ is given and $F=k$, then

$$k = F(x, y, z) \geqq V(x^2 + y^2 + z^2)^m, \quad i.e. \quad |x|, \; |y|, \; |z| \leqq (k/V)^{1/2m},$$

so that there are at most a finite number of integral solutions x, y, z.

Suppose in particular, that this form can be written as

$$F(x, y, z) = aQ(x, y)^m + (z - x - y)G(x, y, z), \tag{6}$$

where $a\neq 0$ is an integer and

$$Q(x, y) = \alpha x^2 + \beta xy + \gamma y^2$$

is a quadratic form in x and y, and $G(x, y, z)$ a form of order $n-1$ in x, y, z, both with integral coefficients. Since

$$F(x, y, x + y) = aQ(x, y)^m, \tag{7}$$

the form $Q(x, y)$ must be positive or negative definite; otherwise there would be real x, y, z not all zero such that $F(x, y, z)=0$.

By the theory of quadratic forms, it is possible to find for every integer $t\geqq 1$ a system of t different prime numbers

$$P_1, P_2, \cdots, P_t,$$

such that the equation

$$Q(x, y) = \mp P_1^{z_1} \cdots P_t^{z_t}$$

has an infinity of integral solutions $x, y, z_1, \cdots, z_t$, where x and y are relatively prime and $z_1\geqq 0, \cdots, z_t\geqq 0$. Hence, by (7), there exist an infinity of different

sets of three relatively prime integers x, y, z, for which the greatest prime divisor of $F(x, y, z)$ is bounded.

It is possible to find positive definite forms $F(x, y, z)$ of every even order $n=2m$, which can be written as a sum (6). For instance, it suffices to take

$$G(x, y, z) = (z - x - y)H(x, y, z),$$

where $H(x, y, z)$ is a positive definite form of order $n-2=2(m-1)$. Or we may take $G(x, y, z)$ arbitrary, but such that $G(0, 0, 1)>0$, and then can make $F(x, y, z)$ positive definite by just choosing for a a sufficiently large positive integer.

In the excluded case that $Q(x, y)$ is indefinite, $F(x, y, z)$ evidently represents an infinity of different integers k in an infinity of different ways.

As there are many forms of the type (6), we may impose on them further conditions, *e.g.*, consider only forms which are quadratic or cubic forms in x^h, y^h, z^h. In the following examples, the sign "$\rightarrow$" means that the right-hand side is derived from the left-hand side by the substitution $z=x+y$.

(a) Quadratic forms in x^2, y^2, z^2. A few examples are given by the identities, in which α is arbitrary:

$$(1 - \alpha)x^4 + \alpha x^2y^2 + \alpha x^2z^2 + \alpha^2y^2z^2 \rightarrow (x^2 + \alpha xy + \alpha y^2)^2;$$
$$\alpha x^4 + \alpha(4\alpha - 1)y^4 + \alpha z^4 + (1 - 2\alpha)x^2z^2 \rightarrow (x^2 + xy + 2\alpha y^2)^2.$$

(b) Quadratic forms in x^3, y^3, z^3. The following identities hold for arbitrary α:

$$(\alpha^2 - 4\alpha + 4)(x^6 + y^6) + (\alpha^2 - \alpha + 1)z^6 + (3\alpha^3 - 16\alpha^2 + 28\alpha - 16)x^3y^3$$
$$- (2\alpha^2 - 5\alpha + 2)(x^3 + y^3)z^3 \rightarrow 3(x^2 + \alpha xy + y^2)^3;$$
$$(\alpha^2 - 3\alpha + 3)x^6 + \alpha^2(y^6 + z^6) + (2\alpha^2 - 3\alpha)x^3(y^3 - z^3) + (3\alpha^3 - 2\alpha^2)y^3z^3$$
$$\rightarrow 3(x^2 + \alpha xy + \alpha y^2)^3;$$
$$\alpha(x^6 + z^6) + (3\alpha^3 - 3\alpha^2 + \alpha)y^6 + (2\alpha - 3\alpha^2)y^3(x^3 - z^3) + (3 - 2\alpha)x^3z^3$$
$$\rightarrow 3(x^2 + xy + \alpha y^2)^3.$$

(c) A quadratic form in x^4, y^4, z^4:

$$x^8 + y^8 + 17z^8 + 14(x^4 + y^4)z^4 \rightarrow 2(2x^2 + 3xy + 2y^2)^4.$$

(d) There is no irreducible cubic form in x^4, y^4, z^4 with rational coefficients, but there are four with coefficients in $K(\sqrt{3})$ which are conjugate in pairs with respect to this field.

Final remark. Analogous to (6), there are positive definite quaternary forms $F(x, y, z, w)$ of every even order $n=2m$ which can be written as

$$F(x, y, z, w) = aQ(x, y, z)^m + (w - x - y - z)G(x, y, z, w),$$

where $a\neq 0$ is an integer, $Q(x, y, z)$ a positive definite quadratic form and $G(x, y, z, w)$ a form of order $n-1$. For forms F of this kind, the equation $F(x, y, z, w)=k$ has evidently at least const. $|k|^{1/n}$ solutions for an infinity of k's.

UNDERGRADUATE MATHEMATICAL RESEARCH*

F. L. GRIFFIN, Reed College

In returning to the scene of my own undergraduate work I find it a pleasant duty to acknowledge my deep indebtedness to that great college teacher, Herbert Ellsworth Slaught. The sense of exploring the unfamiliar, which permeated the initial two years spent under him, was a delightful experience. And he prepared students smoothly for the later years under Professors Moore, Bolza, and Maschke.

Spontaneous mathematical research arising out of the curiosity and reflections of inquiring and gifted young minds is no new phenomenon. Witness Pascal and Galois. More frequently a stimulating teacher has stirred an alert student to study some challenging question. Indeed, some institutions have long made a systematic effort to bring this type of experience to any student capable of profiting by it. In seeking to arouse interest in investigation they have utilized prize contests, departmental club programs, thesis requirements of candidates for honors, and so on. Many colleges now encourage a student to study independently some field new to him and to write a report of an expository or critical nature. From the standpoint of the student this may be called a type of research.

In the present discussion, however, I am using the word "research" in the more technical sense of exploring some new question or re-exploring some old question and getting results previously unfamiliar to specialists in the field. When a student investigates a question and gets results new to him but not actually involving priority, the experience may afford as valuable training for him as if he were the first to explore the matter. But in connection with the discussions of research under way in the present sessions of this Association and of the Society, it seems to me of possible interest to inquire how extensively research in the more technical sense is being carried on by undergraduates of our colleges.

It is, of course, a fair question how generally undergraduates ought to attempt mathematical research. For students who are going on to the graduate school, there is so much fundamental material to be learned as a basis for specialization that it is debatable whether time ought to be used for efforts in research. And whether a student is going on or not, one may wonder how much worthwhile research can be accomplished with the tools available in undergraduate days. Possibly the experience of institutions which have experimented with undergraduate research projects may throw some light on these questions.

Reports from various institutions. Thinking that one source of information might be the colleges and universities which according to published report†

* Read at the summer meeting of the Mathematical Association of America, Chicago, September 1, 1941.

† The American Council on Education, American Universities and Colleges, 1940.

require a thesis of some or all seniors, I sent a circular letter to the heads of the departments of mathematics in about ninety such institutions. The replies, not yet complete, run about as anticipated. Usually the senior thesis has the character of a critical expository essay or report. Seventeen institutions, however, report undergraduate research papers in the technical sense, arising either as theses or from individual initiative,—spontaneous combustion, as it were. The information thus far available runs as follows:

Albion College. In 1912 a paper on the normals to a conic embodied substantial research. Since 1930 research has been carried on extensively.

Beloit College. Two recent honors theses have involved research leading to new results. One dealt with the mathematical relations involved in a Precision Harmonograph; the other developed a variation in Newton's method of approximating the real roots of an equation.

Brooklyn College. The Mathematical Mirror, published annually for nine years by mathematics students of the college, has contained a number of research articles.

University of Buffalo. In the past twelve years there have been about 15 research papers, usually submitted in the Sherk Prize Contest sponsored by the Mathematics Club. Some papers have been presented at the New York State Undergraduate Scientific Congress.

University of California. Research is frequently done by students in going farther into problems which they have begun in some course. The topics have related to hyperspace, line geometry, synthetic projective geometry, higher plane curves, affine geometry, birational transformations, vector geometry and tensor theory, general postulate theory, Boolean algebra, real and complex variable, and problems found in Polya and Szegö's, *Aufgaben und Lehrsätze aus der Analysis*

University of Delaware. Honors theses, one or two each year, have dealt with such topics as semi-regular-continued fractions, Legendre's transformations, hypergeometric functions, and elliptic functions. New results of interest have been obtained in some cases.

Harvard University. For honors a thesis was long optional or required. Undergraduate research papers by A. L. Lowell and Maxime Bôcher were published back in the 1880's. Since 1926 a thesis has been required of students offering a concentration in any field. These theses often contain interesting results not in the literature. A recent thesis on the Chi-Square Tests has been published by the Harvard University Press.

University of Minnesota. For graduation *summa cum laude* a thesis is required. In mathematics a few theses have been of the research type.

University of New Mexico. For several years past a thesis has been required for graduation with honors. In mathematics about one thesis a year has made an original contribution. One of the latest developed a new implication relation in the calculus of propositions. Another studied the asymptotic behavior of certain functions of the Bessel type as the argument becomes infinite.

New York University. Allusion is made to a recent undergraduate publication.

Agricultural and Technical College of North Carolina. Since 1939 thesis work for students majoring in mathematics has been of a research character, directed through the Department of Physics. Abstracts of seven of these theses have been published in collaboration with members of that department.

University of Pennsylvania. In recent years a junior and a senior published important research papers on subjects in which they had become interested through advanced courses. One paper gave a new proof of the quadratic reciprocity law; the other, in the field of algebra, dealt with foundational questions.

Princeton University. Since 1927 a thesis has been required of each senior. The mathematical papers have frequently been of the research type.

St. Lawrence University. Two sophomores recently became interested in differentiators and constructed a machine.

University of Vermont. Since about 1890 a thesis has been required for graduation with honors. In the past ten years several of the papers have been of the research type.

University of Wisconsin. An exceptional student sometimes takes both the bachelor's and the master's degree at the end of four years. One outstanding thesis written by such a student studied the convergence of Newton's method of finding the roots of an algebraic equation when the roots, and possibly the coefficients, are complex numbers.

It would doubtless be of interest if the experience of the preceding institutions, and of others in which undergraduate research has been carried on, might be reported more fully by persons who have the necessary information.

Experience at Reed College. Please be indulgent if I now speak somewhat disproportionately about our experience at Reed College since I am most familiar with that. Since the beginning of the college a thesis has been required of all seniors in all departments, except students graduated in 1919. The thesis always involved some independent investigation; and after a few years, largely due to the interest of the students, it took on pretty generally a research character in the specialized sense. Of the 94 theses in mathematics written in the 26 years, at least 80 have involved research with some novel results; and in 56 of these I believe that the problem itself was new. At least, I have not been able to locate the problems in the literature.

Since a typical thesis, in a condensed form, has forty or fifty pages and would require about thirty minutes for an adequate description, perhaps the best I can do here is to suggest the range of topics treated and in a few instances allude to some results.

Analysis. The largest single group of our research theses falls in the Calculus of Variations, where of 17 theses fifteen have dealt with apparently new problems. Eight of these related to physical problems: *e.g.*, the form of the plane curved wire connecting two points and having minimum moment of inertia about an axis in its plane, or having minimum attraction for a particle; or carry-

ing an electric current through a wire of given length and producing the maximum magnetic force at a point; also, similar problems for surfaces of revolution. Seven other seniors determined the system of geodesics upon some surface, *e.g.*, the surface generated by revolving a hypocycloid of four cusps about a diameter through two cusps. In each case the student determined and plotted typical extremals, discussed their properties, studied the envelope of the one-parameter family through a single point and determined its singularities, if any; and in most cases gave a sufficiency proof for a relative extremum. Parametric equations involving elliptic integrals appeared in each problem, except two which required a graphical and numerical solution of Euler's differential equation.*

One thesis in this field interests me considerably, because it was written by a student whose work the first three years was of such low standing that we reluctantly accepted her as a senior. To our surprise she worked with unusual initiative and with application, and needed less guidance than some students of distinctly higher standing have had. She dealt with an old problem in an elementary way, "The geodesics on the torus," which was treated by a leading mathematician in both his master's and his doctor's theses. (While she was studying it I told her nothing of the earlier investigations, preferring, as advocated by Professor R. L. Moore, to let a student develop his or her own methods before consulting the literature.) She calculated the scales for conformal mapping of the torus and obtained and plotted the geodesics, using approximate integration extensively and accurately. I wonder whether it may be easier to do research of such a specialized type than to organize a broad field clearly, as for example our junior course in modern algebra which ranges from matrix theory to elements of the Galois theory.

Six other theses have discussed troublesome differential equations. A recent one worked with a system of equations relating to a hydraulically operated radial gate in a government dam at Eugene, Oregon. The motions were accurately calculated by numerical integration. This thesis, also, was written by a student whose previous standing had been only average, and whose energies had been devoted chiefly to actuarial mathematics. Another, written some years ago by a top-notch student, did a brilliant job in studying the effect of a shield on a coil carrying currents at radio frequencies. He set up a partial differential equation of order 2, similar to Maxwell's equations, got its solutions as series involving complex trigonometric and Bessel functions, discussed questions of accuracy, and got integral expressions for the back magnetic field and power loss. Still another paper on differential equations, perhaps the most mature thesis job we have had in our department was written this past year, with virtually no suggestions from an instructor. This dealt with a second-order equation which includes Bessel's and Riccati's equations as very special cases. Besides obtaining formal series solutions and discussing transformations, the stu-

* Interestingly enough, a student who wrote one of the latter theses got a job in the Ballistic Section, Bureau of Ordnance, where her work with trajectories has seemed almost a continuation of her thesis. One of her classmates had a similar carry-over in the Coast and Geodetic Survey.

dent defined a class of quasi-periodic functions and obtained 18 theorems relating these to the solution of his differential equation. (The same man in his freshman year had written a paper entirely on his own initiative in which he postulated two classes of elements, C and K, and two operations transforming an element of C into an element of K and vice versa, and deduced a series of theorems relating to repeated application of the operations.)

We have also had seven theses dealing with the calculation of elliptic integrals of type III. One of these used methods apparently not in the literature. The resulting tables, running to four decimal places, give values of $\pi(n, k, \phi)$ for a suitable range of values of n, k, and ϕ.

Two other research theses in analysis generalized the circular and hyperbolic functions of sectoral areas, modifying the circle and equilateral hyperbola or choosing the origin away from the center. Extensive systems of identities, derivatives, etc., were found.

Geometry. Turning to geometry, there is time to say only that we had a sequence of nine theses dealing with conjugate curves* and surfaces. In some of the classes studied, the conjugate of a line or conic is usually a higher plane curve, sometimes a familiar curve such as the cissoid, conchoid, strophoid, limaçon, etc., or perhaps another conic, or the polar of the origin with respect to a conic.

There is a group of four theses dealing with centro-surfaces of conicoids,† which are distinct from Cayley's center-surfaces. The centro-surface in some cases pinches down to a line and then expands again, so that certain plane sections may have nodes or cusps. The determination of the sections usually involves substantial algebraic difficulties.

There have been 15 other research theses in geometry, including generalizations of the conics in three dimensions, the inverse of the envelope problem, a study of the normals to an ellipse (which yielded fifteen new theorems), studies of constructibility by use of higher plane curves, and the projective determination of conics.

Algebra and Combinatory Analysis. In this field there have been five research theses, three of which obtained substantial and apparently new results. In one, criteria were obtained for the algebraic solvability of certain high-degree equations by reduction through a chain of reciprocal equations. In another, criteria as to the nature of the roots of the cubic and quartic were derived from Argand's representation of the roots of the resolvent equations. In 1916 a student who had been reading Ball's Mathematical Recreations and Essays decided to work on an unsolved case of "the schoolgirl problem" (or Euler's problem of the 36 officers): 36 girls are to walk in rows of 6; required, if possible, an arrangement by which each girl shall walk once and only once with each other girl. By an ingenious system of eliminating conflicts it was shown that the desired arrange-

* Defined in Bulletin American Mathematical Society, vol. 30, 1924, p. 15.

† Defined in Bulletin, American Mathematical Society, vol. 36, 1930, p. 483.

ment is impossible; and some related questions were treated. These results were reported to the Society, but were apparently overlooked (as was also a complete subsequent treatment of the general problem by Professor H. F. MacNeish)* when the same problem was treated some fifteen years later for a doctoral dissertation in one of our leading universities.

Further physical problems. Four theses have dealt with problems of light. Two of these have studied the effect of a line source of light, placed in certain positions, upon the performance of a parabolic mirror. Reflected rays diverging from the axis by more than arc sin .01 are considered as "lost"; and for various points on the luminous source, boundary curves are determined which separate regions of the mirror that "lose" light from regions that "save" light. The quantity lost is expressed by a triple integral of complicated form, which is approximated numerically.

Another thesis in this field studied the illumination in the courts of certain office buildings in Portland. By spherical trigonometry and descriptive geometry actual shadows, as contrasted with approximations often used, were constructed for the time of the equinoxes and the solstices. By using a photometer the variation of light in offices in a number of respects was studied.

One of our most original and spontaneous theses proposed a mathematical theory as to the nature of potential energy. The student, who while yet in high school had studied the theory of relativity, drew extensively upon modern physical theory. (He has since taken his doctorate in point-set theory which provides him with something of a contrast!)

Astronomy. A student, who for years had made variable star observations with a telescope of his own, analyzed these to determine the light curves of four stars. The usual methods for such analysis proved to be useless; but consistent patterns were obtained by choosing as the argument the ratio of the time interval since each maximum brightness to the length of the time in that particu'ar "cycle".

Economics. A thesis on a problem of duopoly extended a study of certain economic questions treated by Professor G. C. Evans.† Keeping his linear demand law, but replacing his quadratic cost function by a cubic, the student found various types of maxima occurring in the study of profits. A plaster model was constructed showing some peculiarities of the graphical surface in one case.

Statistics. A thesis submitted in 1930 dealt with lines of best fit, with special reference to the statistical determination of the demand curve for sugar. Three different basic assumptions as to the displacement of points in the scatter diagram from the required line were considered, especially the assumption that the displacements are normal to the line. Using numerical data for sugar, a detailed study was made of the surface which exhibits the sum of the squared deviations as a function of the parameters of the line. Systems of lines of "equally good fit"

* Annals of Mathematics, 2nd ser., vol. 23, 1922, pp. 221–227.

† This MONTHLY, vol. 29, 1922, pp. 371–380.

were discussed, corresponding to points on contours of the surface; also, "indifference curves" whose tangents are such lines of equally good fit. (This thesis, supervised jointly by professors of mathematics and economics, took first prize in the Hart, Schaffner and Marx Economics Essay Competition for Undergraduates that year. Incidentally, it was written, with comparatively little guidance, by a student whose course work had been of only average quality.)

Some General Considerations. I have referred to some 56 of our Reed mathematical theses as having dealt with what appeared to be new problems. There are many such problems available, not sufficiently theoretical or advanced to justify their use for doctoral disserations but much too involved to permit their inclusion as exercises in textbooks. Others suggest themselves in considering possible modifications or generalizations of standard procedures or theories. Some of these theses were reported to the Society, due to lack of a section of the Association in our region.

While novelty is not essential for the student's profit in working upon a problem, it adds interest. The student greatly enjoys the experience of exploring new material, where the conclusions to be expected are not familiar to his instructor and are not availabe in the literature. In selecting a topic from a list of several possibilities the student may, if he likes, first glance over old theses to see what field or general type of investigation might suit him best. Sometimes he has a preference for a type of work very different from any of the suggested topics; and if he does not have a specific topic in mind an effort is made in consultation to formulate one.

One possible value of thesis work is that it gives the student an opportunity to make the most of whatever originality, independence, and spontaneity he may possess. It has long been my view that the tendency in undergraduate instruction is to neglect somewhat the most promising students in order to give adequate attention to the great middle group. In the charming town where I passed much of this past summer, one sees traffic signs near each school, which I suspect might be repunctuated thus as advice to the institution itself: "School, don't kill a child." In college, at least, it should not be necessary to kill the interest of our best students by limiting them to routine course work. Let me close with a quotation from a Reed College Bulletin on the subject of the senior thesis: "Latent powers of self-direction are developed in a more complete and extensive way than is afforded at any other point in the college career, and the student has the satisfaction of achieving a degree of mastery not commonly associated with the completion of a course."

THE ROOTS OF A QUATERNION

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The existence of an mth root of a quaternion α is known,* since the question reduces to the existence of a quaternion root of the equation $\xi^m-\alpha=0$. We are led to inquire how many mth roots there are, and how to find them. The answer to the first inquiry is that there are exactly m distinct mth roots of a quaternion α which is not a real number; if α is real there are infinitely many mth roots unless $m=2$ and α is positive, in which case there are only two square roots, $\pm\sqrt{\alpha}$. Our method of proof gives all roots.

LEMMA. *Any positive integral power of a quaternion $a+bi+cj+dij$ has coefficients of i, j, and ij which are proportional to b, c and d.*

The proof is by induction. Noting that the conclusion is true for the first power, we assume that

$$(a+bi+cj+dij)^{m-1}=A+q(bi+cj+dij).$$

Then we obtain at once

$$(a+bi+cj+dij)^{m}=aA-q(b^2+c^2+d^2)+(aq+A)(bi+cj+dij),$$

which proves the lemma.

Suppose now that α is any quaternion which is not a real number. Then the lemma enables us to say that any mth root of α must have the form $c\alpha+d$, c and d being real. The question is, simply, what real values of c and d can be used so that the equation

$$(c\xi+d)^m-\xi=0, \tag{1}$$

has the quaternion α as a root? Now, designating the trace and norm of α by t and n respectively, we know that α satisfies the equation

$$\xi^2-t\xi+n=0. \tag{2}$$

This is the minimal equation with real coefficients satisfied by α, and hence our problem is to find real values c and d so that the left side of (1) is divisible by the left side of (2). The latter is the case if and only if the roots of (2) in complex numbers are also roots of (1). Since α is not real, the discriminant of (2) is negative, so that the complex roots of this equation can be denoted by the conjugates λ and $\bar{\lambda}$. Taking λ to be the one with positive imaginary coördinate, we can write

$$\lambda=r^m(\cos m\theta+i\sin m\theta),\qquad 0<m\theta<\pi, \tag{3}$$

the m being inserted for convenience. We want to choose c and d so that

* Ivan Niven, Equations in quaternions, this MONTHLY, Vol. 48, 1941, pp. 654–661.

$$(4) \qquad (c\lambda + d)^m = \lambda, \qquad (c\bar{\lambda} + d)^m = \bar{\lambda}.$$

We can ignore the second of these two equations, it being a consequence of the first. Substituting (3) in the first part of (4) and taking mth roots, we obtain

$$(5) \quad cr^m(\cos m\theta + i \sin m\theta) + d = r\left\{\cos\left(\theta + \frac{2\pi k}{m}\right) + i \sin\left(\theta + \frac{2\pi k}{m}\right)\right\}$$
$$(k = 0, 1, \cdots, m - 1).$$

Taking conjugates of both sides of (5), and subtracting the result from (5), we get

$$(6) \qquad 2cr^m i \sin m\theta = 2ri \sin\left(\theta + \frac{2\pi k}{m}\right) \qquad (k = 0, 1, \cdots, m - 1).$$

This can be solved for real c since $\sin m\theta \neq 0$, because of the inequality in (3). Substituting in (5) we obtain

$$(7) \qquad d = r \cos\left(\theta + \frac{2\pi k}{m}\right) - r \cot m\theta \sin\left(\theta + \frac{2\pi k}{m}\right)$$
$$(k = 0, 1, \cdots, m - 1).$$

We prove now that the mth roots of α obtained above are distinct. If the values of c in (6) are all different, clearly the quaternion values $c\alpha + d$ have the same property. If two values of c are equal, so that

$$\sin\left(\theta + \frac{2\pi h}{m}\right) = \sin\left(\theta + \frac{2\pi k}{m}\right), \qquad h \neq k,$$

then we can show that the corresponding values of d in (7) are unequal. For if they were equal, we would have

$$\cos\left(\theta + \frac{2\pi h}{m}\right) = \cos\left(\theta + \frac{2\pi k}{m}\right), \qquad h \neq k.$$

Since the angles involved here are unequal and lie between 0 and 2π, their sum must be 2π, that is,

$$\theta + \frac{2\pi h}{m} + \theta + \frac{2\pi k}{m} = 2\pi.$$

This simplifies to $m\theta = \pi(m - h - k)$, which contradicts the inequality in equation (3). We summarize our results.

THEOREM 1. *Given a quaternion α which is not real, solve the quadratic equation* (2) *in complex numbers (t and n being the trace and norm of α). Denote the modulus and amplitude of the root with positive imaginary part by r^m and $m\theta$. Then the m distinct mth roots of α are $c\alpha + d$, where c and d are given by equations* (6) *and* (7), *the same value of k being used simultaneously in the two equations.*

Henceforth let $\alpha \neq 0$ be real. Apart from real mth roots, any quaternion mth root of α satisfies a quadratic of the form (2), whose left side divides $\xi^m - \alpha$. Hence we search for real values t and n (these no longer denoting the trace and norm of α) so that this divisibility property holds; these values can be constructed as follows. If the real positive mth root of $|\alpha|$ be r, we may designate α by $r^m \cos s\pi$, s being 0 or 1 according as α is positive or negative. Then the complex roots of $\xi^m - \alpha = 0$ are

$$(8) \qquad r\left(\cos \frac{(s\pi + 2k\pi)}{m} + i \sin \frac{(s\pi + 2k\pi)}{m}\right) \qquad (k = 0, 1, \cdots, m-1).$$

Ignoring any real values momentarily, we note that a conjugate pair of these are roots of (2) provided

$$t = 2r \cos \frac{(s\pi + 2k\pi)}{m}, \qquad n = r^2.$$

Since any non-real quaternion root of (2) must have trace t and norm n, it must have the form

$$(9) \qquad r \cos \frac{(s\pi + 2k\pi)}{m} + yi + zj + wij \qquad (k = 0, 1, \cdots, m-1),$$

where y, z and w are any real values satisfying

$$(10) \qquad y^2 + z^2 + w^2 = r^2 - r^2 \cos^2 \frac{(s\pi + 2k\pi)}{m} \qquad (k = 0, 1, \cdots, m-1).$$

Now among the values (8) there may be one or both of the real values r and $-r$. Note that any such values are also given by (9), since (10) implies that $y = z = w = 0$ in this case. Hence all quaternion mth roots of α are given by (9). There are infinitely many roots unless the right side of (10) is zero for every value of k, and this conditions obtains only if $s = 0$ and $m = 2$.

THEOREM 2. *Given any real number $\alpha \neq 0$, let r denote the real positive mth root of $|\alpha|$, and let s be* 0 *or* 1 *according as α is positive or negative. Then the mth quaternion roots of α are given by* (9) *with condition* (10) *imposed, the same value of k being used simultaneously in the two equations.*

DISCUSSIONS AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The Department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A COMMENT ON "DIFFERENTIALS"

DUNHAM JACKSON, University of Minnesota

The note on "Differentials" by Kac and Randolph (this MONTHLY, vol. 49, 1942, pp. 110–112) stresses the idea that when dy is defined in terms of dx the latter is to be regarded as an independent variable. May I suggest that the presentation would gain strength, not in its introductory stages but *after* the dust of construction and the smoke of discussion have been cleared away, by a summary for permanent reference such as the following, for which I make no claim to originality:

If y is a function of x, in symbols $y=f(x)$, *dy is a function of the two variables x and dx*, represented by the formula $dy=f'(x)dx$. Graphically dy is the increment of the ordinate of the tangent line, if x is given an increment Δx equal to dx. If dx is infinitesimal and $\Delta x=dx$ (and if each of the variables x, y has with respect to the other a derivative different from zero), dy differs from Δy by an infinitesimal of higher order (than dx or dy or Δy).

To this is to be added: The use of the differential notation in connection with the mutual dependence of three variables, any one of which is a function of either of the others (and similarly with a larger number of variables), is justified by, and serves to commemorate, the rule for differentiating a function of a function.

DIFFERENTIALS

ALONZO CHURCH, Princeton University

1. I am interested in the note of Kac and Randolph in the February number of the MONTHLY (pp. 110–112), because I agree with them that the usual definition of the differential which they criticize is unsound, and that this unsoundness is within the understanding of the more intelligent beginning student in the calculus—or at least that it is sufficiently near the threshold of his understanding so that the definition causes him difficulty (even if he cannot make explicit the reason for his difficulty).

I would urge, however, that the objection which they make to the usual definition is not sufficient to reveal an unsoundness in it. In effect their objection is that, in the equation

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x},$$

if dx is identified with Δx, the same variable Δx appears in the equation in two different rôles, on the left as a free variable and on the right as a bound variable.* But this must not be considered an error. For example, in a context where π has been defined as

$$\int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}},$$

no one would think of objecting to the equation,

$$\tan (x + \pi) = \tan x,$$

on the ground that π stands for an expression containing x as a bound variable. Likewise, in a context where $D_x y$ has been defined as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

there should be no objection to writing an equation which contains $D_x y$ and at the same time contains a separate occurrence of Δx as a free variable.

The modification of Kac and Randolph in the definition of the differential must therefore be considered as designed to remove a difficulty for the student, rather than as correcting an actual error. Its advantage is that it avoids the possible necessity for an added explanation, which would, at least in effect, have to reproduce the distinction between free and bound variables.

2. On the other hand, there is a more serious objection which applies alike to the definition of Kac and Randolph and to the more usual definition which they wish to replace.

This objection may be formulated in the following terms. Both definitions agree in defining the differential of the independent variable, say x, by taking dx to be a new independent variable—hence it should with equal correctness be possible to use a single letter, say z, to represent this new variable. Then the differential of a dependent variable, say y, is defined by taking dy to be $(D_x y)dx$ —*i.e.*, $(D_x y)z$. But a survey of the more usual purposes for which differentials are employed in the calculus will show that not all of these are adequately served by taking dx and dy to be z and $(D_x y)z$ respectively.

In particular, the use of differentials in connection with integration fails to be provided for. Thus $\int x dx$ becomes simply $\int xz$, and the whole significance of the notion is lost. (What plausibly is the operation $\int$ which, applied to the product of two independent variables x and z, yields $\frac{1}{2}x^2 + C$?)—It should be emphasized that the facility which comes with the use of differentials is more marked in the integral calculus than in the differential calculus, and that any definition of differentials which fails to account for their use with the sign $\int$

* A variable is *free* in a given expression (in which it occurs) if the meaning or value of the expression depends upon determination of a value of the variable; in other words, if the expression can be considered as representing a function with that variable as argument. In the contrary case the variable is called a *bound* (or *apparent*, or *dummy*) variable.

has therefore lost more than half their value. Compare, *e.g.*, any one of the following processes in terms of differentials with the clumsier parallel process which regards the notation $\int \cdots dx$ as indivisible and employs only derivatives without differentials: (1) the integration $\int x\sqrt{4x+3}\,dx$ by the substitution $t^2=4x+3$, using the equation between differentials, $dx=\frac{1}{2}tdt$; (2) the solution of the differential equation $dy/dx=xy$ by multiplying both sides by dx/y and then applying the operation $\int$ to both sides; (3) the discovery by inspection of an integrating factor for a differential equation of the first order and first degree in two (or more) variables.

Another aspect of the foregoing objection lies in the point that the same notation, dx or dy, is given different meanings according as x is independent or dependent variable. This is especially unfortunate because it is often precisely one of the advantages in the use of differentials that various variables are symmetrically treated, without arbitrarily singling out one or more of them as the independent variable or variables (compare, *e.g.*, the equation $ds^2=dx^2+dy^2$ with the corresponding equation for $(ds/dx)^2$, or for $(ds/dy))^2$.

Sometimes a student will ask why the result,

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx},$$

cannot be obtained by simple cancellation of du against du, and the more difficult argument which employs properties of limits thus avoided. On the basis of the usual definition of a differential, the reply is to point out that du has different meanings in its two occurrences; but this immediately reveals the weakness of this usual definition.

If desired, the objection that the usual definition of a differential does not treat the variables symmetrically can be regarded as the fundamental objection. The operation $\int$ must, of course, be the inverse of the operation d, however the latter operation is defined; and if the operation d fails to treat the variables symmetrically, the same lack of symmetry must affect the inverse operation. The difficulty in connection with integration reduces in part to the point that the embarrassment occasioned by the lack of symmetry is more acute in the case of the inverse operation. But the matter is further complicated by the way in which the usual definition of dy introduces a new independent variable z.

Unless some solution of these difficulties can be found, it seems that it would be preferable to introduce differentials in a frankly inaccurate and heuristic manner as small values of the increment or "little bits" of the "variable quantity" involved, rather than to clothe the idea with the deceptive appearance of logical accuracy.

3. There is a method of introducing differentials which suggests itself as a possible remedy, but unfortunately it may not be suitable for use in an elementary course except by devoting a disproportionate amount of time to the

study of parametric equations. The statement of it which follows is at all events not intended to be in form for presentation to the student.

This method is simply to define dx and dy to be $D_\tau x$ and $D_\tau y$ respectively, where τ is an arbitrary parameter.*

This does not contradict the usual statement that differentials are direction numbers of the tangent line. On the contrary it implies that statement. But it also adds a supplement to it which is needed to provide for certain ordinary uses of differentials. In particular the sign $\int$ taken by itself is then naturally understood to mean integration with respect to τ.

The extension of this idea to the differential du of a function u of two independent variables x and y is possible but somewhat cumbrous. It would perhaps be preferable to interpret equations involving du as relative to an arbitrary functional relationship between x and y (*i.e.*, as holding for every such functional relationship which satisfies appropriate conditions).

No very convenient method is provided of introducing second differentials, or of associating differentials with small values of the increment. In fact it seems that, if this definition of differentials is adopted, the notion of a differential should be kept entirely separate from considerations connected with small values of the increment. A special notation to represent $f'(x)\Delta x$ may be unnecessary; if such a notation is introduced, it should be something like $\delta_x f(x)$ or $\delta_x y$ (the subscript being dropped only in cases where it is irrelevant which variable is independent).

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Mathematics in Agriculture. By R. V. McGee. New York, Prentice-Hall, Inc., 1942. 9+189 pages. $4.00.

This text covers many of the topics usually taught in a course in Agricultural Mathematics. The applications of mathematics to agriculture are illustrated by means of interesting, practical problems but this reviewer wonders whether problems of sufficient difficulty are included to challenge the student of exceptional ability.

* There should be no objection from the point of view of rigor to the introduction of an arbitrary parameter, as opposed to a particular parameter. It means that, in a more explicit formulation, function variables (or relation variables) would appear, corresponding to the fixed functions (or relations) which would be used in introducing a particular parameter.

The material in the first three chapters on operations, percentage and equations is of the degree of difficulty usually associated with high school algebra and arithmetic. The application of the material on lengths, areas, and volumes is evident to the agriculturist. The treatment of ratio and proportion is generous but variation is omitted entirely.

The discussion of the right triangle and trigonometry is reduced to the minimum essentials. The treatment of statistics consists of an extensive presentation of the arithmetic, geometric and harmonic means with applications. The term "standard deviation" is defined but very few problems on this topic are given. The chapter on graphs is well done and in the opinion of this reviewer is the best feature of the text.

The chapter on special applications of practical measurements contains much useful information. The author uses the mixture lever instead of the dairyman's rectangle in solving mixture problems. The treatment of exponents, logarithms, and slide rule is conventional. The inclusion of extensive tables, 23 pages, for future reference is a very desirable feature.

Conspicuous by their absence from this text the reviewer notices, for example, variation, progressions, interest and annuities.

FRED ROBERTSON

A Manual of Problems in Statistics. By S. Dayton. New York, Henry Holt and Company. 163 pages. $.95.

This work has been designed as a companion volume of problems to the customary undergraduate text in economic statistics (such as "Statistical Methods," by Frederick C. Mills). In this purpose, it has succeeded admirably. There is a wealth of problems which illustrate clearly the computational elements involved in virtually all the statistical methods in common use. The wide variety of topics handled include algebraic exercises, graphs, description of the frequency distribution, index numbers of prices and physical volume, analysis of time series (secular, seasonal and cyclical variation), simple correlation (grouped and ungrouped data and time series), multiple correlation, the normal curve of error, the Chi-square test, measures of reliability, small samples and analysis of variance. Two appendices give tables of problem data and the chief statistical tables. The data used in the problems are drawn from significant economic material and are presented in tables which are models of good construction.

A few more detailed remarks may be made on these technical aspects. Problem 2c in Section A involves the solution of three simultaneous linear equations; these equations are linearly dependent, which may confuse the student. The section on graphs, though covering a wide range, omits the Lorenz chart which has become prominent in recent years in connection with studies of income distribution. On the other hand, inclusion of the logarithmic standard deviation in describing frequency distributions and of the analysis of variance test for seasonal variation means treatment of two topics of importance which are frequently neglected.

However, some caveats may be entered on a more fundamental level. These may be illustrated by the problems given for describing frequency distributions, which involve calculation of several measures each of central tendency, dispersion and skewness. This disregards the logic of the mathematical theory of statistics (as developed by R. A. Fisher, H. Hotelling, J. Neyman and A. Wald), according to which these measures are used as estimates of parameters of the underlying probability distribution. There can, of course, be only one best estimate in each case. This same disregard of the mathematical assumptions underlying statistical techniques appears in the section on cyclical variation in time series. The essence of the proposed method (which is the standard one in statistics texts) is to divide the observed values by the product of the trend value and the seasonal index. However, the fitting of a trend by least squares involves the assumption that there is no cyclical variation. It might be objected that since there are no mathematically valid methods of dealing with cyclical variation, the suggested technique is at least a good approximation. Prima facie, this is a good argument; but it puts the matter up to practical experience. E. L. Frickey (see "The Problem of Secular Trend," Review of Economic Statistics, 1934) has shown that the average duration of a cycle as determined by standard methods from different trends fitted to pig iron production may range from 3.6 to 45 years (depending on the trend fitted). This would seem to suggest the unreliability of the method. Some attention might be paid to the simpler methods used by the National Bureau of Economic Research (see W. C. Mitchell and A. F. Burns, "The National Bureau's Measures of Cyclical Behavior," Bulletin 57 of the Bureau), which seem to have definite advantages over the standard method suggested by Mr. Dayton.

Two minor points may be disposed of before closing. In discussing correlation, use of the standard error of the correlation coefficient is suggested (Section J, problem 4), although the size of the sample is only 18; further, in using the formula to test the significance, the calculated value is substituted for r, instead of the theoretical value. In discussing the analysis of variance, the method is introduced (as is customary) by a comparison of two variances. The method of treating this problem (making the larger variance always the numerator in the variance ratio) amounts to using a 10 percent level of significance instead of the 5 percent level which is intended.

These remarks must not be interpreted to deny the great contribution this manual can make to the instruction of a statistics class. Most of the observations made on the relation of the material to the fundamental mathematical assumption are directed more against possible abuses by instructors than against the work itself. As a matter of fact, the sections dealing with measures of reliability, small samples and analysis of variance play close attention to the underlying statistical theory; and many recent developments have been incorporated in these sections. In fine, Mr. Dayton's work can be recommended as an invaluable aid in the teaching of elementary economic statistics.

KENNETH J. ARROW

Essentials of Trigonometry with Applications. By D. R. Curtiss and E. J. Moulton. New York and Boston, D. C. Heath and Company, 1942. 8+174+94 pages. $2.25.

The first five chapters of this book are identical with the corresponding chapters of the authors' Brief Course in Trigonometry, except for minor revision (Reviewed in this MONTHLY, vol. 47, page 560, 1940). The chapters on logarithms and on the solutions of plane triangles are well written. The former is commendably brief, without the sacrifice of any essential feature. The section on co-logarithms is starred for possible omission. In the chapter on the solutions of plane triangles the laws of sines and of cosines are proved without the use of projections.

In the part on spherical trigonometry, the right-angle triangle is first discussed, with emphasis on Napier's Rules. Again the laws of sines and of cosines are established without the use of projection. Napier's Analogies are then obtained as consequences of these. The applications include plane surveying, problems in artillery, plane sailing, including use of the unit mil in angular measurement. In spherical trigonometric applications, plane sailing, great circle sailing, determination of time of sunrise and sunset, and of positions on the Earth's surface, of azimuth and hour angle are included.

The tables are Heath's Logarithmic and Trigonometric Tables, prepared by Professor E. J. Oglesby. They include four-place tables of squares, trigonometric tables, radians, and logarithms of numbers. Then follow five-place tables of small angles, trigonometric functions, common and natural logarithms of numbers, and a selection of constants and their logarithms.

The printing and press-work are excellent.

VIRGIL SNYDER

The Principles of Financial and Statistical Mathematics. By Maximillian Philip. Revised Edition. New York, Prentice-Hall, Inc., 1941, 16+335 pages. $3.50.

This book is divided into three parts, Basic Mathematics, Financial Mathematics, and Statistical Methods. The first division contains material which the author believes is essential to students of financial and statistical mathematics. Many teachers will consider that most of the material in this part is too elementary for students who are mature enough for these subjects. The well presented topics on interest and logarithms are essential. The material pertaining to progressions, binomial theorem and approximate calculations should have been placed in this section of the text.

The second part contains subject matter usually presented in text books on financial mathematics, together with a short chapter on life insurance. Newton's method of interpolation, continuous decrease and increase and depreciation are well presented. The three methods of evaluating the price to be paid for a bond are of much interest and should prove helpful to the reader. Students who need the knowledge of the mathematics in most of Part I before proceeding to the

second and third parts will find it extremely difficult to follow what is presented about logarithmic series, the base of natural logarithms and continuous increase and decrease; others should appreciate these ideas.

The problems in this part should help one to apply the methods presented and should furnish the teacher with adequate material for assignments.

The last part contains subject matter pertaining to statistics. The presentations of the normal curve from a ratio and the straight line in relation to different kinds of plotting paper are good. The development of the method of least squares by minimizing a quadratic expression will enable those who have not studied the calculus to understand and appreciate this method. Very little use is made of probability as applied to statistical problems. Many important topics which should be introduced in a beginning course in statistics are omitted. There is a shortage of good problems in this part. Little space has been given to interpretations of results. It is a mistake to omit ideas concerning sampling theory from any test on statistics.

The tables which accompany this book contain square and cube roots, common and natural logarithms, $(1+i)^n$, $(1+i)^{-n}$, $s_{\overline{n}|i}$, $a_{\overline{n}|i}$, $1\div a_{\overline{n}|i}$, $1\div s_{\overline{n}|i}$, $(1+i)^{1/p}$, log $(1+i)$, bond table, e^x, e^{-x}, life expectation, mortality table, net and gross annual premiums, ordinate and areas for the normal curve, log $(n!)$. These tables are well arranged, and are very easy to read.

W. D. Baten

A Mathematician's Apology. By G. H. Hardy. Cambridge, The University Press; New York, The Macmillan Company, 1940. $7+93$ pages. \$1.00.

Here is a little book every mathematician should have in his circulating library. This book should be read and then loaned; first, to every young person with mathematical leaning, then to non-mathematicians, and finally to any mathematician who inadvertently has failed to buy a copy.

The young mathematician will learn, probably to his surprise, that his very youth with its freshness and lack of restraint, is a much greater asset in his chosen field than the greater knowledge and experience that he hopes to accumulate over the years to come. He should realize after reading this book how important it is for him to pour what he has into an all out effort at the very beginning of his carrer. At that time, according to Hardy, he is more likely to win fame and place (but not fortune) for himself than at any later date. An old mathematician, as an old tennis player, may be a fairly young man.

A non-mathematician may be well repaid for two of his hours spent with this book, but he would profit relatively more if he invested five or six. Even the person capable of the remark "Who cares if there is an infinity of prime numbers?" may feel a sense of satisfaction after Hardy has led him to this trough and made him drink. Of course the non-mathematician will puzzle over the sentence "Real mathematics has no effect on war," but it will be good for his soul to be jarred from his notion that school mathematics is the ultimate in

mathematics. He will, of course, not see what lies beyond but must realize that there is something to be seen.

A mathematician needs no apology for mathematics, but he can read this book and after every section feel a deep sense of gratitude to one of his greatest contemporaries for putting the cause so clearly and with such force. He may not have had the nerve to say for himself, but he will be moved when he reads "and there are probably more people really interested in mathematics than in music."

It is not to be expected that all mathematicians will agree with Hardy on every point. There may, for example, be mathematicians who object to Hardy's use of the term "trivial mathematics," and certainly there are "real mathematicians" who would not feel they were criticizing the bible if they threw a little mud at some "real mathematics."

This book is not only about mathematics, it is about ideals, art, beauty, importance, significance, seriousness, generality, depth, young men, old men and G. H. Hardy. It is a book to be read, thought about, talked about, criticized, and read again.

J. F. RANDOLPH

The College Placement Algebra Workbook; *Defense Mathematics*. By A. E. Fulton. Marietta, Georgia, Kennesaw Publishing Co., 1942. 112 pages. $.75.

This little book is a series of tests of from four to some twenty exercises, on perforated sheets to be removed as used. Preceding each test is a short summary of the processes to be employed, largely reducing it to a mechanical procedure. These begin with the most elementary processes of simplification, addition, subtraction, multiplication, division, and factoring. The statements are given without proofs or motivation. When a new symbol is introduced, the practice is confusing, for example, radicals, p. 27. The introduction of imaginaries is at least naive. These and graphs are so brief as to be really harmful. The same statement applies to a solution of a system of simultaneous linear equations, p. 39.

To this early list of tests is added a considerable selection. from actual examination papers given at the Naval Academy, without the summary of the processes involved.

In addition to the incorrect and confusing statements made throughout the book, there is also a persistent suggestion of the advertising purpose of the course. This is probably not more objectionable than in many other cases, but is a lamentable tendency under the present circumstances.

On the whole the above remarks are applicable to a considerable literature on short cuts in mathematical teaching. They are not the opinions of one teacher, but the result of a consensus of opinions of a considerable number of teachers.

VIRGIL SNYDER

NEW BOOKS RECEIVED

Essentials of Trigonometry with Applications. By D. R. Curtiss and E. J. Moulton. New York and Boston, D. C. Heath and Company, 1942. 8+174+94 (Tables) pages. $2.25.

Mathematics in Agriculture. By R. V. McGee. New York, Prentice-Hall, Inc., 1941. 9+189 pages. $4.00.

Tables of the Moment of Inertia and Section Modulus of Ordinary Angle, Channels and Bulb Angle with Certain Plate Combinations. New York, Work Projects Administration, 1941. 13+197 pages. $1.25.

Intermediate Algebra for College Students. By T. S. Peterson. New York and London, Harper and Brothers, 1942. 8+358 pages. $1.85.

Technidata Hand Book. By E. L. Page (Essential Data on Mathematics, Physics, Chemistry, Engineering, Mechanics. For Engineers, Designers, Chemists, Mechanics and Technical Students.) New York, Norman W. Henley Publishing Company, 1942. 64 pages. $1.00.

Exploring Numbers. By H. G. Campbell and F. L. Wren. New York, D. C. Heath and Company, 1942. 7+264 pages. $0.80.

Number Activities. By H. G. Campbell and F. L. Wren. New York, D. C. Heath and Company, 1942. 7+247 pages. $0.80.

Number Experiences. By H. G. Campbell and F. L. Wren. New York, D. C. Heath and Company, 1942. 7+248 pages. $0.80.

Discovering Numbers, by H. G. Campbell and F. L. Wren. New York, D. C. Heath and Company, 1942. 7+280 pages. $0.80.

Supplement to Pandiagonal Magic Squares of Prime Order. By A. L. Candy. Lincoln, Nebraska, A. L. Candy, 1942. 1+30 pages.

The College Placement Algebra Workbook; *Defense Mathematics.* By A. E. Fulton. Marietta, Georgia, Kennesaw Publishing Company, 1942. 112 pages. $.75.

Operational Methods in Applied Mathematics. By H. S. Carslaw and J. C. Jaeger, Oxford, University Press, 1941. 16+264 pages. $5.00.

Mathematics in Daily Use. By W. W. Hart, C. Gregory and V. Schult. Boston, D. C. Heath and Company, 1942. 8+376 pages. $1.32.

Portraits of Famous Physicists with Biographical Accounts. By H. Crew. New York, Scripta Mathematica, 1942. 12 Portraits.

Calculus. By A. L. Nelson, K. W. Folley, and W. M. Borgman, Boston, D. C. Heath and Co., 1942. 10+356 pages. $2.75.

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT AND J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Brown University, Providence, R. I.

THE FILM—A TRIPLE INTEGRAL

E. A. WHITMAN, Carnegie Institute of Technology

In a recent leaflet entitled *Library of the American Mathematical Society,* Professor R. C. Archibald, the Society's long time, enthusiastic, and efficient Librarian said "In the near future films are sure to become very important library adjuncts." Already there are extensive film libraries on many subjects but the number of films on mathematics is distinctly limited. As the National Council of Teachers of Mathematics program on Visual Aids at the recent mathematics meetings at Bethlehem, Pennsylvania, revealed several new films, we may suspect that a considerable number of teachers of mathematics are considering the possibilities of films. Yet the experiences of these are seldom finding their way into print. The belief that an exchange of experiences with films might be desirable leads this writer to describe briefly the production of the film *A Triple Integral.* In addition, the showing of this film at our recent Christmas meetings was followed by inquiries which seemed to indicate interest in the details of its production.

This film resulted from an experiment in the use of the animated picture idea as an aid in class room teaching. There has been no attempt to include any motivating effects. Triple integration is one topic where it seems that the process could especially well be illustrated by a large number of drawings, too many for the teacher to make before a class. For example, suppose a solid has been shown on the blackboard and the *element of volume* selected, say a rectangular parallelepiped with edges Δx, Δy, and Δz. Now the question arises as to what happens if other solids like our chosen element of volume are piled one on top of another to form a *column* parallel to the z-axis. Thus far the work can be shown by drawings on the blackboard or even better yet by supplementing the drawings by use of a model. But the calculus requires the limit of the sum of the elements of volume as Δz approaches zero, Δx and Δy remaining constant. The number of elements of volume is increasing and the height of the column is changing. Here the large number of drawings required to sketch the process makes the task too long for the class room even for those teachers who can do a good job of sketching. Here it would seem that the animated picture might well be used to show the process. When the columns are to be summed to form a *slice* and the limit of this sum is considered, the figures become increasingly complex. At the third integration when the slices are summed the situation is still more complicated. Yet all the figures can be shown in a very short time by the use of a film.

In making the film *A Triple Integral* about five hundred twenty drawings and some forty titles were used. These drawings represented only about one fourth of the drawings made since in developing this topic the process was necessarily one of trial with no previous film to give guidance. The drawings were made by students as a part of a National Youth Administration project and required some six hundred student hours. The work however has been good training both in learning something of the meaning of precision in drawing as well as in preparation for the calculus. The photography in this the fourth trial took some twenty hours. The drawings were made on thin vellum paper instead of the celluloid sheets used commercially. The latter are said to require greater technical skill in drawing. The photography was accomplished by placing the drawings and titles on opal glass with light beneath and camera above. Our camera could take single exposures as well as run continuously. It did not have a rewind feature so it was not possible to make those easy changes between successive titles or between the greater changes in the development of the story. The animation was accomplished by placing different drawings over various constant backgrounds but the character of the film seemed to require much slower animation than commercial movies. The showing time for titles required some experimentation. The common rules for titles anticipate only the needs of very juvenile minds.

The technical difficulties in accomplishing animation are a real stumbling block in this type of work, at least so our experience goes. Our paper drawings would stretch. Our system of indicating where the drawings were to be placed with reference to their backgrounds was not sufficiently accurate to prevent undesirable internal motions. Even this fourth attempt at filming has produced something that one shows only with hesitation. It would seem to be an ideal situation if the mathematician could do the development work and the final set of drawings and the photography could be done in some studio that was equipped and manned for such work. After all our mathematics text books owe much to the art of the printer. Might we not expect then that fine or even acceptable films might require the skill and art of the movie studio?

Such studio production would seem to need either a wealthy patron or an interested sponsor. Sometime the question of sponsoring films will likely come before our mathematics organizations. Then there is also the possibility that some manufacturer may sponsor some films in order to increase the need for equipment in somewhat the same way that manufacturers of radio receiving sets have sponsored radio broadcasts. Studio production would make films generally available, and only under such conditions of availability can films have wide enough circulation to take them out of the hobby class.

Meanwhile it may be desirable to promote the exchange of experiences between those interested in trying to produce films in the field of mathematics. To this end the film *A Triple Integral* will be gladly sent to any one interested in it. To those who have inquired about having it generally available, the only reply at present is that this must await future developments.

CLUB REPORTS, 1941–1942

Pi Mu Epsilon, University of Oklahoma

Expanding horizons was the topic of Dr. J. O. Hassler and Balfour Whitney at the first meeting of the chapter held at the university observatory. Other programs were devoted to a discussion of *Arabic mathematics, yesterday and today* by Richard Dulaney who spent two years in Arabia with an engineering company, and to *Elementary groups* by Stuart Lee. The annual mathematics contest sponsored by the society was won by Abdurrahman Durakel who had been sent to study at the university by the Turkish government. Mr. Durakel was presented with Courant's *Differential and Integral Calculus.* Professor O. H. Hamilton of Oklahoma Agricultural and Mechanical College was guest speaker at the banquet and initiation and spoke on *Mathematics as an avocation.* Part of the entertainment at the banquet was furnished by problems and puzzles which appeared on the printed program. Officers were: Director, Harold Huneke; Vice-Director, Stuart Lee; Secretary, Marian Wright; Treasurer, Phyllis Barclay; Sponsor, Dr. Dora McFarland.

Delta Zero Mathematics Club, Mississippi Delta State Teachers College

This club was organized in September 1940 under the sponsorship of Dr. J. A. Ward. A constitution with by-laws was composed and approved. Topics for programs held during the year were: *The use of matrices in secret codes*; *Five solutions of an equation—Factoring, completing the square, formula, geometric drawing and slide rule*; *Famous men in mathematics—Newton, Archimedes, Pythagoras*; *Interesting facts about fourth dimensional cubes* (with models); *Three ways of trisecting angles*; *Three ways to take the square root*; *Speed and force of a bullet shot from different angles*; *Aeronautical movies*, showing the connection between mathematics and aviation. Officers were: President, William MacDonald; Vice-President, Christine Douglas; Secretary-Treasurer, Connie Scott; Reporter, Ranny Williams.

Pi Mu Epsilon, Oregon State College

Subjects and speakers at the six meetings held during the year were: *Works of Gauss* by Dr. Henri Scheffe, *Duo-decimal system* by Jean King, *Graphical solutions of equations* by Karl Steinbrugge, *Cantor's theory of trans-finite numbers* by Dr. Andrew Sobczyk, *The isograph* by Calvin Gross, *Numerical solution of a pursuit problem* by Robert Beagles, *Heaviside's unit function* by Helen Murdock, *The mathematical relationships of the frequencies of musical tones in scales and chords* by Eugene Grant. The chapter awards a scholarship annually to a student registered in mathematics and also a yearly prize in mathematics to the Willamette Valley Science Conference. Director, Eugene Grant; Vice-Director, J. Peterson; Secretary, Annabelle Berg; Treasurer, Professor G. A. Williams.

Mathematics Club, Women's College, University of Delaware

The first meeting was devoted to a discussion of the development of the old Greek curves of double motion. Topics and speakers at later meetings were: *Applications of mathematics to astronomy* by Professor R. W. Jones, *Mathematical Logic* by Ann Hamilton, *Types of proof of the Pythagorean theorem with illustrations* by Dr. G. H. Wilson of the Department of Physics. At the annual banquet Dr. C. J. Rees discussed five equivalent definitions of the sine function. President, Grace Shockley; Vice-President, Katherine Mitchell; Secretary-Treasurer, Alice Ward; Faculty Adviser, Edith A. McDougle.

Mathematics Club, Brown University

This club held its usual series of six monthly meetings from November to April. These were planned in advance and announced on a printed program. Undergraduate speakers at four of the meetings included Ellen Swanson on *Coincidences and probability*, Harvey Spear on *Election by proportional representation*, Donald Hall on *One sided surfaces*, Albert Acorn on *The bridges of Koenigs-*

berg, Tamara Bachman on *Curiosities in numbers*, Paul Tamarkin on *Minimal surfaces and soap films*, Arthur Long on *Euler Phi-function* and Herbert Maass, Jr. on *Mathematics in aeronautics.* Faculty speakers included Professor J. A. Clarkson of the University of Pennsylvania who spoke on *Curves of constant width*, and Professor J. S. Frame who demonstrated and discussed *Stencils for detecting primes.* Each year a picture is taken of the club and added to a collection of such pictures which date, with a few interruptions, back to the founding of the club in 1915.

Pi Mu Epsilon, University of Missouri

Activities for the year included five program meetings, a picnic, social meeting in which mathematical tricks and puzzles were the source of the evening's entertainment, and an initiation banquet at which 26 new members were admitted. Topics discussed were: *Dimensionality—three and otherwise* by Dr. Herman Betz, *Some applications of statics to geometry* by L. M. Kelly, *Mechanical description of curves* by Olen Nance, *Hyperbolic functions* by Dr. B. E. Gillam, *Elementary concepts from number theory* by W. E. Ferguson. Director, D. F. Abell; Vice-Director, G. E. Brown; Corresponding Secretary Dr. L. M. Blumenthal; Treasurer, Dr. B. E. Gillam.

Mathematics Club, University of Cincinnati

Articles from the MONTHLY were used for discussion at several of the meetings. *Soap films and minimal surfaces* were discussed by Adolph Goodman and Ruth O'Donnell. *Wine carrying and river crossing in a general case* was the topic of Robert Buck. Other titles were: *Pigs is pigs—a discussion of three standard problems of the Diophantine type* by Everett Yowell, *Polynomials simultaneously orthogonal on two or more circles* by Stanley Lawwill, *The number theory of quaternions* by Earl Swafford, *Regular polyhedra* by Robert Buck, *Symmetry* by Keyser Kunz, *Measurement in education* by Harvey Weitkamp, *The un-necessity of the ruler* by Ruth O'Donnell, *Elementary functional equations* by W. E. Restmeyer. President, Adolph Goodman; Vice-President, Robert Buck; Secretary-Treasurer, Mary Connor.

Mathematics Club, Hunter College of the City of New York

During the week set aside for dedication ceremonies for the new Hunter College Building, the club sponsored a Bureau of Vital Statistics. They showed pictorially the various data on the new building, the students and activities pursued. Programs included the following papers: *Alternating current developed by means of complex numbers* by Frieda Axelrod, *Invariants in the analytic geometry* by Professor C. C. MacDuffee, *P-adic numbers* by Shirley Orlinoff, *Transformations* by Professor Jewell Hughes Bushey, *Mathematical tricks, puzzles, and fallacies* by Dora Feldman, *Darkening at the limb of the sun* by Tecla Combariati, *Propositional functions* by Louise Miller, *Undergraduate mathematical publications* by Professor Lao G. Simons. President, Tecla Combariati; Vice-President, Charlotte Gertler; Secretary, Phyllis Monderer; Treasurer, Hortense Schindler; Faculty Adviser, Dr. L. A. Aroian.

Harvard Mathematical Club, Harvard University

Speakers and their topics for the year included: *The principle of sufficient reason* by Professor G. D. Birkhoff, *Introduction to the prime number theorem* by the Rev. R. E. O'Connor, *Parallelism, geodesics and related topics* by Caro Lippman, *Milne's kinematical relativity* by John Breakwell, *Some curious properties of numbers* by Dr. A. Whiteman, *Transfinite arithmetic* by Professor Garrett Birkhoff, *The elusive Plücker characters* by Professor Oscar Zariski, *The Fibonacci sequence* by E. C. Gras, *Radix and number systems* by G. F. Forbes, *Some questions in analysis* by H. Pollard, *An elementary fixed point theorem* by Professor Alfred Tarski, *Bertrand's postulate* by C. Price, *Some paradoxes in measure theory*, by J. Eisenstein, *Is it worthwhile to study mathematics?* by Professor George Sarton. Radcliffe College students are admitted as associate members of the club. President, E. C. Gras; Vice-President, Murray Lampert; Secretary, Julian Eisenstein; Treasurer, Alfred Putnam; Faculty Adviser, Professor D. V. Widder.

Pi Mu Epsilon, Columbia University

At the beginning of the year, the committee on reorganization which had been elected at the close of the previous year, sent out letters to a select group of students. Replies were received from over twenty five and seventeen of these were ultimately inducted into the society during the year. Bi-weekly meetings were held at which the following subjects were presented: *The mathematical theory of gases* by Professor L. P. Siceloff, *Mathematics and semantics* by Leon Henkin, *Contour integration* by Lawrence Annenberg, *Dimensions* by Professor P. H. Smith, *Tschirnhaus transformations* by Richard Brown, *The summation of series* by Bernard Gelbaum, *Transfinite numbers* by Kenneth Miller, *Mathematics and economics* by Harry Schwartz, *Hyperbolic geometry* by Dr. W. Strodt. Additional meetings were also devoted to the solution of problems and preparation for the Putnam and the Metropolitan New York *Pi Mu Epsilon* contest. Director, Leon Henkin; Secretary-Treasurer, Ulrich Strauss; Faculty Adviser, Professor L. P. Siceloff.

Mathematics Club, The University of Kansas

Semi-monthly meetings were held throughout the year. Printed programs were issued at the beginning of the year and papers were presented as follows: *The foundations of mathematics* by Bruce Crabtree, *Numbers* by Kenneth Barnett, *The first quadrature of a curvilinear surface* by Clark Moots, *Descartes and the invention of analytic geometry* by Arthur Ames, *Some famous problems of modern mathematics* by Professor G. B. Price, *Newton, Leibnitz and the invention of the calculus* by John Tweed, *Limits* by Merle DeMoss, *The theory of numbers* by Calvin Foreman, *Probability* by L. R. Shobe, *A problem in biophysics* by Wellesley Dodds, *Non-Euclidean geometry* by Ralph Burson, *The Lorentz transformation* by Arthur Peters, *Iteration method of solving equations* by Professor H. E. Jordon, *Generalized electrical equations and the M.K.S. system* by Professor Hessler. President, Bruce Crabtree; Vice President, Merle DeMoss; Secretary-Treasurer, Kenneth Barnett; Faculty Adviser, Professor G. B. Price.

Mathematics Club, University of Rochester

This club was reorganized early in the year after a lapse of several years. A quiz program in the nature of an Information Please contest was conducted at a mid-year meeting. Papers presented at the remaining meetings included: *The flow of heat and the application of Fourier analysis* by David Van Horne, *Quadratic Diophantine equations* by Robert Mann, *Linear Diophantine equations—the problem of the monkey and cocoanuts—and similar problems* by Helen Nyquist, *Nomograms and their uses* by George Monroe, *The Minkowski geometry of numbers* by David Falkoff. President, Robert Mann; Secretary-Treasurer, Catherine Reid.

Pi Mu Epsilon, University of Arkansas

During the year, this chapter held seven regular meetings, two picnics and two initiation banquets. At the latter, twenty six new members were admitted to the organization. Papers were given as follows: *Gambling* by Howard Head, *Proof that the proverbial "dog did catch the rabbit"* by Robert Morse, *Apportionment of representatives in Congress* by Landon Brown. A mathematics contest was held at one of the banquets. President, John Turner; Vice President, Bobbie Alfrey; Secretary, Robert Hobson; Treasurer, Harry Clayton; Faculty Adviser, Dr. V. W. Adkisson.

Mathematics Club, Wayne University

At the semi-monthly meetings held during the year the following papers were discussed: *Configuration symbols in plane projective geometry* by Jack Swartz, *The use of latent squares in finding regression lines* by Clifford Simms, *The theorem: If any odd perfect number exists, it must have at least six prime factors* by Robert Coveyeau, *Trisection of angles by various means* by Jack Swartz, *The simplification of the equation of the conic* by Eva Rossman. President, Jack Swartz: Vice-President, Rollin Woodward; Secretary, Betty Pickering; Faculty Adviser, Dr. D. C. Morrow.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 526. *Proposed by R. C. Yates, Louisiana State University*

Find the locus of P if the angles formed by the tangents from P to two fixed circles are equal.

E 527. *Proposed by V. Thébault, San Sebastián, Spain*

Show that the sum of the radii of the circles C_1, C_2, C_3 of E 457 [1941, 637] is equal to the diameter of the incircle, and that the sum of the radii of the three analogous circles whose centers are exterior to the segments A_iI is three times as great.

E 528. *Proposed by R. A. Rosenbaum, Reed College*

Prove and generalize the identity

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k^2}\cdot e^{-k} = e^{-1/2}.$$

E 529. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

Construct an irregular hexagon which shall be both inscriptible and circumscriptible.

E 530. *Proposed by P. D. Thomas, Southeastern State College, Okla.*

Is there a sphere orthogonal to the six radical spheres determined by four given spheres whose centers are not coplanar? (The radical sphere of two spheres is the locus of a point whose two powers have zero sum.)

SOLUTIONS

The ith Root of -1

E 491 [1941, 635]. *Proposed by E. T. Frankel, Albany, N. Y.*

Prove that $\sqrt[\sqrt{-1}]{-1}=23\frac{1}{7}$, approximately.

Solution by I. Opatowski, University of Minnesota

The ith root of the complex number $z=e^{\theta i}=e^{(\theta+2k\pi)i}$ is $e^{\theta+2k\pi}$ where k is an arbitrary integer. When $z=-1$, this becomes

$$\sqrt[i]{-1} = e^{(1+2k)\pi}.$$

Since $e^{\pi} = 23.14069 \cdots$, the value sought by the proposer is that corresponding to $k=0$.

Also solved by R. K. Allen, Aaron Bakst, W. B. Brown, R. E. Crane, William Douglas, Thor Eriksson, Howard Eves, Edward Fleisher, G. S. Heller, J. F. Kenney, C. N. Mills, C. C. Oursler, P. W. A. Raine, W. L. Roberts, E. P. Starke, Alan Wayne, and the proposer.

Four- and Six-digit Squares

E 492 [1941, 635]. *Proposed by V. Thébault, San Sebastián, Spain*

Find a four-digit square which remains a square when two zeros are intercalated between the thousands digit and the hundreds digit.

Solution by Alan Wayne, Rhodes School, New York

From Barlow's Tables, there are only seven 6-digit squares whose second and third digits are zeros. These lead at once to two 4-digit squares satisfying the conditions, namely

$$2704 = 52^2 \qquad (200704 = 448^2),$$

$$7569 = 87^2 \qquad (700569 = 837^2).$$

Also solved by W. E. Buker, M. L. Constable, Thor Eriksson, Evelyn Hesseltine, E. P. Starke, and the proposer. (The larger solution alone was found by D. H. Browne and William Douglas.)

A Tetrahedron and Four Spheres

E 492 [1941, 635]. *Proposed by N. A. Court, University of Oklahoma*

Given a tetrahedron $ABCD$ and a point M, prove that the tangent planes, at M, to the four spheres $MBCD$, $MCDA$, $MDAB$, $MABC$, meet the respective faces BCD, CDA, DAB, ABC in four coplanar lines.

Solution by Howard Eves, Allen Academy, Bryan, Texas

Given a tetrahedron $ABCD$ and a point M, we can easily prove that the planes through M, parallel to the faces BCD, CDA, DAB, ABC, meet the respective spheres $MBCD$, $MCDA$, $MDAB$, $MABC$ in four cospherical circles. For the perpendiculars to the four circles, at their centers, coincide with the perpendiculars to the faces BCD, CDA, DAB, ABC at their circumcenters. Since these concur (at the circumcenter of the tetrahedron), it follows that the four circles (through M) are cospherical.

The desired theorem can be derived from this by inversion, with center M.

Also solved by the proposer.

A Diophantine Equation

E 494 [1941, 635]. *Proposed by Henry Scheffé, Reed College*

Show that, for every positive integer N, the equation

$$x^2 + y^2 + 2xy - 3x - y + 2 = 2N$$

has a unique solution in positive integers x, y. Give a method for finding it, without successive trials. Generalize the problem to n unknowns $x_1, x_2, \cdots, x_n$, satisfying

$$F(x_1, x_2, \cdots, x_n) = 2^{\alpha-1} N,$$

where F is a special polynomial of degree $\alpha = 2^{n-1}$, with integral coefficients.

Solution by W. B. Carver, Cornell University

The equation

$$x^2 + y^2 + 2xy - 3x - y + 2 = 2N \tag{1}$$

may be written as $\binom{t}{2} + y = N$, where $t = x + y - 1$. Hence all its solutions in integers are given by

$$x = \binom{t+1}{2} - N + 1, \qquad y = N - \binom{t}{2}, \tag{2}$$

where t is an arbitrary integer. If now we require that x and y be *positive* integers, we must have

$$\binom{t}{2} < N \leqq \binom{t+1}{2}.$$

For a given positive integer N, this defines a unique positive integer t, namely the greatest t for which $\binom{t}{2} < N$. Substituting this in (2), we have the unique solution of (1) in positive integers.

To generalize the problem, we define a function of n variables by the recurrence formula

$$f_i(x_1, \cdots, x_i) = f(f_{i-1}(x_1, \cdots, x_{i-1}), x_i), \qquad (i = 3, 4, \cdots, n), \tag{3}$$

where $f(x, y) = f_2(x, y) = \frac{1}{2}(x^2 + y^2 + 2xy - 3x - y + 2)$.

It is readily seen that the function

$$F(x_1, \cdots, x_n) = 2^{\alpha-1} f_n(x_1, \cdots, x_n) \tag{4}$$

is a polynomial of degree $\alpha = 2^{n-1}$ with integral coefficients. Writing the diophantine equation

$$F(x_1, \cdots, x_n) = 2^{\alpha-1} N \tag{5}$$

as

$$f(f_{n-1}(x_1, \cdots, x_{n-1}), x_n) = N,$$

we see that it is satisfied by unique positive integral values of f_{n-1} and x_n, say $f_{n-1}(x_1, \cdots, x_{n-1}) = N_1$ and $x_n = c_n$. We now wish to find $x_1, \cdots, x_{n-1}$ to satisfy $f_{n-1} = N_1$, or

$$f(f_{n-2}, x_{n-1}) = N_1;$$

and we know that this has a unique solution, say $f_{n-2} = N_2$ and $x_{n-1} = c_{n-1}$. It is

clear that this process continues until we have values for $x_n, x_{n-1}, \cdots, x_3$, and want values of x_1 and x_2 to satisfy

$$f_2(x_1, x_2) = N_{n-2},$$

which is the equation (1) itself. Hence equation (5) gives the required generalization, F being the polynomial defined by (3) and (4).

Putting $n=3$, we find that the equation

$$\begin{aligned}(x+y)^4 - 6x^3 - 14x^2y - 10xy^2 - 2y^3 + 4z(x+y)^3 \\ + 7x^2 - y^2 + 2xy + 4z^2 - 12xz - 4yz + 6x + 2y + 4z = 8N\end{aligned}$$

has, for any positive integer N, one and only one solution for x, y, z in positive integers.

Also solved by G. B. Huff and the proposer.

Barycentric Coördinates

E 495 [1941, 635]. *Proposed by Daniel Arany, Budapest, Hungary*

If x, y, z are the barycentric coördinates of a point Q with respect to a triangle ABC, show that, for any point P in the same plane,

$$xAP^2 + yBP^2 + zCP^2 = xAQ^2 + yBQ^2 + zCQ^2 + (x+y+z)PQ^2.$$

Solution by Howard Eves, Allen Academy, Bryan, Texas

Let A', B', C' be the projections of A, B, C on the line QP, and let k be the line through Q perpendicular to QP. Then we have

$$\begin{aligned}AP^2 &= AQ^2 + PQ^2 - 2QP\cdot QA',\\ BP^2 &= BQ^2 + PQ^2 - 2QP\cdot QB',\\ CP^2 &= CQ^2 + PQ^2 - 2QP\cdot QC'.\end{aligned}$$

Multiplying these equations by x, y, z, respectively, and adding, we have

$$\begin{aligned}\sum (xAP^2) &= \sum (xAQ^2) + \sum xPQ^2 - 2QP\cdot \sum (xQA') \\ &= \sum (xAQ^2) + \sum xPQ^2 - 2QP\cdot \sum (x\alpha),\end{aligned}$$

where α is the distance from k to A, *etc.* But, from a fundamental property of barycentric coördinates, we know that $\sum(x\alpha)=0$. Hence the theorem.

(This is easily generalized to any system of points $A, B, C, \cdots$ and their mean center Q for multiples $x, y, z, \cdots$. See, *e.g.*, art. 55 in McClelland's *Geometry of the Circle.*)

Also solved by the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4040. *Proposed by N. A. Court, University of Oklahoma*

Given n points $A_1, A_2, \cdots, A_n$ in space, let G denote their centroid, k^2 the sum of the squares of the $n(n-1)/2$ segments determined by the given points, and P_{ij} the power of a given point P with respect to the sphere having for diameter the segment A_iA_j; show that

$$\sum P_{ij} = n(n-1)PG^2/2 - k^2/2n.$$

4041. *Proposed by Cezar Coşniţă, Focşani, Roumania*

Integrate the following differential equations

$$y' = \frac{x^2 - y^2 + 1}{x^2 - y^2 - 1}, \qquad y' = -\frac{y(2x + y - 1)}{x(x + 2y - 1)}.$$

Determine the integral curves passing through the origin.

4042. *Proposed by Henry Scheffé, Princeton University*

Prove that if A is a fixed positive definite hermitian matrix and X is a variable non-negative hermitian matrix (rank = index), then the minimum value of the determinant $|A+X|$ is $|A|$ and is attained if and only if $X=0$.

4043. *Proposed by H. F. Sandham, Trinity College, Dublin*

Prove that the angle in which the major auxiliary circle of a conic inscribed in a triangle cuts the nine point circle, is equal to the angle which the foci of the conic subtend at the inverse of one of them in the circumcircle. Complete this result and deduce that the minor auxiliary circle of the conic which has the Brocard points as foci, touches the nine point circle, and the major auxiliary circle cuts the latter in an angle which is the complement of three times the Brocard angle of the triangle.

4044. *Proposed by V. Thébault, San Sebastián, Spain*

Determine the straight lines such that the circumsphere of the pedal tetrahedron of each of its points with respect to any given tetrahedron $ABCD$ passes through a fixed point P.

Examine the case for which $ABCD$ is orthocentric and P is the foot of one of its altitudes.

SOLUTIONS

Perspective Triangles (Tetrahedrons)

3988 [1941, 213]. *Proposed by N. A. Court, University of Oklahoma*

The symmetrics of a given straight line (plane) with respect to the sides (faces) of a given triangle (tetrahedron) form a second triangle (tetrahedron) perspective to the first, and the center of perspectivity is equidistant from the sides (faces) of the second triangle (tetrahedron).

Solution by H. W. Eves, Allen Academy, Bryan, Texas

We shall prove the theorem for the plane case; a parallel proof can be given for the solid case. Let ABC be the given triangle with sides a, b, c and let l be the given line. Let the reflections of l in a, b, c be a', b', c', giving the second triangle $A'B'C'$. Now triangles ABC and $A'B'C'$ are centrally perspective because they are axially perspective (on l). All that remains to be proved, then, is that AA', BB', CC' are bisectors of interior or exterior angles of triangle $A'B'C'$. To this end consider triangle $b'c'l$. In this triangle b and c are interior or exterior angle bisectors because of the reflection property. It follows that AA' must be a third angle bisector, bisecting the interior or exterior angle at A'. Similar remarks now hold for BB' and CC' and the theorem is proved.

Solved also by the proposer in a similar manner giving the detailed proof for the tetrahedron and remarking that the proof for the triangle is analogous.

Cones and Spheres

3989 [1941, 213]. *Proposed by N. A. Court, University of Oklahoma*

Three given spheres with non-collinear centers are touched by a (fourth) sphere in the points P, Q, R, and (p), (q), (r) are great circles, in parallel planes, on the three given spheres. Show that the three cones $P(p)$, $Q(q)$, $R(r)$ have a circle in common.

Solution by H. W. Eves, Allen Academy, Bryan, Texas

Given two spheres S_M and T tangent at a point M. Let (m) be a great circle of S_M. Then it is a simple matter to show that the cone $M(m)$ cuts T in a great circle (t) whose plane is parallel to the plane of (m). The theorem now readily follows. For let S_P, S_Q, S_R be the three given spheres and T the fourth tangent sphere. Then, since the planes of (p), (q), (r) are parallel, the three cones $P(p)$, $Q(q)$, $R(r)$ all cut T in the great circle (t) whose plane is parallel to that of (p), (q), (r). This proves the theorem. Note that the condition of non-collinearity of the centers of S_P, S_Q, S_R is not necessary.

Solved also by G. A. Yanosik and the proposer.

Editorial Note. The solution by the proposer is similar to the above and gives essentially the following argument:

Since M is the point of contact of the two spheres S_M and T, it is a homothetic center for the two spheres; and in the homothetic relation thus defined the two centers are corresponding points. To the plane of the great circle (m) there

corresponds a diametral plane of T; and hence the cone $M(m)$ cuts the sphere T in points of this diametral plane of T.

The solution by Yanosik is analytic and much more complicated than the above simple synthetic proof.

Envelope of Spheres

3992 [1941, 214]. *Proposed by V. Thébault, San Sebastián, Spain*

Show that the envelope of a variable sphere (S) which has its center on a quadric surface of revolution (Q) and which is orthogonal to a sphere (Σ) tangent to (Q) along a circle (C) is composed of two spheres passing through (C).

Solution by G. A. Yanosik, New York University

Since the entire configuration here involved is one of revolution about a given axis, the problem may be solved by working in any plane section taken through the axis of revolution.

Let the Y-axis be the axis of revolution, and let the plane $z=0$ be the plane of section.

Take the section of (Q) as

$$x^2 = ay^2 + by + c.$$

The diameter of circle (C) will join the points $(\pm\sqrt{c}, 0)$.
The section of (Σ) will now be

$$x^2 + y^2 - by - c = 0.$$

Now the section of the variable sphere (S) will be given by

$$(x - h)^2 + (y - k)^2 = R^2,$$

subject to

$$h^2 = ak^2 + bk + c,$$

and to

$$R^2 + c = h^2 + k^2 - bk.$$

These three relations lead to

$$[(b - 2y)^2 - 4ax^2]k^2 + 2[(b - 2y)(x^2 + y^2 + c) - 2bx^2]k + [(x^2 + y^2 + c)^2 - 4cx^2] = 0.$$

The parameter being k, the envelope will now be found to be

$$a[x^2 + y^2] + [b \pm \sqrt{(1 + a)(b^2 - 4ac)}]y - ac = 0,$$

which equations represent two circles passing through the points $(\pm\sqrt{c}, 0)$. This establishes the truth of the final statement in the problem.

Editorial Note. The proposer stated that the theorem of the problem results immediately from the similar theorem for the plane; and he also remarked that,

when the quadric surface is a paraboloid of revolution, the envelope of (S) is the plane of the circumference of (C). No proofs were given.

Some properties of the envelope of the plane figure are easily found when the conic is replaced by any curve (Q) which is tangent to a fixed circle (C) at the point T. A circle (S) with its center S on (Q) is orthogonal to (C), and we consider the envelope of (S). A given circle (S) and a similar neighboring circle intersect in two points which are inverses with respect to (C). The limiting positions E_1 and E_2 of these two points on (S) are points of the envelope, and it is clear that E_1 and E_2 are inverses with respect to (C) and that the midpoint P of E_1E_2 is on the pedal curve of (Q) with respect to C, the center of (C). If E_1 and E_2 are real and distinct points, the point P must lie outside the circle (C); when S is at T, the three points P, E_1, E_2 must coincide in T. Thus the envelope is invariant under inversion with respect to (C) and it must contain the point T, which may be a singular point for the envelope. If (Q) cuts (C) in a point, the two curves not being tangent at the point, the corresponding point P for the intersection is inside (C), and E_1 and E_2 are imaginary. If a portion of (Q) lies inside (C), then each of the points of this portion yields only imaginary points for the envelope, except for points, such as T, where this portion may be tangent to (C). An example of this is the case where (Q) is an ellipse inside (C) and tangent to (C) at one or two points.

Suppose now that a certain portion of (Q), proceeding in a given sense from T, is such that the tangent at each of its points contains no interior point of (C). Then the corresponding parts of the envelope are real. For in this case P lies outside (C), and the circle (CS) with the diameter CS passes through P and cuts (C) in the real points M_1 and M_2. The points C, M_1, P, M_2 are necessarily in this order, the circle (S) must contain P in its interior, and E_1 and E_2 must be real points. Thus, if (Q) is an ellipse containing the circle (C) in its interior so that the two are tangent at one or two points, then all the points E_1 and E_2 are real.

If proceeding from T, say to the right, a portion of (Q) is separated from (C) by the common tangent at T, for example if (C) lies below the tangent, and the tangent at each point S has an increasing inclination to the common tangent, then for each S on this portion the tangent to (Q) cuts (C) in two real points, the point P is inside (C), and E_1 and E_2 are imaginary. This continues until we reach a point S_1 on (Q) such that the tangent at S_1 is also tangent to (C) at C_1, and then for S_1 the three points P, E_1, E_2 coincide in C_1. For points on this portion of (Q) beyond S_1, to a certain extent the corresponding points P lie outside (C) and inside the corresponding circles (S), and the corresponding points E_1 and E_2 are real. Thus if (C) is tangent to the two branches of a hyperbola, the envelope has no real points except the two points of contact.

If (Q) is a parabola tangent to the circle (C) at two distinct points, the envelope is real and passes through these two points of contact. From the above solution we see that one part of the envelope is the straight line of the common chord; hence the other part is the inverse of this straight line, which is the circle

through C and the two points of contact. The equation of this circle in the notation of the solution is

$$2b(x^2 + y^2) + (4c - b^2)y - 2bc = 0, \qquad a = 0, \qquad bc \neq 0.$$

The equation of the envelope in the solution results after discarding the factor x^2. The equation of the axis on the conic is $x=0$; and there cannot be more than four points of the envelope on this axis, and these points must be also on the two circles of the envelope. Hence the factor x^2 should be discarded.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to C. O. Oakley, Haverford College, Haverford, Pennsylvania.

The University of Michigan is offering a new course in mathematics under Professor H. C. Carver, who has been serving as a cadet in the Army Air Force since February. The course is designed to speed up the training of navigators for the air force and is being given during the first eight weeks of the summer session beginning June 15. From June 29 to August 21, also at the University of Michigan, Professor Jerzy Neyman of the University of California is offering an advanced course in the theories of testing hypotheses and of estimation, and he also joins with Professor C. C. Craig in conducting the regular seminar in mathematical statistics for advanced students.

The graduate school of New York University has instituted a program beginning June 23, 1942, the main purpose of which is to enable college graduates to obtain their master's degree in scientific fields by February 1943, so that they may become available as scientific workers or specialized personnel for the war effort. During the summer two half-courses in advanced calculus including vector analysis are given by Professor Friedrichs and Dr. Robbins, and a full course on topics in mathematical physics and applied mathematics by Professor Courant. In addition there is an advanced seminar in research problems.

Fordham University is offering during the summer session an evening course including topics in algebra, solid geometry and plane and spherical trigonometry to prepare prospective candidates for training as officers in the armed forces.

Assistant Professor C. B. Allendoerfer of Haverford College has been promoted to an associate professorship.

At Massachusetts Institute of Technology Assistant Professor P. D. Crout has been promoted to an associate professorship and Dr. Eric Reissner to an assistant professorship.

Dr. J. A. Daum of the A. and M. College of Texas is now a Second Lieutenant, U. S. Army.

Dr. W. B. Fite, Davies professor of mathematics at Columbia University, retired in June.

Assistant Professor J. S. Frame of Brown University has been appointed associate professor and head of the department at Allegheny College.

Dr. J. L. Gibson, professor of mathematics at the University of Utah since 1904 and dean of the School of Arts and Sciences since 1915, retired in June 1941. He continues as president of the Utah Conservation and Research Foundation, an office he has held since 1937.

After forty-seven years of service at Williams College, Professor J. G. Hardy retired at the end of this academic year.

Miss Will Lipscombe, assistant professor at the University of Akron, has been promoted to an associate professorship.

Dr. C. T. McCormick of Fort Hays Kansas State College has been made professor and head of the department.

M. L. Manning, research engineer with the Westinghouse Electric and Manufacturing Company, has been appointed an associate professor of electrical engineering at Illinois Institute of Technology.

Professor J. S. Petersen, Jr., is on leave of absence from Brescia College, New Orleans, and is serving in the U. S. Army Signal Corps.

At the College of the City of New York, Professor F. G. Reynolds retired in June, and Associate Professor Maximilian Philip succeeds him as head of the department of mathematics.

Professor P. R. Rider of Washington University has been appointed an exchange professor at the National University of Mexico for fourteen months beginning August 1, 1942. This appointment is under the Convention for the Promotion of Inter-American Cultural Relations, and under the auspices of the Division of Cultural Relations of the U. S. Department of State. He is to lecture on mathematical statistics.

At Louisiana State University, Professor S. T. Sanders retired June 30, 1942, after thirty-five years of service there, and Professor W. V. Parker succeeds him as head of the department of mathematics.

Assistant Professor C. V. L. Smith of Lafayette College is on leave of absence and is a lieutenant (jg) of class D-V(S), U. S. Naval Reserve. In March through June he attended the Naval Training School of Radio Engineering at Bowdoin College.

Dr. Roy MacKay, associate professor at New Mexico State College, died May 12, 1942, at the age of thirty-eight. He had been a member of the Association for eleven years.

W. F. Reynolds, Chief, Section of Triangulation, Division of Geodesy, U. S. Coast and Geodetic Survey, died May 1, 1942, at the age of sixty-one. He had been a member of the Mathematical Association for fifteen years.

Maria M. Roberts, Professor Emeritus of mathematics at Iowa State College, died on April 12, 1942, at the age of seventy-four. She had taught at Iowa State College for fifty years, and was a charter member of the Mathematical Association.

THE NATIONAL MATHEMATICS MAGAZINE

A campaign is being conducted to increase materially the number of subscribers of the *National Mathematics Magazine*, published at Baton Rouge, Louisiana. This publication, under the editorship of Professor S. T. Sanders, has long received financial support from Louisiana State University. Much of this support is presently to be withdrawn. In working out new financial arrangements an enlarged subscription list is essential. Many mathematicians who do not at present receive the journal will agree that it has an important place in American mathematics and will wish to come to its aid by becoming subscribers. The subscription price is $2.00 per year.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fifth Summer Meeting, Poughkeepsie, N. Y., September 7–9, 1942.
Twenty-seventh Annual Meeting, New York, N. Y., December 30–31, 1942.

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, State College, Pa., Oct. 1942
ILLINOIS
INDIANA, Notre Dame, April 9–10, 1943
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI, fall, 1942
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, Jan. 30, 1943
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 28, 1942
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE, fall, 1942
WISCONSIN, Milwaukee, May 7, 1943

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 450 Ahnaip Street, Menasha, Wisconsin, or 97 Elm Street, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, 97 Elm Street, Oberlin, Ohio, to reach him five weeks before the change becomes effective.

OFFICERS OF THE ASSOCIATION

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THE MIGRATION OF MATHEMATICIANS

ARNOLD DRESDEN, Swarthmore College

In 1785, Joseph Priestley was elected to membership in the American Philosophical Society. In June 1794, at the age of 61, he came to America and settled in Northumberland, Pennsylvania. At its meeting of June 20, 1794, the American Philosophical Society appointed a committee to prepare a congratulatory address to Dr. Priestley. At the same meeting the committee reported a draft, which was adopted "and the officers of the Society with as many of the other members as can conveniently attend are directed to meet at the Hall tomorrow afternoon at 1 o'clock in order to present the same." In the minutes of the meeting of July 18, 1794 it was "reported that a number of the officers and members . . . waited on Dr. Priestley 'and presented the address.'" It is as follows:

To Joseph Priestley, LL.D., &c.

The American Philosophical Society, held at Philadelphia, for Promoting Useful Knowledge, offer you their sincere congratulations on your safe arrival in this country. Associated for the purposes of extending and disseminating those improvements in the sciences and the arts, which most conduce to the substantial happiness of man, the Society felicitate themselves and their country, that your talents and virtues have been transferred to this Republic. Considering you as an illustrious member of this institution, your colleagues anticipate your aid, in zealously promoting the objects which unite them; as a virtuous man possessing eminent and useful acquirements, they contemplate with pleasure, the accession of such worth to the American Commonwealth; and looking forward to your future character of a citizen of this your adopted country, they rejoice in greeting, as such, an enlightened Republican.

In this free and happy country, those unalienable rights, which the Author of Nature committed to man as a sacred deposite, have been secured. Here, we have been enabled, under the favour of Divine Providence, to establish a government of laws and not of men; a government, which secures to its citizens equal rights and equal liberty; and which offers an asylum to the good, to the persecuted, and to the oppressed of other climes.

May you long enjoy every blessing which an elevated and highly cultivated mind, a pure conscience, and a free country are capable of bestowing.

By order of the Society,
DAVID RITTENHOUSE *Pres.*

PHILADA, June 20th 1794

It is the spirit of this congratulatory address which I should like to recapture in this brief report on the mathematicians from abroad who have come to America in consequence of the events of the last ten years. It is meant to be congratulatory both to the newcomers and to us who were here to receive them, having come at birth or later. American mathematicians would like to wait on the good who have sought asylum from persecution and oppression, whose worth has acceded to the American Commonwealth. And this welcome is to be extended not only to those of high eminence in the profession, but to all who intend to make their contribution to American life as professional mathematicians. It is but natural that great honor is accorded to those who have already made contributions of great value and whose association with American mathematics gives more than promise for the future. But it is a truism that

rank and file men are needed as well as captains, colonels and generals. American mathematicians are "associated for the purposes of extending and disseminating" the beauty and value of their science and they "felicitate themselves and their country" that so many "talents and virtues" have become united with them.

Thus no attempt is made to evaluate the significance of the individuals whose names appear; in each group the names are placed in alphabetical order. While it is a regrettable fact that in 1942 it is still necessary to open an asylum to the persecuted and oppressed, we have good reason to congratulate ourselves that the shortage of teachers of mathematics, needed for the instruction of members of the armed forces, may be met, at least in part, by using the talents of those who have come, whatever be the level of eminence to which these talents entitle them.

No one should minimize the difficulties caused by the arrival during a short period of time of a large number of Priestleys. It requires all the resources of administrative statesmanship which we can command to make practically effective the welcome which we want to give them. This problem constitutes a challenge to the strength of our organization, and it is a source of satisfaction that to such a large extent we have succeeded in solving it.*

The record which follows indicates how far we have succeeded. It will be seen that several of the newcomers have not yet been placed; and that many more have not found situations in which they can contribute their gifts and accomplishments most effectively. Perhaps this report will help to bring about more satisfactory placements. There is no doubt that the increase in our resources is an event of the first magnitude in the history of American mathematics. If properly utilized it should give an impetus to the development of American mathematics, whose effects will carry us forward to great heights.†

As with most historical events of importance, there were forebodings of what was to come, for some years before 1933. There are many who can be looked upon as forerunners of the great migration which began in 1933. The names of Bohnenblust, von Kármán, Landé, von Neumann, Radó, Schoen-

* Through the work of the Emergency Committee in Aid of Displaced Foreign Scholars, of the Carl Schurz Memorial Foundation, of the Rockefeller Foundation, of the New School for Social Research, and of the officers of the American Mathematical Society, a great many mathematicians, unfortunately not all, have been placed in permanent positions. The author is indebted to these organizations, also to Dean Richardson of Brown University, to Professor H. Weyl of the Institute for Advanced Study in Princeton, and to many other individuals, for helping him in obtaining the information on which this account is based. In many cases the information is incomplete; it is hoped that it is substantially correct. It is quite likely, particularly for the border fields, that persons have been mentioned who do not consider themselves mathematicians. Such errors of commission are to be regretted, but less so than inadvertent omissions. The author will be grateful for the receipt of corrections and of supplementary data.

† Two articles which have an interesting bearing on the subject of this report have appeared recently in the American Scholar, Vol. 11, No. 3 (Summer, 1942), viz.: B. W. Huebsch, Cross-fertilizations in letters (pp. 304–314), and Herbert Solow, Refugee Scholars in the United States (pp. 374–378).

berg, Seidel, Shohat, Struik, Tamarkin, Uspensky, Wigner, Wintner, Zariski, have long been familiar to American mathematicians. They are men who have become completely absorbed in the scientific life of the country and who have enriched it in a significant way.

1933

The same can be said about those who arrived in 1933, the first year of the period with which we are concerned. They are:

Eugen Altschul after receiving the Ph.D. at Freiburg in 1912, was engaged in business, became lecturer in statistics and economic theory at the University of Frankfurt a/M, and spent a half year at the London School of Economics before coming to this country. He has been a lecturer at the University of Minnesota since his arrival, has undertaken a number of special studies, and is now on leave in order to carry on research for the War Production Board.

Felix Bernstein after many years of work at Göttingen, spent the years from 1933–36 at Columbia University and has since then been professor of Biometry at New York University.

S. Bochner came from Munich to Princeton after a short time spent in England.

Richard Brauer has been at the University of Toronto since 1935 (visiting professor at the University of Wisconsin for half of the year 1940–41, holder of a Guggenheim Fellowship 1941–42), having spent a year at the University of Kentucky and a year at the Institute for Advanced Study,* after coming from Königsberg.

Albert Einstein's career, both here in and Europe, and his contribution to American life are too well known to require mention here.

Fritz Herzog completed his studies for the Ph.D. at Columbia University in 1935 and has been since 1939 an instructor at Cornell University.

Hans Lewy came to this country from Göttingen, spent the years 1933–35 at Brown University and has since then been at the University of California.

Walther Mayer has been at the I.A.S. since his arrival in this country.

Emmy Noether came to Bryn Mawr College from Göttingen and remained there until her earthly career was cut short in 1935.†

Otto Szász came from Frankfurt a/M, spent the years until 1936 at the Massachusetts Institute of Technology, at Brown University and on a lecture tour. Since 1936 he has been at the University of Cincinnati.

Hermann Weyl came from Göttingen to the I.A.S., after having spent the first weeks of his permanent residence in this country as the Cooper Foundation lecturer at Swarthmore College.

* The "Institute" will hereafter be referred to as I.A.S.

† Bryn Mawr College has published the memorial address delivered by Hermann Weyl on April 26, 1935.

Thus we see that the group which came in 1933 has a perfect record for thorough assimilation to American scientific life.

1934

An equally good record can be reported for the party which came in 1934.

Hans Bethe, mathematical physicist, left Tübingen in 1933, spent the next year in England and has been at Cornell Uiversity since his arrival in this country.

Felix Bloch, mathematical physicist in Leipzig until 1932 has been at Stanford University since 1934.

Richard Courant left the directorship of the Mathematical Institute at Göttingen in 1933, spent the year 1933–34 at Cambridge University and has been at New York University from 1934 on.

George Gamow, mathematical physicist at Leningrad until 1933 has been professor at the George Washington University since 1934.

Hans Rademacher, after earlier appointments at Berlin and Hamburg, had been professor at Breslau since 1925. He came to this country as visiting professor at the University of Pennyslvania. His position at that university was later made permanent.

Gabor Szegö came from Königsberg to Washington University in St. Louis, and went from there in 1938 to the chairmanship of the department of mathematics at Stanford University.

Stefan Warschawski had spent some years at Göttingen and a year at Utrecht before coming to the United States. A year at Columbia was followed by two years at Cornell, and two years at Rochester, before he became assistant professor at Washington University, St. Louis, in 1939.

Max Zorn, student of Artin's in Hamburg, held assistantships there and at Halle. A fellowship at Yale, until 1936, was followed by an appointment at the University of California at Los Angeles.

1935

The number of arrivals varied but slightly during the following three years. More and more the I.A.S. became a reception center for refugee mathematicians. The presence on its staff of Einstein and Weyl acted as a magnet for German physicists and mathematicians. Added to this, the great personal interest of Oswald Veblen and other members of the staff, and the active participation in measures necessitated by the events in Europe of the director of the Institute, Abraham Flexner, and his successor in 1939, Frank Aydelotte, made the Institute a natural way station for the victims of fascist and nazist persecution and for others whose emigration to this country was more nearly voluntary. In 1935 came the following persons:

Reinhold Baer was in Halle until 1933, in England for two years, at the I.A.S. during the years 1935–37. This was followed by a year at the University

of North Carolina and finally by a permanent appointment at the University of Illinois in 1938.

Herbert Busemann remained at the University of Göttingen for some years after receiving the doctorate, spent the years from 1936–39 at the I.A.S., the next year partly at Swarthmore College and partly at Johns Hopkins. Since 1940 he has been at the Illinois Institute of Technology.

Max Herzberger, theoretical physicist, who came from Jena, where he had been associated with the Zeiss Co., has become a member of the research staff of the Eastman Kodak Co.

Fritz John received the Ph.D. from Göttingen in 1933 and spent the years 1934–35 at Cambridge University. Upon arrival in this country, he became visiting professor at the University of Kentucky. He has now a permanent appointment at that institution.

Bela A. Lengyel came from the Polytechnic Institute in Budapest. He has held a fellowship at Harvard, was statistician in Worcester, Mass., during 1938 and he has been, since 1939, an instructor at the Rensselaer Polytechnic Institute.

Lothar Nordheim, theoretical physicist, in Paris after leaving Göttingen, was at Purdue University from 1935 to 1937. Since then he has been at Duke University.

O. F. G. Schilling completed his studies at Marburg in 1934, spent two years at the I.A.S. (1935–37) and two years at Johns Hopkins (1937–39). At present he holds a permanent appointment at the University of Chicago.

Leo Szilard, physicist from the Kaiser Wilhelm Institute, came first to New York University and is now at Columbia University.

E. Teller, theoretical physicist formerly at Göttingen, spent a year at London University and is now professor at George Washington University.

1936

The year 1936 was one of comparative quiet; it brought to this country a smaller number of mathematicians than any other year during the nine-year period except 1934.

P. G. Bergmann, theoretical physicist, holds the degree of Ph.D. from Prague (1936) and was an assistant at the I.A.S. for four years. He is now a member of the faculty of Black Mountain College.

W. Z. Birnbaum was on the staff of the Institute for Mathematical and Statistical Research (500 Fifth Avenue, New York), spent a year at New York University and has had an appointment at the University of Washington since 1938.

Olaf Helmer studied in Berlin until 1934, took the degree of Ph.D. in London in 1936 and then spent a year visiting a number of colleges and universities in this country and lecturing. An assistantship at the University of Chicago, 1937–38, was followed by an appointment at the University of Illinois.

W. Hurewicz holds a doctor's degree from Vienna and lectured for many years at the University of Amsterdam. He spent the years 1936–39 at the I.A.S. and has been at the University of North Carolina since 1939.

H. A. Jordan is associate professor of mathematics in the Graduate School of Georgetown University.

Eric Reissner continued at the Massachusetts Institute of Technology, from 1936 on, the studies which were begun in Berlin. In addition to the degree of Eng.D. from Berlin (1936), he holds the degree of Ph.D. from M. I. T. (1938). After having been an assistant and instructor, he is now an assistant professor of mathematics at that institution.

Hertha Sponer, theoretical physicist at Göttingen, spent two years at Oslo and has been at Duke University since 1936.

S. Ulam obtained the degree of D.Sc. at Lwów in 1933 and carried on further studies in Zürich and in Cambridge. After spending part of a year at the I.A.S., he was a member of the Society of Fellows at Harvard University (1936–39) and a lecturer at Harvard during 1939–40. Since 1941 he has been a member of the faculty at the University of Wisconsin.

1937

During 1937, a somewhat larger group arrived in this country including some very well known mathematicians; this group consisted of the following persons:

E. Artin came from Hamburg, was at Notre Dame University until 1938 and went from there to a permanent position at the University of Indiana.

Valentin Bargmann, theoretical physicist, holds the doctorate from Zürich (1936) and has been at the I.A.S. since his arrival in this country. From 1940 on he has been assistant to Einstein.

Kurt Friedrichs came from a professorship at Braunschweig, having previously lectured at Aachen and at Göttingen. He became visiting professor at New York University, where he has been on a permanent appointment since 1940.

Hans Hertz was a student at Hamburg from 1934 to 1937. He continued his studies in astronomy at Yale, where he received the Ph.D. degree in 1941. He has been an assistant at the Yale Observatory from 1940 on and he held a Sterling Research fellowship during the year 1941–42.

Leopold Infeld, theoretical physicist, had been a professor at Lwów until 1936. He spent a year at the I.A.S. and obtained a permanent appointment at the University of Toronto in 1938.

Michael Lotkin supplemented his work at Kiel, where he received a doctor's degree, by courses at New York University and he is now teaching at the Tilden High School in Brooklyn.

Karl Menger came to Notre Dame University after having lectured at the University of Vienna for over 10 years.

Erich Rothe holds the Ph.D. from Berlin (1927), lectured at Breslau until 1935, has been at Wm. Penn College (Oskaloosa, Ia.) since 1937.

Feodor Theilheimer holds the Ph.D. from Berlin (1936) and taught in secondary schools there for a year. He has been engaged in private tutoring in St. Louis, 1937–41, and has taken part during the past year in the advanced course in mechanics at Brown University.

1938

The next year, 1938, brought a very considerable rise of the tide.

F. L. Alt was engaged in actuarial work in Vienna, where he had received the doctor's degree. He is at present statistician with the Institute of Applied Econometrics (500 Fifth Avenue, New York).

Alfred Bloch had been engaged in secondary school work in Germany, after having studied at Strassburg and Munich and having passed the "Staatsexamen" in mathematics and physics in 1922. He has been teaching, chiefly German language, at the University of Utah.

Claude Chevalley holds a doctorate from Paris and has lectured at Strassburg and at Rennes. He was at the I.A.S. during the year 1938–39 and has been since then at Princeton.

Paul Erdös has a Ph.D. from Budapest (1934) and the D.Sc. degree from Manchester (1938). He spent four years in England before coming to this country, was at the I.A.S., 1938–40, and has held a fellowship at the University of Pennsylvania since 1940.

Philipp Frank, theoretical physicist at Prague for many years, is now at Harvard University.

Kurt Gödel had been at the I.A.S., while still on the faculty of the University of Vienna, during the year 1933–34 and part of 1935–36. He is now a member of the I.A.S., and lectured at Notre Dame University during part of the year 1938–39.

Eduard Helly is a Ph.D. from Vienna (1907), taught in the secondary schools and lectured at the University of Vienna. He has held important positions in the actuarial field and as consultant with financial institutions. He taught at the Paterson Junior College, 1940–42 and has now a position at Monmouth Junior College (Long Branch, N.J.).

Mark Kac is a Ph.D. from Lwów (1937), where he was a research assistant, 1935–37. He was also engaged in actuarial work in Europe. He held a fellowship at Johns Hopkins during the year 1938–39 and has been an instructor at Cornell University from 1939 on.

Gerhard Kalisch completed in 1941 at the University of Chicago the studies begun in Europe. A fellowship at Chicago, 1939–41, was followed by a year at the I.A.S. and by an appointment at the University of Kansas.

Jacob Klein, a Ph.D. from Marburg (1922), lectured at the University of Prague, 1934–35, and held a special fellowship in Berlin, 1935–37. He is now teaching at St. John's College.

Eugen Luckacs holds a doctor's degree from Vienna (1930). He was engaged in actuarial work in Vienna and Trieste, and taught in the schools and in the

people's university in Vienna. According to available information, he lives at present in Baltimore.

H. B. Mann, Vienna Ph.D. (1935), has taught in secondary schools and privately in Vienna. After a number of positions and private tutoring, he has now a temporary research associateship at Columbia University.

Karl Meissner, physicist, formerly at Frankfurt a/M, spent two years at Worcester Polytechnic Institute and is now at Purdue University.

Paul Nemenyi holds the D.Sc. degree from Berlin (1922) and has lectured on various engineering subjects at technical schools in Berlin and in Copenhagen. He lectured at a number of places in this country, was engaged in hudraulic research in Iowa and has been an instructor at Colorado State College since 1941.

Jerzy Neymann came from an active career in statistical work in Polish universities to the University of California in Berkeley.

Hans Reissner, holder of degrees from Berlin and Aachen, was professor at the technical universities of Aachen (1906–12) and Berlin (1912–36). He is now research professor of Engineering at the Illinois Institute of Technology.

Helene Reshovsky received the degree of Ph.D. in Vienna in 1930 and was a teacher of mathematics and physics in a "Realgymnasium" in that city. During the years 1939–41 she taught the same subjects at Dana Hall School (Wellesley, Mass.). At present she holds a similar position at the Baldwin School (Bryn Mawr, Pa.).

M. A. Sadowsky has been at the Illinois Institute of Technology since his arrival in this country. A holder of the doctorate from Berlin-Charlottenburg (1927), he taught at this institution from 1926–31, at the University of Minnesota, 1931–33 and at Leningrad and Novocherkassk, 1934–37.

Catharine Stern took the Ph.D. degree at Breslau in 1918, her major interest being mathematics and science. She has been largely concerned with educational questions, bearing on the training of children, both in Europe and here. She is associated with the Child Study Association of America and the American Psychological Association. She is included in this report because her work may have an important influence on the teaching of mathematics in our schools.

Abraham Wald received the degree of Ph.D. in Vienna in 1930, was assistant of Karl Menger, 1930–34, member of the Institute of Business Cycle Research (Vienna) during the years 1934–38. After holding a fellowship of the Cowles Commission during the summer of 1938, he became successively research associate in mathematical statistics (1938–40), lecturer (1940–42), and assistant professor at Columbia University.

1939

The peak of the immigration of mathematicians was reached in 1939, a year after the start of increased persecutions in Europe, the year in which the war broke out. Many of those who came in 1939 had of course made preparations long in advance, so that this high tide reflects events of the preceding years

probably to a larger extent than those of 1939. The reader will recognize in the list which follows several names which were well known among American mathematicians for a long time, and which have acquired added importance for our scientific development.

Alfred Basch obtained the doctorate from the Vienna Technical University and has combined teaching at technical schools in Dresden, Prague and Vienna with engineering practice. He taught at Holy Cross College (Worcester, Mass.) during the years 1939–42. At present he has a position in the Paterson Junior College and, during the summer of 1942, in the Harvard Summer School of Engineering.

Gustav Bergmann, theoretical physicist (Ph.D., Vienna, 1928) and a doctor of jurisprudence (Vienna, 1935), was a lecturer and assistant in mathematics at the University of Vienna, and practised law in Vienna during the years 1935–38. He has been at the University of Iowa since 1939.

Stefan Bergmann has a doctor's degree from Berlin (1922), has taught at Berlin and at the Technological Institute of Tomsk. He lectured at the Massachusetts Institute of Technology, 1939–40, and is at present visiting lecturer in applied mathematics at Brown University.

Alfred T. Brauer was assistant to I. Schur at the University of Berlin, 1926–35 and took his degree there in 1928. He has been at the I.A.S. during the years 1939 to 1942, contributing a great deal to the formation of the library of the Institute's school of mathematics. He has now a temporary appointment at the University of North Carolina.

Walther Bruns was known in the German educational world as Walther Jacobsthal. He occupied important positions as director and administrator of schools in Berlin and as a lecturer on mathematical education. He is at present living in Syracuse, N. Y.

Max Chameides was a teacher in the schools and in the "Volksuniversitat" in Vienna.

Samuel Eilenberg received the Ph.D. degree at Warsaw in 1936, spent the next three years in Paris and Cambridge, and has been at the University of Michigan since 1939, obtaining appointment to the regular faculty in 1941.

Willy Feller, Ph.D. Göttingen, was at the University of Kiel, 1928–33, and at Stockholm during the years 1934 to 1939. Since his arrival in this country, he has been at Brown University.

Guido Fubini, professor at the University of Turin for over 30 years, has been at the I.A.S. since 1939.

Hilda Geiringer holds a Ph.D. from Vienna (1918) and has had wide experience in probability, statistics and various other fields of applied mathematics, at the Universities of Berlin, Brussels and Istanbul. She has been on a temporary appointment at Bryn Mawr College, 1939–42, and has filled a part-time position at Swarthmore, 1941–42. During the summer of 1942, she is on the staff of the School of Mechanics at Brown University.

Michael Golomb received the Ph.D. in Berlin in 1934 and spent the next five

years in Yugoslavia. Since coming to this country he has been at Cornell University as a research assistant and as a part-time instructor. In September 1942 he goes to Purdue University as instructor.

Ernst Hellinger, professor at Frankfurt a/M from 1914 to 1938, has been a visiting lecturer at Northwestern University since 1939.

Walter Jacoby studied at the University of Berlin and was a teacher in the high schools of that city.

Georg Jaffé, professor of theoretical physics at Leipzig, 1905–26, and at Giessen, 1926–33, is at present on the faculty of Louisiana State University.

Arthur Korn, professor at Berlin since 1913 and active in various fields of applied mathematics, is now at Stevens Institute of Technology.

Friedrich Kottler, professor of theoretical physics at Vienna, 1919–38, was a member of the staff of the Eastman Kodak Company, 1939–40.

Gustav Kürti, Ph.D. in physics from Vienna, was at the University of Rochester and spent the year 1941–42 at the Massachusetts Institute of Technology.

Gustav Land (formerly Gustav Deutschland), Berlin Ph.D. in Astronomy (1908), was active in astronomy and in meteorology at the observatories in Berlin, Königsberg and Leipzig until 1919. He was engaged in statistical work until 1938. After a short period at the Nautical Almanac Office in London, he became Research Associate at the Sproul Observatory (Swarthmore), 1939–41. Since then he has been research assistant and instructor at the Yale Observatory.

Gerhard Lewin, physicist with a Ph.D. from Prague (1930), was engaged in industry and business in Berlin, 1930–33, and in Prague, 1933–37. Since 1939, he has been connected with the R.C.A. laboratory in Harrison, N. J.

Karl Loewner received the Ph.D. in Prague (1917) and lectured in Berlin (1922–28), in Cologne (1928–30) and in Prague (1930–38). Since 1938 he has been visiting professor at the University of Louisville.

H. T. Ludloff was a privat-dozent at the University of Breslau. He spent some time at Thiel College (Greenville, Pa.) and is now in the department of physics at City College, N. Y.

Richard von Mises was for many years (1920–33) professor of applied mathematics at Berlin, later at Istanbul (1934–39). He is now visiting professor at the Graduate School of Engineering of Harvard University, lecturer at the Massachusetts Institute of Technology, and during 1941–42, visiting professor at Brown University.

Otto Neugebauer, privat-dozent (1926–32) and later professor at Göttingen (1932–34), spent the years 1934–39 at the University of Copenhagen, continuing there the editorship of the Zentralblatt für Mathematik, which he had started in 1931. Since 1939, he has been professor at Brown University and editor of Mathematical Reviews.

Gottfried Noether, nephew of Emmy Noether, carried on in this country the studies begun in Breslau and continued in Tomsk, at Ohio State Univer-

sity (1939–40) and at Illinois (1940–41). He is now in the U. S. Army.

I. Opatowski had spent the years 1932–35 at the University of Turin and was later employed by the Fiat Co. (1936–38). He was an instructor at the University of Minnesota (1939–42) and is now at the Armour Research Foundation, Illinois Institute of Technology.

Anita Riess studied at Heidelberg, Leipzig and Marburg, and received the Ph.D. from the latter university in 1921. She has been active as an educational administrator and as a teacher in secondary schools and normal schools in Leipzig and in Hamburg. She is at present a visiting lecturer at Wellesley College.

Peter Scherk received the Ph.D. degree at Göttingen in 1935. The year 1939–40 was spent at the Taft school. The next year he held a fellowship at Yale and during 1941–42 he was at the University of Indiana.

Olaf Schmidt was a student at the University of Copenhagen and a teacher in the secondary schools. He has been at Brown University since his arrival in this country as an instructor and as a research assistant to Professor Neugebauer.

Alfred Seckel, active as teacher and as educational administrator (Staatsexamen in philosophy, mathematics and physics) in Magdeburg and in Freiburg, had a position at Canisius College (Buffalo) from 1939 till 1941. He is now at the Chestnut Hill Academy (Philadelphia).

Wolfgang Sternberg was professor at Heidelberg during the years 1920–27 and at Breslau during the years 1927–33. At present he is living in New York City.

Alfred Tarski, professor at Warsaw, 1925–39, has been a visiting lecturer at Harvard University (1939–41) and at New York City College. He held a Guggenheim Fellowship during the year 1941–42 and has been called to fill a temporary vacancy at the University of California at Berkeley for 1942–43.

Otto Treitel was a secondary school teacher in Mannheim and in Heidelberg. He has been associated with Milton College.

Wolfgang Wasow was a student at Göttingen (1930–33) and at Paris (1933–34). He taught in German schools in Italy during the years 1935–38. A part of the year 1939 was spent at the Choate School (Wallingford, Conn.), the next year at Goddard Junior College (Plainfield, Vt.). During the years 1940–42 he has held a fellowship at New York University and since 1941 he has taught at the Connecticut College for Women.

Alexander Wundheiler received the degree of Ph.D. at the University of Warsaw in 1932 and lectured there from 1927–39. He held a special lectureship at the Massachusetts Institute of Technology during the second semester of 1940 and taught mathematics at Tufts College during the following year. Since February 1942 he has been an instructor in the physics department at New York City College.

Antoni Zygmund, professor at Warsaw (1922–30) and at Wilno (1930–39), has been on the faculty of Mt. Holyoke College since 1939. He has also been a lecturer at the Massachusetts Institute of Technology and a visiting professor at the University of Michigan.

1940

The next year, 1940, showed a considerable decrease, due to a large extent very probably to the increasing difficulties of communication and of transportation. The arrivals during 1940 include the following:

Felix Adler, theoretical physicist with a Ph.D. from Zürich (1938), spent the years 1938–40 at the Collège de France. He was at the I.A.S. during the year 1941–42 and has a temporary appointment at the University of Wisconsin for 1942–43.

Hugo Basch, an engineer holding diplomas from the Vienna Technical University, was engaged in important civil engineering enterprises in Austria. He has been associated with engineering firms in Philadelphia since 1940 and lives in Moorestown, N. J.

Lipman Bers received the Ph.D. from Prague in 1938 and spent the years 1938–40 in Paris. He took part in the work of the advanced mechanics courses at Brown University in 1941.

Paul Boschan, trained in electrical engineering and in actuarial work, received the degree of Ph.D. in Vienna in 1934 and was engaged in research work with the Foundation for Visual Education in Vienna and also in actuarial work. Since shortly after his arrival in this country he has been associated with the Institute of Applied Econometrics in New York.

Hans Fried, Ph.D. Vienna (1924), Staatsexamen (1927), was a teacher in Vienna from 1927 to 1938. He spent the year 1939–40 in England. After some months in New York, a half year at the Haverford Workshop and the summer of 1941 at Brown University, he became a research assistant at the Sproul Observatory in the autumn of 1941.

E. J. Gumbel was professor at Heidelberg during the years 1923–33 and at the University of Lyons from 1933 to 1940. He is now living in New York City and is associated with the New School for Social Research.

T. Koopmans, trained as a mathematical physicist and as a mathematical economist, received the doctorate from Leiden in 1936. He lectured at the Commercial University in Rotterdam, and was research associate at the Rotterdam Institute for Economics and economic specialist at the League of Nations. Since 1940, he has lectured at New York University, he has been a research associate at Princeton University and economist with the Penn Mutual Life Insurance Co. (Philadelphia). At present he holds a similar position with the Combined Shipping Adjustment Board in Washington, D. C.

Wolfgang Pauli, professor of theoretical physics in Zürich, 1928–40, has been at the I.A.S. from 1940 on.

Bruno Pontecorvo, physicist, formerly at the University of Rome, has been associated with the Well Surveys Co., at Tulsa, Okla.

Georg Pólya was professor at Zürich from 1928 to 1940. He has spent the two years, 1940–42, at Brown University, with a lectureship at Smith College during a part of the last of these years. He has now been called to a permanent appointment at Stanford University.

Arthur Rosenthal was professor at Munich (1912–22) and at Heidelberg (1922–35) and spent the year 1939–40 in Holland. After a year at the University of Michigan, he has now been appointed to a visiting professorship at the University of New Mexico.

C. L. Siegel was professor at Frankfurt a/M, 1922–37, and at Göttingen, 1938–40. Since his arrival in this country, he has been at the I.A.S.

Heinz Simon, who was a teacher of physics and mathematics in Frankfurt a/M, 1925–39, is now at the Southern Union College (Wadley, Ala.).

Andrew Vasonyi received the degree of Ph.D. at Budapest in 1938 and spent the years 1938–40 in Paris. He was at the Haverford Workshop, 1940–41, at the summer school for advanced mechanics at Brown University, 1941 and 1942, and pursued his studies at Harvard University during the year 1941–42.

Alexander Weinstein has lectured at the Universities of Zürich, Hamburg and Breslau (1928–33). The years 1933–40 were spent in France. After some time in New York City, he received a temporary appointment at the University of Toronto in 1941.

Frantisek Wolf, after having received a doctor's degree at Brno (1928), was a teacher in the schools and a privat-dozent at the Charles University in Prague. The year 1937 was spent in Cambridge and the years 1938–40 in Sweden. A temporary appointment at Macalester College (St. Paul, Minn.) was followed in 1942 by an instructorship at the University of California at Berkeley.

1941

During the year 1941, a number of long delayed immigrations were accomplished, bringing to friendly surroundings some who had been driven from their homes a year or more before. The group included the following:

Ferdinand Beer received the degree of Ph.D. from the University of Geneva in 1937 and had taught in the schools of Paris. He has a position at the Goddard Junior College, in Plainfield, Vt.

L. N. Brillouin, D. Sc., University of Paris, 1920, professor of theoretical physics at the Sorbonne and at the Collège de France, 1928–40, and general director of broadcasting in France, 1939–41, was visiting professor at the University of Wisconsin during 1941–42 (he had been guest lecturer there during the second semester of 1927–28). For the year 1942–43, he has been appointed to the faculty of the school for advanced mechanics at Brown University.

L. Corbeillier, student of the Ecole Polytechnique in Paris (1911–13), holder of a Ph.D. from Paris (1926), as well as of an A.M. in Philosophy, was engaged in engineering work in France and in work at the Ecole Polytechnique. He is now a lecturer on electronics in the Officers Training Course at Harvard.

Max Dehn was professor at Kiel (1911–13), at Breslau (1913–21) and at Frankfurt a/M (1921–35). He spent the year 1939–40 as professor at the Norwegian Technical Institute in Trondheim and came to the United States by way of Siberia. After a professorship of mathematics and philosophy at the University of Idaho, in Pocatello, he was appointed visiting lecturer at Illinois Institute of Technology.

Jacques Hadamard, for many years professor at the Collège de France, is now living in New York City. He lectures occasionally at Columbia University and at neighboring institutions.

Herbert Jehle, D.Eng. from Berlin (1933), collaborator on the *Fortschritte* until 1936, spent the year 1937–38 at Southampton University College, and the years 1938–40 at Brussels. He is now in the physics department of Harvard University.

S. Mandelbrojt, D.Sc. (Paris, 1923), has lectured at the universities of Lille (1928–29) and Clermont-Ferrand (1930–38) and became professor at the Collège de France in 1938. He is now visiting professor at the Rice Institute, where he had also spent the year 1926–27.

Helen Polanyi was a student at the Zürich Technical University in 1915–17. She was at the University of Vienna in 1933–35. At present she is on the faculty of Bennington College.

Willy Prager, D.Eng. (Darmstadt, 1926) has been professor at Darmstadt, Göttingen, Karlsruhe and Istanbul. He is now professor of applied mathematics at Brown University.

Fritz Reiche was professor of theoretical physics in Berlin (1913–21), in Breslau (1921–33) and in Prague (1933–35). He is now living in New York City, associated with the New School for Social Research.

Raphael Salem had training in Paris as an engineer and as a mathematician, receiving the degree of engineer in 1921 and the degree of Sc.D. in 1940. For several years he was engaged in banking. At present he is a lecturer in mathematics at the Massachusetts Institute of Technology.

Hans Samelson, mathematical economist with the degree of D.Sc. from Zürich, is now at the I.A.S.

André Weil, D.Sc. (Paris, 1929) has lectured at Aligarth University (Delhi), 1930–32 and at Strassburg, 1933–40. He spent the year 1941–42 at Haverford College and has now been appointed at Lehigh University.

1942

The list of names is brought to a conclusion by recording the arrival during the first half of 1942 of the following:

E. Kogbetliantz, D.Sc. (Paris, 1923), who lectured in Paris (1921–27), at the University of Teheran (1933–38) and again in Paris (1938–40). He is now joining the faculty of Lehigh University.

Sylvia Nowinska was trained at the University of Lausanne and was for several years assistant at the Paris-Meudon Observatory. She is at present living in New York City.

It has not been possible to give more than very brief indications of the scientific careers, both in this country and abroad, of the mathematicians who have come to this country during the years since 1933. Still less has it been possible to speak of their scientific work, of their plans and hopes, or to have

them record their impressions of the new environment, their prospects and their difficulties. Because this has not been feasible, let me presume to speak for them, even though my arrival in this country falls in a very much earlier period; not however in my own words, but rather in those of Priestley, who replied in the following manner to the congratulatory address presented to him by the American Philosophical Society:

To the Members of the American Philosophical Society of Philadelphia.

GENTLEMEN:—It is with peculiar satisfaction that I receive the congratulations of my brethren of the philosophical Society in this city, on my arrival in this country. It is, in great part, for the sake of pursuing our common studies without molestation, tho', for the present, you will allow, with far less advantage, that I left my native country, and have come to America; and a Society of philosophers, who will have no objections to a person on account of his political or religious sentiments, will be as grateful as it will be new to me.

My past conduct, I hope, will show, that you may depend upon my *zeal* in promoting the valuable objects of your institution, but you must not flatter yourselves, or me, with supposing, that, at any time of life, and with the inconvenience attending a new, and uncertain settlement, I can be of much service to it.

I am confident, however, from what I have already seen of the spirit of the people of this country, that it will soon appear that republican governments, in which every obstruction is removed to the exertions of all kinds of talents, will be far more favourable to Science, and the arts, than any monarchical government has ever been. The patronage to be met with there, is ever capricious, and as often employed to bear down merit as to promote it; having for its real object not science, nor anything useful to mankind, but the mere reputation of the patron, who is seldom any judge of science. Whereas a *Republic* which neither flatters, nor is to be flattered, will not fail in due time to distinguish true merit, and to give every encouragement that is proper to be given in the case. Besides, by opening, as you generously do "an asylum to the persecuted and oppressed of all climes," you will in addition to your own native stock, soon receive a large accession of every kind of merit, philosophical not excepted, whereby you will do yourselves great honour, and secure the most permanent advantage to the community.

J. PRIESTLEY

PHILADA, June 21st 1794.

Perhaps it is in the great scheme of things, that a seed, produced by the distressing events which have occurred during the last ten years, planted in the fertile soil of American science, will grow into a powerful tree, to add beauty and strength to the future life of mankind.

THE APRIL MEETING OF THE OHIO SECTION

The twenty-seventh annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Thursday, April 2, 1942, with an afternoon session, a dinner, and an evening session. Professor Louis Brand, chairman of the Section, presided.

Fifty-six persons registered attendance, including the following forty-five members of the Association: G. E. Albert, W. E. Anderson, F. R. Bamforth, Grace M. Bareis, I. A. Barnett, H. M. Beatty, Henry Blumberg, Louis Brand, C. T. Bumer, V. B. Caris, E. H. Clarke, Rufus Crane, Wayne Dancer, O. L. Dustheimer, D. H. Erkiletian, Jr., M. P. Fobes, T. M. Focke, L. R. Ford, R. C. Hildner, Margaret E. Jones, H. W. Kuhn, Lincoln LaPaz, H. W. Linscheid, E. S. Manson, C. G. Maple, Florentina Mathias, Emma J. Olson, C. R. Pettis, Jesse Pierce, H. S. Pollard, Tibor Radó, S. E. Rasor, Maxwell Reade, R. B. Rice, N. S. Risley, K. C. Schraut, H. E. Stelson, C. W. Topp, W. R. Van Voorhis, R. W. Wagner, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, P. M. Young.

The Section was fortunate in having Professor L. R. Ford of Illinois Institute of Technology as the guest speaker. The following officers were elected for the coming year: Chairman, C. T. Bumer, Kenyon College; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, G. E. Albert, Ohio State University; Member of Program Committee, A. C. Ladner, Denison University. It is expected that the next meeting will be held on Thursday, April 1, 1943, at the Ohio State University.

The following six papers were presented:

1. "Non-metric differential invariants" by Professor Louis Brand, University of Cincinnati.

2. "Vector inequalities" by Professor Tibor Radó, Ohio State University.

3. "Mathematical methods in non-linear problems" by Dr. C. A. Ludeke, University of Cincinnati, introduced by the chairman.

4. "Application of tensor analysis to the investigation of geodesic lines on an ellipsoid of revolution" by Professor E. S. Manson, Ohio State University.

5. "The error in using simple interpolation in tables of finance" by Professor H. E. Stelson, Kent State University.

6. "A million ways to solve equations" by Professor L. R. Ford, Illinois Institute of Technology.

In addition to these papers, a portion of the afternoon session was devoted to a symposium in which the members compared experiences as to the effect of the present emergency upon nature and content of courses offered.

Abstracts of some of the papers follow:

1. A relation between two tensor invariants in n-space, generalization of the divergence and curl of three-dimensional vector analysis, and independent of the metric fundamental tensor g_{ij} and the affine connection Γ_{ij}^k, was established by Professor Brand by setting up a one-to-one correspondence between covariant tensors of order $m(0 \leqq m \leqq n)$ and contravariant tensors of order $n-m$,

their duals. It was proposed to define curl P as an absolute, covariant, alternating tensor of order $n-1$, if P is an absolute covariant tensor of order $n-2$; while rot P, defined as the dual of curl P, is a contravariant vector density.

2. An inequality such as the inequality of Schwartz is easily proved while others are more complicated in appearance and proof. For many of these latter, simple proofs can be given based on theorems concerning the area of a triangle in terms of various data. One must first recognize a given inequality as a special case of a more general inequality involving vectors. Professor Radó presented several examples and suggested that such applications of theorems be used as stimulating exercises even though the inequalities derived be beyond the scope of instruction.

3. Dr. Ludeke presented four methods used in solving certain types of non-linear differential equations. The first, known as the method of isoclines, and applicable to non-linear equations of the form $dv/dx=\beta(x)\, v^{n-1}+k(x)/v=f(x, v)$, where $v=dx/dt$ and $f(x, v)$ is single valued and continuous: this method obtains a first integral between x and v by plotting the family of curves $f(x, v)=$ constant, and attaching to them the corresponding slope of the integral curve. The integral between x and t is determined by a second graphical integration. A second method, applicable to the preceding equation with $\beta(x)=0$, is used to find only the frequency of the solution by direct integration over a quarter cycle in steps small enough to ensure approximate linearity of $k(x)$. A third method, used in solving the equation $md^2x/dt^2+\beta_n(dx/dt)^n+kx=P\sin\alpha t$, replaces the term $\pm\beta_n(dx/dt)^n$ by an equivalent term $\beta_1'(dx/dt)$; the equivalence being determined by the dissipative work of each per cycle. The equation is then linear and the solution is well known. A fourth method, applicable to equations of the form $md^2x/dt^2+k(x)=P\sin\alpha t$, assumes the solution in the form $x=\sum_{n=1,3,5,\cdots}^{\infty} a_n \sin n\alpha t$, and $k(x)$ in the form $\sum_{n=1,3,5,\cdots}^{\infty} A_n \sin n\alpha t$; and then proceeds to determine the a_i so that when the A_i are calculated from them the series $\sum_{n=1,3,5\cdots}^{\infty} A_n \sin n\alpha t$ gives the correct form $k(x)$.

4. Professor Manson discussed the application of the equations of a geodesic obtained by tensor analysis,

$$d^2t_\alpha/ds^2+\{\mu\nu, \alpha\}\frac{dx_\mu}{ds}\frac{dt_\nu}{ds}=0$$

to the two dimensional case of the geodesic on the surface of a spheroid of revolution. Equations were developed showing the relation between the longitude θ, and the latitude ϕ, and also between the distance s, and ϕ. Professor Manson showed how these equations may be used in computing the distance along a geodesic between two points of known latitude and longitude.

5. Formulas for determining the error in using simple interpolation in tables of finance were derived by Professor Stelson. The cases considered were interpolation for the time, rate, and amount. Algebraic expressions were derived showing the maximum error and the average error in each case. Approximate rules for the accuracy of interpolation were presented.

RUFUS CRANE, *Secretary*

THE ANNUAL MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the annual convention of the Oklahoma Education Association in Oklahoma City on Friday morning, February 13, 1942. Professor C. E. Springer, chairman of the Section, presided.

Forty-seven representatives of high schools and colleges attended the meeting, including the following sixteen members of the Association: E. F. Allen, Joseph Barnett, Jr., J. C. Brixey, N. A. Court, A. H. Diamond, H. L. Hall, O. H. Hamilton, J. O. Hassler, E. E. Heimann, J. E. LaFon, Dora McFarland, W. C. Randels, S. W. Reaves, W. T. Short, C. E. Springer, B. S. Whitney.

At the business session the following officers were elected: Chairman, O. H. Hamilton, Oklahoma A. and M. College; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following six papers:

1. "The bisectral center of four spheres" by Professor N. A. Court, University of Oklahoma.
2. "Equi-continuous collections of continuous transformations" by Professor O. H. Hamilton, Oklahoma A. and M. College.
3. "A comparison of the effectiveness of mathematics teaching in different types of Oklahoma colleges" by Professor J. O. Hassler.
4. "Conformal mapping of contours in the study of potential flow" by Professor A. H. Diamond, Oklahoma A. and M. College.
5. "On new methods in differential equations with applications to the structural analysis of airplanes" by Professor Stefan Bergmann, Brown University, introduced by the Secretary.
6. "Quadratic forms which represent all integers, except those in certain geometric progressions" by Professor S. B. Townes, University of Oklahoma, introduced by Professor Springer.

Abstracts of the papers follow.

1. Professor Court defined the "sect" v^2 of a point V for a sphere (A, a) as the square of the distance VA of the point V from the center A of (A, a) increased by the square of the radius a of (A, a). The sphere (V, v) having V for center and v for radius bisects the sphere (A, a). He pointed out the degree of parallelism between "sect" and "power." He showed that, given four spheres (A), (B), (C), (D), there is a sphere (V) bisecting those spheres, and that the sphere (V), the orthogonal sphere (U) of the given spheres, and the sphere $(O)=ABCD$ determined by the centers of the given spheres, form a coaxal pencil. The sphere (V) also bisects the sphere having for center the centroid G of the tetrahedron $(T)=ABCD$ and for the square of its radius one-fourth of the sum of the squares of the radii of the given spheres increased by one-sixteenth of the sum of the squares of the edges of (T).

2. Professor Hamilton discussed the definitions of a continuous transformation on a set of points, homeomorphism, and an equi-continuous collection of

homeomorphisms. He gave a proof of the theorem: If T is a homeomorphism of a compact continuum N, lying in a Euclidean space of n dimensions, into itself; and if the integral powers of T form an equicontinuous collection of homeomorphisms, then there can be defined on N a metric function, $D(A, B)$, which is left invariant by the homeomorphism T.

3. Professor Hassler reported the results of an investigation of the grades in mathematics made at the University of Oklahoma by students transferring there from other colleges in the state. He classified the students according to the nature of the colleges from which they came, and made a comparison of the effectiveness of the teaching of mathematics in the different types of colleges.

4. The study of potential flow about biplane wing sections in a two-dimensional stream leads to the problem of conformal mapping of a doubly connected domain into a ring. The plane in which the doubly connected domain lies was first mapped by Professor Diamond into the interior of a rectangle by means of a function defined by an elliptical integral of the first kind. Finally he mapped the rectangle into a ring by the exponential function.

5. Professor Bergmann discussed the mathematics involved in the structural analysis of airplanes.

6. Professor Townes discussed forms $ax^2+by^2+cz^2+dw^2$ which represent all integers except those in certain geometrical progressions. For example, $x^2+y^2+7z^2+7w^2$ was found to represent all integers except $7^k\cdot 3$ and $7^k\cdot 6$, $k\geqq 0$.

J. C. BRIXEY, *Secretary*

THE NINETEENTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The nineteenth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the Heidelberg Hotel in Jackson, Mississippi, on Friday and Saturday, March 6–7, 1942.

The attendance was about sixty-five, including the following twenty-eight members of the Association: P. H. Anderson, T. A. Bickerstaff, Leora Blair, W. H. Bradford, H. E. Buchanan, R. L. Coker, W. E. Cox, D. S. Dearman, W. L. Duren, Jr., Virginia I. Felder, F. C. Gentry, H. S. Kaltenborn, H. T. Karnes, C. G. Killen, A. C. Maddox, Dorothy McCoy, B. E. Mitchell, I. C. Nichols, W. V. Parker, H. L. Quarles, S. T. Sanders, Jr., H. F. Schroeder, C. D. Smith, V. B. Temple, J. F. Thomson, B. A. Tucker, J. A. Ward, F. L. Wren.

Sessions were held Friday afternoon and evening and Saturday morning. Professor B. A. Tucker, chairman of the Section, presided at all sessions. The annual dinner was held on Friday evening. The visiting speaker, Professor F. L. Wren of George Peabody College for Teachers, delivered an address at the dinner and also spoke at the Saturday morning session.

At the business session on Saturday morning it was decided to hold the meeting at Delta State Teachers College, Cleveland, Mississippi, in 1944. The fol-

lowing officers were elected for 1942–43: Chairman, J. A. Ward, Delta State Teachers College; Vice-Chairman for Louisiana, S. T. Sanders, Jr., Southwestern Louisiana Institute; Vice-Chairman for Mississippi, B. O. Van Hook, Millsaps College; Secretary-Treasurer, F. C. Gentry, Louisiana Polytechnic Institute.

The following ten papers were presented:

1. "A solid of revolution" by Professor J. A. Ward, Delta State Teachers College.

2. "Poristic polygons" by Professor H. E. Buchanan, Tulane University.

3. "Fire insurance, a one-, a three-, or a five-year policy?" by Professor I. C. Nichols, Louisiana State University.

4. "The problem of Apollonius" by Mrs. Alta H. Samuels, Hinds Junior College, introduced by the Secretary.

5. "The twelve squares ascribed to a triangle" by Professor B. E. Mitchell, Millsaps College.

6. "A note on Heaviside operators" by Professor J. F. Thomson, Tulane University.

7. "Distributions in stratified sampling" by Dr. P. H. Anderson, Louisiana State University.

8. "Boundary values for probabilities in problems of two variables" by Professor C. D. Smith, Mississippi State College.

9. "The next step forward" by Professor F. L. Wren, George Peabody College for Teachers.

10. "A contraction method for determinant expansion" by Professor F. L. Wren, George Peabody College for Teachers.

Abstracts of some of these papers follow:

1. Professor Ward showed how to obtain the median curve of the solid of revolution obtained by rotating a rectangular parallelopiped about a diagonal. He showed that there are eight different types of median curves, depending on the ratios of the edges, and displayed graphs of several of them.

2. After reproducing a part of Jacobi's paper in order to get an intelligible start Professor Buchanan proved the following theorems: (1) The diagonals of the quadrilateral formed by the points of tangency of a poristic polygon are mutually perpendicular, and intersect at a fixed point on the line of centers of the two circles; (2) The external diagonal of a poristic quadrilateral is a fixed line perpendicular to the line of centers; (3) Every poristic quadrilateral determines another poristic quadrilateral which is inscribed in the smaller of the original circles and circumscribed about a third circle whose center can be found by a geometric construction.

3. Using the assumptions first, that insurance is paid in advance; second, that discounts of half a year and of one whole year are given on three-year and five-year policies respectively; and, third, that the fire rate itself remains unchanged for the duration of the periods under discussion, Professor Nichols showed that (1) a one-year policy is consistently higher than a three-year policy.

(2) A three-year policy is cheaper than a five-year policy for a rate of interest of 4% or higher, the difference becoming greater as the rate increases. (3) For a rate of interest of $3\frac{1}{2}$% or less the five-year policy is cheaper than the three-year policy, the rate at which these two policies are equal being slightly under 4%. Therefore, in general practice, the three-year policy should be written.

4. Mrs. Samuels showed that nine of the ten cases of the problem of Apollonius may be reduced by inversion to three cases, namely, *PPL*, *PCC*, and *CCC*. She then solved these three cases by inversion, emphasizing their simplicity. In each solution, through the mechanics of inversion, three points through which the required circle must pass are easily located, thereby reducing each problem to *PPP*, the first and simplest of the ten cases of the problem of Apollonius.

7. Dr. Anderson compared the random sample distribution of a statistic with the stratified sample distribution of the same statistic. The statistics considered were the mean, the standard deviation, and the student's ratio. The samples were taken from a rectangular, a normal, a skewed, and a general population. This paper appeared in the *Annals of Mathematical Statistics*, March 1942.

8. Beginning with the fundamental concept of Bernoulli, Professor Smith compared five types of probability. Consider the joint occurrence of two characteristics of an item $U_{i,j}$. If the characteristics fall in column (i) and row (j) of the xy-plane and $M_n(j, i)$ is the moment about the y-axis, $P_j \leqq \mu_n(j, i)/\lambda^n$, where P_j is the probability that the item $U_{i,j}$ falls farther from the x-axis than a distance λ. A similar bound with respect to the y-axis gives a rectangle within which $U_{i,j}$ does not fall. In case of correlated characteristics a boundary parallelogram was found. The bounds were reduced by cases where the frequency does not increase with λ. For frequency surfaces, a boundary circle and a boundary ellipse were given.

9. Professor Wren raised and discussed the following questions: What is the next step forward in the mathematics program in the elementary and secondary schools of the United States? The present emergency has given a great deal of emphasis to the importance of mathematics. It has also thrown the spotlight on some of the weaknesses of the mathematics program of the past and present. As we project our thinking into the forthcoming peace era, what are the implications as to curriculum, methods of teaching, and preparation of teachers? What should the Association, Council, and Society be doing now in the planning of this peace-time program for mathematics?

10. Professor Wren proved the theorem which establishes the contraction method for the evaluation of determinants. He then applied this technique to find the quotient of two determinants, to solve systems of simultaneous linear equations, to find the common root of two equations of the form $a_0X^3+a_1X^2+a_2X+a_3=0$ and $b_0X^2+b_1X+b_2=0$, and to find the moments of the point binomial $(p+q)^n$.

W. V. PARKER, *Secretary*

THE TWENTY-EIGHTH ANNUAL MEETING OF THE KANSAS SECTION

The twenty-eighth meeting of the Kansas Section of the Mathematical Association of America was held at Fort Hays Kansas State College, Hays, on Friday and Saturday, March 27–28, 1942, in conjunction with the meetings of the Kansas Academy of Science and the Kansas Association of Teachers of Mathematics. There were three sessions at all of which Professor C. F. Lewis, vice-chairman of the Section, presided. Professor C. V. Bertsch, elected chairman a year ago, is now connected with American University, Washington, D. C.

There were forty in attendance, including the following twenty-five members of the Association: Wealthy Babcock, Florence L. Black, B. H. Buikstra, E. E. Colyer, R. D. Daugherty, Lucy T. Dougherty, Paul Eberhart, F. D. Faulkner, W. H. Garrett, Edison Greer, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, Anna Marm, P. S. Pretz, G. B. Price, C. B. Read, J. A. G. Shirk, D. T. Sigley, G. W. Smith, W. T. Stratton, Gilbert Ulmer, E. B. Wedel, A. E. White.

The officers elected for the coming year are: Chairman, C. F. Lewis, Kansas State College, Manhattan; Vice-Chairman, Paul Eberhart, Washburn Municipal University of Topeka; Secretary, Anna Marm, Bethany College. The decision on the time and place for the next meeting was left to the executive committee.

The Committee on the Third Placement Test, given at the beginning of the 1941–42 school year to Freshmen in most of the colleges and junior colleges of the state, reported through the chairman, Professor Gilbert Ulmer, and Professor O. J. Peterson. There was general discussion on the test in which many of those present took part. The Committee was asked to continue the Test work in 1942–43.

Professor Emma Hyde gave a tribute to the memory of Professor U. G. Mitchell, expressing the feeling in all our hearts, as we met for the first time without his presence—thankfulness for the years during which he has been our leader and our inspiration, and sadness for the parting now.

The following eight papers and reports were presented:

1. "Continued fractions of quaternions" by E. G. Swafford, Fort Hays, Kansas State College, introduced by Professor Colyer.

2. "A system of linear differential equations with a regular singular point" by F. D. Faulkner, Kansas State College.

3. "Some properties of Sine Z as the inverse of an integral" by B. H. Buikstra, Kansas State College.

4. "Some formulas in analytic geometry" by Professor G. B. Price, University of Kansas.

5. "Mathematics used by the Armed Forces" by Professor D. T. Sigley, Kansas State College.

6. "Some circles related to the triangle" by Professor G. W. Smith, University of Kansas.

7. "Is mathematics an exact science?" by Professor C. B. Read, University of Wichita.

8. "Report of the committee on the Placement Test" by Dean Gilbert Ulmer and Professor O. J. Peterson.

Abstracts of most of the papers follow.

1. Mr. Swafford showed that any right-handed quotient of two quaternions of the Hurwitz integral domain H can be represented by a finite regular right-handed continued fraction of quaternions of H by using an euclidean algorithm to determine a greatest common right divisor of the two quaternions.

2. Mr. Faulkner discussed in detail the solutions of the system of differential equations,

$$t(dX_1/dt) = \theta_{11}(t)X_1 + \theta_{12}(t)X_2,$$
$$t(dX_2/dt) = \theta_{21}(t)X_1 + \theta_{22}(t)X_2,$$

where the θ's are functions of t holomorphic at the origin, at least one of which is not zero at the origin.

4. Professor Price considered the formulas for lengths, areas, volumes, . . . expressed in terms of certain determinants of the coördinates of the points, and showed how they followed a simple and definite pattern thus explaining to an analytic geometry class how mathematics grows by generalization. A second set of formulas, related to the above, gives the distance from a point, line, plane, . . . determined by one point, two points, three points, . . . in m-dimensional euclidean space.

5. Professor Sigley presented some observations on the status of mathematical development in the United States at the present time, and on the relation of this development to the war. He enumerated typical problems, problems that men in the armed forces and related industries must solve, classified according to mathematical principle involved. It seems that present offerings in mathematical curricula are sufficient for proper training of the personnel of the armed forces. More time could be devoted profitably to these subjects, with additional emphasis placed on the study of charts, graphs, tables, interpolation, and other phases of graphical and numerical methods.

6. By means of a set of charts, Professor Smith discussed the geometry of a plane triangle; especially the circles that are associated with a triangle, such as the circumscribed, the inscribed, and the escribed circles, the inexcircles, the exexcircles, the nine-point circle, the isogonic circles, the Spieker circle, the circles of Droz-Farney, the Fuhrmann and the Brocard circles, several of the Tucker circles, the circles of Apollonius, the Canon circles and the Blanc circles. Although the triangle has been studied for a long time and by a great many investigators, it still presents many problems, several of which Professor Smith pointed out.

7. Professor Read called attention to several instances in which texts and reference books in mathematics contain definitions or statements which are contradictory. The plea was made that, if there is a disagreement as to definition of terms, the author might at least call attention to the fact that there is variation in usage. Among illustrations used, were: the meaning of factorial n for

fractional values of n; definition of the principal values of the inverse trigonometric functions; inconsistencies in number classification; confusion in the use of terms *total force* and *total pressure*; definitions of an asymptote; use of the term *rate of change*; definition of a mantissa. The last three mentioned have been discussed in the "Questions, Discussions and Notes" department of this MONTHLY, October 1939, December 1939, and March 1941.

LUCY T. DOUGHERTY, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The thirty-first regular meeting of the Iowa Section of the Mathematical Association of America was held at Iowa Wesleyan College, Mount Pleasant, Iowa, on Friday and Saturday, April 17–18, 1942, in conjunction with the fifty-sixth annual meeting of the Iowa Academy of Science. Dean O. C. Kreider, chairman of the Section presided. He was relieved by Professor Fred Robertson for part of the session on Saturday morning.

The attendance was about twenty-six including the following seventeen members of the Association: J. W. Beach, J. O. Chellevold, L. M. Coffin, N. B. Conkwright, A. W. Davis, C. W. Emmons, Cornelius Gouwens, L. A. Knowler, O. C. Kreider, R. B. McClenon, F. M. McGaw, E. E. Moots, H. V. Price, Fred Robertson, W. J. Rusk, E. R. Smith, Roscoe Woods.

On Friday evening the members and friends of the Association and the Iowa Academy of Science had a joint dinner. The officers of the section elected for 1942–1943 are as follows: Chairman, N. B. Conkwright, State University of Iowa; Vice-Chairman, H. E. Ellingson, Luther College; Secretary-Treasurer, Cornelius Gouwens, Iowa State College.

A resolution expressing the appreciation of the members of the Section for the hospitality and courtesy extended to them by the host, Iowa Wesleyan College, and by Professor G. E. King of the department of mathematics, was adopted at the business meeting.

The following resolution on the closing of the earthly career of Miss Maria M. Roberts of Iowa State College was presented and passed:

Miss Maria M. Roberts entered Iowa State College as a student in the Fall of 1887, and upon her graduation was retained on the staff of the department of mathematics as instructor. Her student career was so satisfactory and notable that this was a worthy recognition of her abilities. Thereafter for more than fifty years she was connected with the college as teacher or as administrator in connection with the Junior College.

Miss Roberts was a true teacher, inspiring guide, and helpful friend to the host of youth who had the privilege of coming under her influence. As co-author with Miss Julia Colpitts, Miss Roberts made in their text in analytical geometry a valuable contribution to the teacher's tools. The Association has lost a faithful and helpful member, and Iowa State College a worthy alumna and teacher. A

large number of former students everywhere will join in spirit in this resolution of memory and esteem.

The following thirteen papers were read on the Association program, the last four by title:

1. "Summation of certain series of constants by the method of residues" by Professor J. J. L. Hinrichsen, Iowa State College, introduced by the Secretary.

2. "The number of isomeric alkenes" by R. E. Carr, Iowa State College, introduced by the Secretary.

3. "The adequacy of the frequency definition of probability" by Professor E. S. Allen, Iowa State College, introduced by the Secretary.

4. "Trends in teaching" by Professor P. G. Robinson, Iowa State College, introduced by the Secretary.

5. "Spherical trigonometry as an emergency course" by Professor R. B. McClenon, Grinnell College.

6. "Survey courses" by Dean O. C. Kreider, Ellsworth Junior College.

7. "Report and discussion of survey courses by the Section Committee" by Professor Fred Robertson, Iowa State College, Chairman.

8. "Report of the representative on the Board of Governors for Region Number 9, Iowa—Minnesota—Wisconsin" by Professor Cornelius Gouwens, Iowa State College.

9. "Systems of linear equations with coefficients subject to error" by Dr. A. T. Lonseth, Iowa State College. (Read by Professor Gouwens.)

10. "Boolean matrices" by Professor E. W. Chittenden, State University of Iowa. (Read by title.)

11. "A test for the nature of the roots of a cubic equation" by Professor E. E. Watson, introduced by the Secretary. (Read by title.)

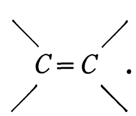
12. "The asymptotic distribution of means of stratified samples" by Professor A. T. Craig, Iowa State University. (Read by title.)

13. "Gamma function expansions" by Professor Henry Van Engen, State Teachers College. (Read by title.)

Abstracts of some of the papers follow:

1. Professor Hinrichsen reviewed the theory of residues and its application to the summation of certain convergent series of real constants. This method was illustrated by summing the series $\sum_{n=1}^{\infty}(1/1+n^2)$ and more generally $\sum_{n=1}^{\infty}(1/1+n^{2p})$, where p is a positive integer. The usefulness of the method is restricted in that the nth term of the series must be an even function of n.

2. Mr. Carr described a method for finding the number of isomeric alkenes. The alkenes are considered to be built up by attaching four alkyl radicals to the four available bonds of a basic skeleton



The alkenes are classified in 30 types, chosen according to the degree of similarity

between the attached alkyl groups. Formulas developed by Mr. Carr for each type enable one to compute the numbers of different varieties of isomers.

3. Recently several writers have named instances of probabilities of isolated events—probabilities, according to them, incapable of description in terms of frequency limits. Professor Allen, analyzing them, held that every numerical probability can be regarded as the limit of relative frequency of success in actual or imagined sequences of trials.

4. Dr. Robinson gave a refreshing discussion on trends in teaching, particularly in the direction of thinking on the part of the student as contrasted with the emphases on skills.

5. Professor McClenon gave an outline of the content of a fundamental course in spherical trigonometry and exhibited interesting analogies between formulas in spherical trigonometry and in plane trigonometry.

6. Dean Kreider stated that the various survey courses may be classified as of the review type, the eclectic, the analytical, the cultural, the historical or the psychological. The main characteristics of survey mathematics should be breadth, psychological organization, and value. The keynote should be to make mathematics really and obviously worth while to students and to encourage students rather than to eliminate them. This paper will appear in the *Proceedings of the Iowa Academy*.

7. Professor Robertson offered evidence showing the need for survey courses, described types which have been offered and exhibited recent texts in the field. An interesting discussion developed.

8. Dr. Gouwens presented a report on the state of the Mathematical Association of America and an account of the business meetings of the Board of Governors during the past year.

9. Dr. Lonseth presented formulas for maximum errors in the solutions of a system of n linear equations in n unknowns where the coefficients are subject to small errors.

10. Professor Chittenden discussed the algebra of matrices whose elements belong to a Boolean algebra. If the Boolean algebra is finite, the resulting algebra of matrices is an interesting form of non-commutative distributive algebra. Tables of incidence relations in topology are well known examples of such matrices in case the basic Boolean algebra has only two elements. He defined addition and multiplication of such matrices in the usual manner and rank in terms of linear dependence.

CORNELIUS GOUWENS, *Secretary*

REMARKS ON THE ABEL-DINI THEOREM

T. H. HILDEBRANDT, University of Michigan

1. The Abel-Dini Theorem* may be stated as follows: If d_n is a sequence of positive numbers such that the infinite series $\sum_n d_n$ is divergent and if $s_n = \sum_{m=1}^{n} d_m$, then the infinite series $\sum_n \frac{d_n}{s_n^{1+\alpha}}$ is convergent for $\alpha > 0$, and divergent for $\alpha \leqq 0$. A corresponding result holds for convergent series, excepting that the remainders replace the sums, viz., if c_n are positive numbers such that $\sum_n c_n$ is convergent and if $r_n = \sum_{m=n+1}^{\infty} c_m$, then $\sum_n \frac{c_n}{r_{n-1}^{1+\alpha}}$ is convergent for $\alpha < 0$ and divergent for $\alpha \geqq 0$. Of the two theorems, the second is slightly easier to demonstrate. For the divergence feature it is sufficient to consider the case where $\alpha = 0$, make the substitution $c_n = r_{n-1} - r_n$ and make the observation that

$$\sum_{n=k}^{m} \frac{c_n}{r_{n-1}} = \sum_{n=k}^{m} \frac{r_{n-1} - r_n}{r_{n-1}} > \sum_{n=k}^{m} \frac{r_{n-1} - r_n}{r_{k-1}} = 1 - \frac{r_m}{r_{k-1}}$$

which approaches 1 in m for fixed k. For the convergence theorem we take $\alpha = -\beta$, with $0 < \beta < 1$ and note that

$$\frac{c_n}{r_{n-1}^{1+\alpha}} = \frac{r_{n-1} - r_n}{r_{n-1}^{1-\beta}} = r_{n-1}^{\beta}\left[1 - \frac{r_n}{r_{n-1}}\right] \leqq \frac{1}{\beta} r_{n-1}^{\beta}\left[1 - \left(\frac{r_n}{r_{n-1}}\right)^{\beta}\right]$$
$$= \frac{1}{\beta}\left[r_{n-1}^{\beta} - r_n^{\beta}\right]$$

since $x^{\beta} \leqq 1 + \beta(x-1)$ for $0 < \beta < 1$ and $x \geqq 0$. Then since $\sum_n (r_{n-1}^{\beta} - r_n^{\beta})$ converges it follows that $\sum_n c_n / r_n^{1-\beta}$ converges for $0 < \beta < 1$ and consequently for $\beta > 0$. The Abel-Dini theorem, where $\sum_n d_n$ diverges, now follows by making the substitution $r_n = \frac{1}{s_n}$ so that $c_n = \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{d_n}{s_{n-1}s_n}$. Then $\frac{c_n}{r_{n-1}^{1+\alpha}} = \frac{d_n}{s_{n-1}^{-\alpha} s_n}$. It is interesting to note, however, that the convergence part of the theorem now yields the Pringsheim modification of the Abel-Dini theorem, viz., if $\sum_n d_n$ is divergent then $\sum_n \frac{d_n}{s_{n-1}^{\alpha} s_n}$ is convergent for $\alpha > 0$.† Obviously one can get a "Pringsheim" modification for $\sum_n c_n$ convergent by starting with the Abel-Dini theorem and setting $s_n = \frac{1}{r_n}$.

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* Cf., for instance, Knopp, Infinite Series, 1928, pp. 290–293.

† Cf. Knopp: loc. cit., p. 300.

If in the proof that $\sum \frac{c_n}{r_{n-1}^{1-\beta}}$ is convergent for $\beta>0$, we should consider instead $\frac{c_n}{r_n^{1-\beta}}$, we find that the inequality $x^\beta - 1 \leqq \beta[x^\beta - 1]$ would need to be applied in the reverse form, yielding nothing. It raises the question whether the theorem is valid in this form. The answer is in the negative.

For this purpose we take $r_n = 1/n^{n^n} = n^{-n^n}$. Then $c_n = (n-1)^{-(n-1)^{n-1}} - n^{-n^n}$ and $c_n/r_n^{1-\beta} = (n-1)^{-(n-1)^{n-1}} \cdot n^{(1-\beta)n^n} - n^{-\beta n^n}$, the last term of which obviously approaches zero, but the first term of which for $0<\beta<1$ is obviously greater than $(n-1)^{(1-\beta)n^n-(n-1)^{n-1}}$ which approaches infinity for all $0<\beta<1$; i.e., it is possible for $\sum_n c_n/r_n^\alpha$ to diverge for all $\alpha>0$. We might note, however, that if $r_n = 1/n^n$ then $\sum_n c_n/r_n^{1-\beta}$ converges for $\beta>0$. Similar results hold for the divergent series, *i.e.*, there exist divergent series $\sum_n d_n$ such that $\sum_n d_n/s_{n-1}^\alpha$ is divergent for all $\alpha \geqq 0$.

2. The Abel-Dini theorem gives at once the properties of the harmonic series $1/n^{1+\alpha}$ by taking $d_n = 1$; as a matter of fact it might be considered a generalization of this. If we in turn apply the Abel-Dini theorem to $d_n = 1/n$ we get the statement: $\sum \frac{1}{n\left(\sum_1^n \frac{1}{m}\right)^{1+\alpha}}$ is convergent if $\alpha>0$ and divergent if $\alpha \leqq 0$.

Since $\lim_n \frac{\log n}{\sum_1^n \frac{1}{m}} = 1 > 0$, this is equivalent to $\sum_n \frac{1}{n \log^{1+\alpha} n}$ is convergent if $\alpha>0$ and divergent if $\alpha \leqq 0$. The next steps $\sum_n \frac{1}{n \log n (\log \log n)^\alpha}$ and so on, seem to require more complicated consideration.* These results suggest the question whether it is possible to prove similar theorems about $\sum_n \frac{d_n}{s_n(\log s_n)^{1+\alpha}}$, $\sum_n \frac{d_n}{s_n \log s_n (\log \log s_n)^{1+\alpha}}$ and so on.

We note that since $s_n \to \infty$ implies $\log s_n \to \infty$, $\log s_n - \log s_{n-1}$ are the terms of a divergent series. Hence $\sum_n \frac{\log s_n - \log s_{n-1}}{(\log s_n)^{1+\alpha}}$ with $s_0 = 1$ is convergent if $\alpha>0$ and divergent if $\alpha \leqq 0$. Now the series $\sum_n \frac{d_n}{s_n(\log s_n)^{1+\alpha}}$ will have the same convergence and divergence properties as $\sum_n \frac{\log s_n - \log s_{n-1}}{(\log s_n)^{1+\alpha}}$ if the ratio of the corresponding terms is bounded from zero and infinity. This ratio is equal to

$$\frac{d_n}{s_n(\log s_n - \log s_{n-1})} = \frac{1 - s_{n-1}/s_n}{\log (s_n/s_{n-1})}.$$

* Cf., however, Knopp, loc. cit., pp. 292–293.

Now the function $\dfrac{x-1}{\log x}$ for $0<x<1$ increases monotonically from 0 to 1. Hence we have the theorem:

THEOREM I. *If* $s_n=\sum_{m=1}^{n}d_m$ *where* d_n *are positive and the terms of a divergent series and if* s_{n-1}/s_n *is bounded from zero in* n (*or* d_n/s_n *is bounded from unity*), *then* $\sum_n \dfrac{d_n}{s_n(\log s_n)^{1+\alpha}}$ *is convergent for* $\alpha>0$, *and divergent for* $\alpha\leqq 0$.

Since for the convergences of a series $\sum_n u_n$ it is sufficient that the ratio of the terms u_n/a_n with $\sum_n a_n$ convergent be bounded we can assert:

THEOREM Ia. *If* $d_n>0$ *are the terms of a divergent series and* $s_n=\sum_{m=1}^{n}d_m$, *then* $\sum_n \dfrac{d_n}{s_n(\log s_n)^{1+\alpha}}$ *is convergent for* $\alpha>0$.

However, the divergency for $\alpha\leqq 0$ cannot be asserted for all d_n, particularly if $\lim d_n/s_n=1$. We note the following instances. If $s_n=n^n$, then $s_{n-1}/s_n\rightarrow 0$. However, $\sum_n d_n/s_n(\log s_n)^{1+\alpha}=\sum_n \dfrac{n^n-(n-1)^{n-1}}{n^n}\,\dfrac{1}{n^{1+\alpha}(\log n)^{1+\alpha}}$ is obviously divergent for $\alpha\leqq 0$. On the other hand, if $s_n=n^{n^n}$, then $d_n/s_n(\log s_n)^{1+\alpha}$ is less than $\dfrac{1}{n^{(1+\alpha)n}\log^{1+\alpha} n}$ and consequently gives rise to a convergent series for $\alpha>-1$.

The following extension to iterated logarithms holds:

THEOREM II. *If* $d_n>0$ *are the terms of a divergent series, and* $s_n=\sum_{m=1}^{n}d_m$ *and if* s_{n-1}/s_n *is bounded from zero* (*or* d_n/s_n *from unity*), *then*

$$\sum_n \frac{d_n}{s_n\log s_n\,(\log\log s_n)^{1+\alpha}}$$

is convergent for $\alpha>0$ *and divergent for* $\alpha\leqq 0$.

By the Abel-Dini theorem, summing from a value of n_0 so that $\log s_n\neq 0$ and $\log\log s_n\neq 0$ for $n\geqq n_0$, the series $\sum_n \dfrac{\log\log s_n-\log\log s_{n-1}}{(\log\log s_n)^{1+\alpha}}$ will be convergent for $\alpha>0$ and divergent for $\alpha\leqq 0$. The ratio of the terms of this series and of the series of the theorem can be transformed as follows:

$$\begin{aligned}\frac{d_n}{s_n\log s_n\log(\log s_n/\log s_{n-1})}&=\frac{s_n-s_{n-1}}{s_n[\log s_n-\log s_{n-1}]}\cdot\frac{1-\log s_{n-1}/\log s_n}{\log(\log s_n/\log s_{n-1})}\\&=\frac{1-s_{n-1}/s_n}{\log(s_n/s_{n-1})}\cdot\frac{1-\log s_{n-1}/\log s_n}{\log(\log s_n/\log s_{n-1})}\,.\end{aligned}$$

Since s_{n-1}/s_n is bounded from zero, the first term of the product is bounded from zero and infinity. Further, if s_{n-1}/s_n is bounded from zero, then

$$\frac{\log s_{n-1}}{\log s_n} = 1 + \frac{\log (s_{n-1}/s_n)}{\log s_n}$$

obviously approaches 1 as n and s_n approach infinity. Hence the product of these terms is bounded from zero and infinity. Obviously we can assert the convergence of $\sum_n \frac{d_n}{s_n \log s_n(\log \log s_n)^{1+\alpha}}$ for any $\alpha > 0$.

One might inquire whether the divergence of $\sum_n \frac{d_n}{s_n \log s_n}$ with $\frac{s_{n-1}}{s_n}$ approaching zero is sufficient to guarantee the divergence of $\sum_n \frac{d_n}{s_n \log s_n(\log \log s_n)}$. The example $s_n = n^n$ gives

$$\frac{d_n}{s_n \log s_n \log \log s_n} = \frac{n^n - (n-1)^{n-1}}{n^n} \cdot \frac{1}{n[\log^2 n + (\log n) \log \log n]}$$

and shows that the reply is negative.

Obviously the reasoning of Theorem II can be continued yielding the result: If the ratio s_{n-1}/s_n is bounded from zero, then the series

$$\sum_n \frac{d_n}{s_n \log s_n \cdots [(\log)^k s_n]^{1+\alpha}} \quad \text{where } (\log)^k s_n = \underbrace{\log \log \cdots \log}_{k \text{ times}} s_n$$

is convergent if $\alpha > 0$ and divergent if $\alpha \leqq 0$, the series starting with an n_0 such that for $n \geqq n_0$ and $k \geqq 1$, $(\log)^k s_n$ does not vanish.

3. The harmonic series $a_n = 1/n$ with the property that $\sum_n a_n^{1+\alpha}$ is a convergent series for $\alpha > 0$ and divergent series for $\alpha \leqq 0$ suggests the question: Do similar properties hold if $a_n = d_n/s_n$, $d_n > 0$, $\sum_n d_n$ divergent? No general theorems can be proved because:

If $D_n > 0$ is any series such that $D_1 = 1$ and $D_n < 1$, $n > 1$, and $\sum_n D_n$ is divergent, then there exists another divergent series $\sum_n d_n$ such that $d_n/s_n = D_n$. For $d_n/s_n = (s_n - s_{n-1})/s_n = D_n$ gives

$$s_n = \frac{s_{n-1}}{1 - D_n}.$$

Then

$$s_n = \frac{s_1}{\prod_{m=1}^{n} (1 - D_m)}.$$

Since $s_n > s_{n-1}$ we obtain at once $d_n > 0$, in terms of $s_1 = d_1$. Obviously $s_n \to \infty$.

In the case of the harmonic series $a_n = 1/n$, the number 1 is the dividing line between the numbers α such that $\sum a_n^\alpha$ converges and diverges, 1 being on

the divergent side. Do there exist series where the dividing number is on the convergent side; *i.e.*, do there exist sequences $a_n>0$ such that $\sum_n a_n^\alpha$ is convergent for $\alpha \geqq \alpha_0$ and divergent for $\alpha<\alpha_0$? Obviously if the statement is true for $\alpha_0=1$, it can be obtained for any α_0 replacing a_n by a_n^{1/α_0}. For $\alpha=1$ we have the instance $a_n=\dfrac{1}{n\log^{1+e} n}$ where $e>0$. Obviously $\sum_n a_n$ is convergent and so $\sum_n a_n^\alpha$, $\alpha>1$, will also be convergent. On the other hand, $\sum_n \dfrac{1}{[n\log^{1+e} n]^{1-\alpha}}$ is divergent for $\alpha>0$. This follows by comparison with $\dfrac{1}{n^{1-\beta}}$, $0<\beta<\alpha$ and the fact that $\lim_{x\to\infty}\dfrac{x^e}{\log x}=\infty$ if $e>0$. A similar trick enables us to prove the following theorem of which the result just mentioned is a special case:

THEOREM III. *If $d_n>0$ is such that $\sum_n d_n$ is divergent, then there exists a sequence c_n such that $0<c_n\leqq d_n$ for all n, $\sum_n c_n$ is convergent but $\sum c_n^\alpha$ is divergent for $\alpha<1$.*

By way of proof we let $D_n=d_n$ if $d_n\leqq 1$, and $D_n=1$ if $d_n\geqq 1$. Let $S_n=\sum_{m=1}^n D_m$. Then obviously $S_n\to\infty$, and $D_n/S_n\to 0$. Let $c_n=\dfrac{D_n}{S_n(\log S_n)^{1+\beta}}$, $\beta>0$. Then by Theorem Ia, $\sum_n c_n$ is convergent, and $c_n\leqq d_n$. On the other hand, since $D_n\leqq 1$, we have for $\alpha>0$

$$\left[\frac{D_n}{S_n(\log S_n)^{1+\beta}}\right]^{1-\alpha} \geqq \frac{D_n}{S_n^{1-\alpha}[\log S_n]^{(1-\alpha)(1+\beta)}}$$

which by comparison with $D_n/S^{1-\gamma}$, with $0<\gamma<\alpha$, turns out to be divergent.

We have the obvious corollary:

If $d_n>0$, is any sequence such that $\sum_n d_n^e$ is divergent and $0<f<e$, then there exists a sequence $0<c_n\leqq d_n$ such that $\sum_n c_n^f$ is convergent but $\sum_n c_n^g$ is divergent for $g<f$.*

Obviously the Abel-Dini theorem gives:

If $d_n>0$, is any sequence such that $\sum_n d_n^e$ is divergent and $0<f<e$, then there exists a sequence $d_n'\leqq d_n$ such that $\sum_n d_n'^g$ is divergent for $g\leqq f$, but $\sum_n d_n'^g$ is convergent for $g>f$.

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* This and the succeeding corollary are due to H. H. Goldstine, who obtained them in proving a theorem on classes of sequences due to E. H. Moore.

ON SEMI-CONTINUITY*

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1. The concept. Semi-continuity is one of many fundamental conceptions that were introduced into Analysis around the year 1900. The following incomplete survey will be concerned primarily with applications of the idea of semi-continuity. The extent of the literature makes the inclusion of a comprehensive bibliography prohibitive, and for our modest purposes a detailed bibliography is irrelevant anyway. Our objective is merely to illustrate how semi-continuity, originally a mere abstraction, became in the course of time an indispensable tool for many classical problems in Analysis, Geometry, and Potential Theory. We restrict ourselves therefore to two remarks concerning literature. The well-known standard treatises on Real Variables and Point-set Theory (Carathéodory, Hahn, Hobson, R. L. Moore) contain comprehensive information on semi-continuity itself, while further details and references concerning applications may be found in two surveys by the present author (*On the problem of Plateau* and *On subharmonic functions*, both in the Ergebnisse der Mathematik series).

For brevity, we shall write l.s.c. and u.s.c. for lower semi-continuous and upper semi-continuous respectively. Let $f(x)$ be a function defined in a closed interval

$$I: a \leqq x \leqq b.$$

Then $f(x)$ is l.s.c. at a point x_0 of I if

$$f(x_0) > -\infty \quad \text{and} \quad f(x_0) \leqq \underline{\lim}\, f(x_n)$$

for every sequence $x_n \to x_0$. Similarly, $f(x)$ is u.s.c. at a point x_0 of I if

$$f(x_0) < +\infty \quad \text{and} \quad f(x_0) \geqq \overline{\lim}\, f(x_n)$$

for every sequence $x_n \to x_0$. If $f(x)$ is l.s.c. (u.s.c.) at every point of I, then $f(x)$ is l.s.c. (u.s.c.) in I.

The conditions involving the values $\pm\infty$ are needed to cover certain applications. More explicitly, we require that $-\infty < f(x) \leqq +\infty$ if $f(x)$ is l.s.c. and $-\infty \leqq f(x) < +\infty$ if $f(x)$ is u.s.c.

For the sake of immediate illustration, the following simple remark may be used. Let $g(x)$ be continuous at a point x_0 of I. Let us take a number $k \neq g(x_0)$. Define

$$f(x) = g(x), \qquad x \neq x_0,$$
$$f(x_0) = k.$$

Clearly, $f(x)$ is discontinuous at x_0. On the other hand, $f(x)$ is semi-continuous there. Specifically, $f(x)$ is l.s.c. at x_0 if $k < g(x_0)$, and u.s.c. at x_0 if $k > g(x_0)$.

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* Presented, by invitation, at the annual meeting of the Mathematical Association of America, in Bethlehem, Pa., January 1, 1942.

If a function $f(x)$ is both l.s.c. and u.s.c. at x_0, then it is continuous at x_0 and conversely. Thus the property of continuity may be thought of as the logical sum of the properties of lower and upper semi-continuity. We expect therefore that semi-continuous functions will enjoy about half of the properties of continuous functions. Several simple and important theorems bear out this expectation. Two typical theorems of this type will be stated now. The simple proofs are omitted. Such theorems may possibly be used to good advantage as exercises in familiarizing beginners with the idea of continuity itself.

If $f(x)$ is l.s.c. (u.s.c.) in I, then $f(x)$ is bounded from below (above) in I.

If $f(x)$ is l.s.c. (u.s.c.) in I, then it takes on there a minimum (maximum).

The class of semi-continuous functions is of course much more comprehensive than the class of continuous functions. Still, in a sense to be explained now, semi-continuous functions are quite close to continuous functions. Let us use the term L-sequence (U-sequence) to refer to a monotonically increasing (decreasing) sequence of functions. We have then the following theorem.

If $f_n(x)$ is an L-sequence (U-sequence) of continuous functions in I, then lim $f_n(x)=f(x)$ is l.s.c. (u.s.c.) in I. Conversely, if $f(x)$ is l.s.c. (u.s.c.) in I, then we have an L-sequence (U-sequence) of continuous functions $f_n(x)$ such that $f(x)=\lim f_n(x)$ in I.

Thus semi-continuous functions are simply the limits of monotonic sequences of continuous functions. It is then natural to consider the limit functions of monotonic sequences of semi-continuous functions. It turns out that these limit functions comprise all the functions that seem to be needed in applications, namely all measurable functions.* Thus semi-continuous functions are, in a sense, stepping stones between continuous and measurable functions, As a consequence semi-continuous functions may be used to construct a most beautiful and pedagogically sound theory of measurable functions and their integrals.

The conception of semi-continuity can be extended, in an obvious manner, to functions of several variables and more generally to functionals. We state presently the general definition; examples will follow in a moment.

Let $\eta=f(\xi)$ be a functional, defined on a range X, with values in a range Y, Then $f(\xi)$ is l.s.c. (u.s.c.) at ξ_0 in X if

$$f(\xi_0) \leqq \underline{\lim}\, f(\xi_n) \quad \text{for} \quad \xi_n \to \xi_0,$$

or

$$f(\xi_0) \geqq \overline{\lim}\, f(\xi_n) \quad \text{for} \quad \xi_n \to \xi_0$$

respectively. This definition presupposes first that in both ranges X, Y the conception of convergent sequence and that of limit inferior (limit superior) is suitably determined, and second that in the range Y an order relation is given.

We shall discuss now a few outstanding examples of semi-continuous functions and functionals.

* The precise theorem reads as follows. Every measurable function is the limit, almost everywhere, of a U-sequence of l.s.c. functions and of an L-sequence of u.s.c. functions.

2. Potentials and convex functions. We restrict ourselves, for definiteness, to the two-dimensional case. In the xy-plane, let there be given a negative mass-distribution (the reader may think of a distribution of negative electric charges). Such a distribution gives rise to a potential $u(x, y)$. What are the characteristic mathematical properties of this potential? This question is of course of great interest. By virtue of a most important coincidence, the answer to this question is identical to the answer to the following, entirely different, question that originates in pure mathematics. Let $f(x)$ be a function that is convex from below in an interval I. Then, in every sub-interval of I, the graph of $f(x)$ is below its chord. Since the chord is given by a linear function $h(x)=\alpha x+\beta$, a convex function $f(x)$ may also be described as a *sublinear function* in the sense that, in every sub-interval, $f(x)$ is dominated by the linear function $h(x)$ with the same boundary values. In view of the importance of convex functions of a single variable, we may ask for a useful generalization of this idea to functions of two (or several) variables. Now, a linear function $h(x)=\alpha x+\beta$ may also be described as a solution of the differential equation

$$\frac{d^2h}{dx^2} = 0.$$

As a natural generalization, we have then, in the two-dimensional case, the *harmonic functions* which arise as solutions of the differential equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

Thus we arrive at the definition of *subharmonic functions*, defined by the property that in every sub-region of its range of definition, a subharmonic function is dominated by the harmonic function with the same boundary values. Now, how is all this related to semi-continuity? First of all, we find that the potentials $u(x, y)$, referred to above, are subharmonic functions. As far as applications in pure mathematics are concerned, we may include in the definition of a subharmonic function any continuity requirements to suit ourselves. However, if all the potentials $u(x, y)$ of negative mass-distributions are to be subharmonic functions and conversely, if every subharmonic function is to be, essentially, a potential of this type, then we have no choice in this matter. It turns out that we have to require precisely the property of upper semi-continuity.

3. Arc-length and area. In the preceding example, semi-continuity was in a sense forced upon the Analyst by an outside application. Similar situations arise if we undertake to apply Analysis to certain simple and fundamental problems in Geometry. Let X designate the class of continuous curves in three-space, say. Let ξ designate an individual curve and $f(\xi)$ the arc-length of ξ, where $f(\xi)$ may be $+\infty$. Simple examples show that $f(\xi)$ is not a *continuous* functional. That is, $\xi_n \to \xi_0$ generally does not imply that $f(\xi_n) \to f(\xi_0)$. But, if convergent sequences of curves are suitably defined, then it is found that $f(\xi)$ is a l.s.c. functional.

Similar statements hold if X stands for the class of continuous surfaces and $f(\xi)$ for the area of the surface ξ. In fact, this semi-continuity property is used as a desideratum in defining the area of a surface. Unfortunately, the area of a surface cannot be defined, in analogy with the arc-length, as the limit of the areas of inscribed polyhedra, as simple examples show. The generally accepted definition of the area $f(\xi)$ of a surface ξ is given by the formula.

$$f(\xi) = \text{gr.l.b.} \varliminf \phi(\pi_n), \tag{1}$$

where the greatest lower bound is taken with respect to all sequences of polyhedra π_n, inscribed or not, converging to ξ in an appropriate sense, and $\phi(\pi_n)$ designates the elementary area of π_n. The purpose and effect of this definition is to make the area a l.s.c. functional.

We wish to call attention on this occasion to a simple, natural and apparently quite difficult problem. As regards the definition of the area by formula (1), it may be argued that it would be more natural to state that definition in terms of *inscribed* polyhedra. Let us designate by $f^*(\xi)$ the quantity obtained by restricting, in the formula (1), the polyhedra π_n by the requirement that each π_n be inscribed in the surface ξ. The problem we wish to call attention to requires to determine whether $f(\xi) = f^*(\xi)$. Only partial results are known, and in every instance the contrast between the simplicity of the problem and the complexity of the methods is striking. Still, a little reflection shows that the problem would be solved if we could prove that the functional $f^*(\xi)$ is also l.s.c.

4. Calculus of Variations. The analytic formulation of the geometrical facts stated in sections 2 and 3 leads to important questions in Analysis. Let us consider, for definiteness, a sequence of smooth curves given by equations

$$y = g_n(x), \qquad 0 \leqq x \leqq 1.$$

Suppose that $g_n(x) \to g(x)$ uniformly in this interval, where $g(x)$ is again a smooth function. The arc-lengths of all these curves are then given by the familiar integral formula, and the remarks in section 3 lead to the relation

$$\int_0^1 [1 + g'(x)^2]^{1/2} dx \leqq \varliminf \int_0^1 [1 + g_n'(x)^2]^{1/2} dx. \tag{2}$$

Let us observe that our assumptions do not imply any degree of approximation in terms of derivatives. Hence there is nothing obvious about (2) from the point of view of Analysis. We are thus led to the following general question. Let a curve ξ be given by equations of the form

$$x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad 0 \leqq t \leqq 1.$$

Let $F(x, y, z, \lambda, \mu, \nu)$ be a function of six variables. We define a functional

$$f(\xi) = \int_0^1 F(x(t), y(t), z(t), x'(t), y'(t), z'(t)) dt.$$

Under what conditions will this functional $f(\xi)$ be l.s.c.? The same question arises in connection with double integrals depending upon a surface ξ. The questions thus raised are important in Calculus of Variations, where we deal with the problem of minimizing such integrals. We mentioned above that a l.s.c. function, defined on a closed interval, reaches a minimum there. We can expect then to obtain existence theorems in Calculus of Variations in cases when the variational integral is a l.s.c. functional. This simple idea is the foundation of the so-called *direct method* in Calculus of Variations. In this manner, semi-continuity takes its place amongst the fundamental conceptions of a classical field in Analysis.

5. Point sets. As a last example, let us consider the following situation. Let X designate a point-set, in the xy-plane for definiteness. Let X be partitioned, in some definitely given manner, into mutually exclusive subsets for which we shall use the generic notation η. We denote the class of all the sets η by Y. We define now a functional $\eta = f(\xi)$, for points ξ in X, by the agreement that $\eta = f(\xi)$ is the unique set η of the class Y that contains the point ξ. Under what conditions shall we say that this functional $f(\xi)$ is u.s.c.? Checking back, we see that we have to state first various definitions. Since ξ stands for a point in the xy-plane, the relation $\xi_n \to \xi_0$ will stand for ordinary convergence. Next, we define $\overline{\lim}\ \eta_n$ as the set of points (x, y) with the property that every neighborhood of (x, y) is entered by infinitely many sets of the sequence η_n. Finally, if S_1, S_2 are two point-sets, we shall interpret the relation $S_1 \leqq S_2$ as meaning that S_1 is a subset of S_2, and we shall write $\subset$ instead of $\leqq$. Applying the general definition stated in section 1, we shall say that the functional $\eta = f(\xi)$, defined at the beginning of the present section, is u.s.c. if for every ξ_0 in X

$$f(\xi_0) \supset \overline{\lim}\ f(\xi_n),$$

whenever $\xi_n \to \xi_0$. The prevalent practice, though, is to describe the class Y of subsets η as an *upper semi-continuous collection.* The wording of the definition is of course irrelevant so long as the logical content is the same. The point is that semi-continuity, applied in this manner in point-set theory, admits of many significant applications.

The preceding rapid survey lends support to the statement that a mathematical conception may originate as a mere abstraction, and still develop into a tool of great usefulness in a surprising variety of applications. To avoid giving undue aid and comfort to the specialist dealing in mere abstractions, we want to conclude by a complementary statement: it would seem that only those mere abstractions survive that do develop into tools of great usefulness in a surprising variety of applications.

THE RATIONAL CANONICAL FORM OF A MATRIX*

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1. Introduction. There are several different methods in the literature for the computation of the rational canonical form of a matrix.† We shall give a method here which is (1) valid in an arbitrary field $\mathfrak{K}$, (2) exceedingly simple, and (3) based on elementary transformations. Our result is obtained by exploiting the same lemma which Albert has used in simplifying the computation of the inverse of a non-singular matrix.‡ A description of the general reduction process is given in Section 2 and a particular numerical example is treated in Section 3.

2. The reduction process. Two matrices A and B with elements in $\mathfrak{K}$ are said to be similar (in $\mathfrak{K}$) in case $B=PAP^{-1}$ with P a non-singular matrix with elements in $\mathfrak{K}$. Now a non-singular matrix is the product of a finite number of elementary transformation matrices, § and the lemma used by Albert then shows that A is similar to B if and only if we can obtain B from A by a finite number of *elementary similarity transformations* of the following types.

Type (i). *Interchanging the ith and jth rows and the corresponding columns.*

Type (ii). *Adding $c\epsilon\mathfrak{K}$ times the ith row to the jth row and subtracting c times the jth column from the ith column.*

Type (iii). *Multiplying the ith row by a nonzero element $a\epsilon\mathfrak{K}$ and multiplying the ith column by a^{-1}.*

We shall denote these transformations by e_{ij}, $p_{ij}(c)$, and $r_i(a)$, respectively.

Let us call a matrix of the form

$$(1)\qquad \begin{pmatrix} 0 & 1 & 0 & \cdots 0 & 0 \\ 0 & 0 & 1 & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdots\cdot & \cdot \\ 0 & 0 & 0 & \cdots 0 & 1 \\ a_m & a_{m-1} & a_{m-2} & \cdots a_2 & a_1 \end{pmatrix}$$

a *canonical block*. We shall show how a matrix $A=(a_{ij})(i,j=1,\cdots,n)$ may be reduced by transformations of types (i)–(iii) to a matrix $B=\operatorname{diag}\{B^{(1)},\cdots,B^{(t)}\}$ where $B^{(\alpha)}(\alpha=1,\cdots,t)$ are canonical blocks. The computation of the rational canonical form of B may then be carried through via elementary similarity trans-

* Presented to the American Mathematical Society, Dec. 29, 1941, under the title Elementary similarity transformations and the rational canonical form of a matrix.

† C. C. MacDuffee lists the contributions prior to 1933 in The Theory of Matrices, Ergebnisse der Mathematik, vol. 2, 1933, p. 424. See also E. T. Browne, On the reduction of a matrix to canonical form, this MONTHLY, vol. 47, 1940, pp. 437–450.

‡ A rule for computing the inverse of a matrix, this MONTHLY, vol. 48 1941, pp. 198–199. See also A. A. Albert, Introduction to Algebraic Theories, Chicago, 1941, p. 42; and G. Birkhoff and S. Mac Lane, A Survey of Modern Algebra, New York, 1941, p. 279.

§ A. A. Albert, Introduction to Algebraic Theories, p. 44.

formations. However, it is much simpler to resort to a computation of the invariant factors of $xI-B$. Let $f_\alpha(x)(\alpha=1, \cdots, t)$ be the non-trivial invariant factor of $xI-B^{(\alpha)}$. Let $d(x)$ be the greatest common divisor of $f_1(x), \cdots, f_t(x)$. Let $B_1^{(t)}, B_1^{(t-1)}$ be canonical blocks corresponding to $d(x)$ and to $f_t(x)\cdot f_{t-1}(x)/d(x)$ (See Albert, *Introduction to Algebraic Theories*, p. 105). Then $B_1=\operatorname{diag}\{B^{(1)}, \cdots, B^{(t-2)}, B_1^{(t-1)}, B_1^{(t)}\}$ is similar to B. Repeat this process on the matrix $\operatorname{diag}\{B^{(1)}, \cdots, B^{(t-2)}, B_1^{(t-1)}\}$ and continue repetitions until the rational canonical form of B is obtained.

Let us return, then, to the problem of reducing $A=(a_{ij})(i, j=1, \cdots, n)$ to diag $\{B^{(1)}, \cdots, B^{(t)}\}$ with each $B^{(\alpha)}(\alpha=1, \cdots, t)$ a canonical block. First, if some $a_{1j}(j>1)$ is nonzero, we may apply e_{2j} to obtain a matrix (a'_{ij}) with $a'_{12}\neq 0$. Application of $r_2(a'_{12})$, followed by $p_{12}(a'_{11})$, $p_{32}(a'_{13}), \cdots, p_{n2}(a'_{1n})$ yields a matrix (a''_{ij}) whose first row is $(0, 1, 0, \cdots, 0)$. If then some $a''_{2k}(k>2)$ is nonzero, we may apply e_{3k} to obtain a matrix (a'''_{ij}) with $a'''_{23}\neq 0$. Application of $r_3(a'''_{23})$ followed by $p_{13}(a'''_{21})$, $p_{23}(a'''_{22})$, $p_{43}(a'''_{24}), \cdots, p_{n3}(a'''_{2n})$ produces a matrix whose first two rows agree with (1). This process may be repeated until we have reduced the matrix A to its rational canonical form of just one canonical block, in which case we have attained our goal; or until we reach a matrix of the form

$$\begin{pmatrix} B_1 & 0 \\ C & D \end{pmatrix}$$

in which B_1 is a $q\times q$ canonical block. Using transformations of type (ii), we can replace this matrix by one of the same form in which the last $q-1$ columns of the matrix corresponding to $C=(c_{rs};\ r=1, \cdots, n-q,\ s=1, \cdots, q)$ are all zero. How this is done will be clear if we show how to "remove" the element c_{1q}. To do this, multiply the $(q-1)$st row by c_{1q} and subtract it from the $(q+1)$st row and add c_{1q} times the $(q+1)$st column to the $(q-1)$st column. Note that this transformation is $p_{q-1,q+1}(-c_{1q})$ and that it leaves the matrix B_1 unaltered. When the last $q-1$ columns of the matrix C have been replaced by zeros, it may be that each element in the first column of the matrix corresponding to C is zero, and then we have reduced A to the matrix diag $\{B^{(1)}, D^{(1)}\}$, with $B^{(1)}=B_1$ and $D^{(1)}$ the matrix corresponding to D. If so, we apply the above process to the matrix $D^{(1)}$. If not, some element c'_{1s} of the matrix $C^{(1)}=(c'_{rs})$ corresponding to C is nonzero. Apply $r_{q+s}(1/c'_{1s})$, followed by $p_{q+s,q+1}(-c'_{11}), \cdots, p_{q+s,q+s-1}(-c'_{1,s-1})$, $p_{q+s,q+s+1}(-c'_{1,s+1}), \cdots, p_{q+s,n}(-c'_{1,n-q})$, $e_{q+1,q+s}$ to obtain a matrix

$$\begin{pmatrix} B_1 & 0 \\ C^{(2)} & D^{(2)} \end{pmatrix}$$

in which the only nonzero element of $C^{(2)}=(c''_{rs})$ is $c''_{11}=1$. In this case it is always possible to obtain a matrix with a larger canonical block in the upper left-hand corner. The argument falls into two cases according to whether or not there is some nonzero element d''_{s} $(s>1)$ of the matrix $D^{(2)}=(d''_{us};\ u, s=1, \cdots, n-q)$. Since the argument is similar in the two cases, we shall give the

details of proof only in the more involved case in which there is some nonzero element $d''_{1s}(s>1)$. Apply $e_{q+2,q+s}$ $r_{q+2}(d''_{1s})$, followed by $p_{q+3,q+2}(d''_{13}), \cdots, p_{n,q+2}(d''_{1n-q})$ to obtain a matrix

$$\begin{pmatrix} B_1 & 0 \\ C^{(2)} & D^{(3)} \end{pmatrix}$$

in which the first row of $D^{(3)}$ is $(\beta, 1, 0, \cdots, 0)$. Now apply $e_{1,q+1}, e_{2,q+1}, \cdots, e_{q,q+1}$ to obtain the matrix

$$(2)\qquad \begin{Bmatrix} \beta & 1 & 0 & \cdots 0 & 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 & \cdots 0 & 0 & 0 & 0 \cdots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots 0 & 1 & 0 & 0 \cdots 0 \\ 0 & a_q & a_{q-1} & \cdots a_2 & a_1 & 0 & 0 \cdots 0 \\ * & 0 & 0 & \cdots 0 & 0 & * & * \cdots * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ * & 0 & 0 & \cdots 0 & 0 & * & * \cdots * \end{Bmatrix}$$

Here the asterisks designate possibly nonzero elements. Applying $p_{q+2,2}(1)$ introduces new elements into the second row; but it is easy to remove these elements, new elements being introduced in the third row. Then these elements and the subsequent elements may be similarly removed without affecting the element 1 in the nth row and the $(n+1)$st column. The same process may be employed to remove β. We are thus assured a larger canonical block in these cases. Since the matrix A has only a finite number of rows and columns, our process must ultimately cease and we have reduced the matrix A to the desired form.

3. An example. As an illustration of our reduction process of Section 2 we shall reduce the matrix

$$A = \begin{Bmatrix} -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & -1 \\ 0 & 5 & 3 & -1 \end{Bmatrix}$$

to canonical form. Note that A is singular and that it has the form (for $q=1$) which led to (2). We apply e_{12}, $p_{32}(1)$, and $p_{12}(2)$ to obtain successively

$$\begin{Bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 3 & 0 & 2 & -1 \\ 5 & 0 & 3 & -1 \end{Bmatrix}, \quad \begin{Bmatrix} 2 & 1 & 0 & 0 \\ 3 & -1 & 3 & -1 \\ 3 & 0 & 2 & -1 \\ 5 & 0 & 3 & -1 \end{Bmatrix}, \quad \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 5 & 1 & 3 & -1 \\ 3 & 0 & 2 & -1 \\ 5 & 0 & 3 & -1 \end{Bmatrix}.$$

Application of $r_3(3)$, $p_{43}(-1)$, and $p_{13}(5)$ gives

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 5 & 1 & 1 & -1 \\ 9 & 0 & 2 & -3 \\ 5 & 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 5 & 1 & 1 & 0 \\ 4 & 0 & 1 & -1 \\ 5 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 5 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Use of $p_{23}(1)$, $r_4(-1)$, and $p_{14}(-1)$ yields

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 4 & 2 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 4 & 2 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Finally, we use $p_{24}(4)$ and $p_{34}(2)$ to find

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 2 \end{pmatrix}.$$

The last matrix obtained is the rational canonical form of A.

A MAXIMUM PROBLEM

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If an impurity appears at a point P in a solid, the question may be asked: What plane section cutting off P will leave a maximum volume of pure material? For example, the problem would seem to have a certain practical aspect for the gem-cutter. There is a simple geometric way to characterize the answer.

Given a set M of a finite number of bounded solids, fixed in position in three-space and a point P in the space; then a plane p through P divides M into two subsets on the volume of one of which attention is fixed by the label V_p. If this volume is the same as that of M, the plane p determines an absolute maximum for the function V. Otherwise consider the two cases I and II according as P is not, or is, in the same plane as a plane face of one of the solids of M. Let the plane p have in common with the set M the area A_p. Using these terms it is possible to state the following theorem:

THEOREM. *In case I a necessary condition that the plane p serve to maximize the function V is that P be the centroid of the area A_p.*

For the proof let the plane p be fixed; let L be a variable line in p through P; let q be a plane intersecting p in L and making an angle θ with the plane p;

then the volume V_q corresponding to V_p may be described as $V(\theta, L)$. Let the line L be fixed. Let L divide the area A_p into areas A_{p1} and A_{p2} and divide the corresponding area A_q into areas A_{q1} and A_{q2} where A_{p1} and A_{q1} are to be separated by an acute angle θ. Let $\bar{x}_{p1}$, $-\bar{x}_{p2}$, $\bar{x}_{q1}$, $-\bar{x}_{q2}$ be the distances of the centroids of A_{p1}, A_{p2}, A_{q1}, A_{q2}, respectively, from the line L. Denote the portion of M between A_{p1} and A_{q1} by $V_1(\theta, L)$ and the portion between A_{p2} and A_{q2} by $V_2(\theta, L)$.

Let us investigate the existence of the derivatives $\partial V_1/\partial\theta$ and $\partial V_2/\partial\theta$ for $\theta=0$ and any fixed L. Let the angle θ be taken small, say, less than $\Delta\theta$. Since we are considering case I, $A_{q1}(\theta)$ is a continuous function, and therefore $\bar{x}_{q1}(\theta)$ and $A_{q1}(\theta)\bar{x}_{q1}(\theta)$ are continuous. Thus if s and S indicate the minimum and maximum values of $A_{q1}(\theta)\bar{x}_{q1}(\theta)$ in the interval $-\Delta\theta\leqq\theta\leqq\Delta\theta$, we know that s and S may be made to differ from $A_{p1}\bar{x}_{p1}$ by an arbitrarily small number simply by taking $\Delta\theta$ sufficiently small. Consider the inequality:

$$s\Delta\theta \leqq \Delta V_1 \leqq S\Delta\theta$$

where the bounding approximations are the volumes of solids of revolution expressed by the Theorem of Pappus. Then divide by $\Delta\theta$ and let $\Delta\theta$ approach 0 to show that $\partial V_1/\partial\theta$ exists at $\theta=0$ and has the value $A_{p1}\bar{x}_{p1}$. In similar fashion $\partial V_2/\partial\theta=-A_{p2}\bar{x}_{p2}$.

Consider the total variation $\Delta V=V_q-V_p=\Delta V_1-\Delta V_2$. Hence

$$\frac{\partial V}{\partial\theta}=\frac{\partial V_1}{\partial\theta}-\frac{\partial V_2}{\partial\theta}=A_{p1}\bar{x}_{p1}+A_{p2}\bar{x}_{p2}.$$

Then since $\partial V/\partial\theta$ exists at $\theta=0$, a necessary condition that p serve to maximize $V(\theta, L)$ is that $\partial V/\partial\theta=0$; but this is to assert that the line L passes through the centroid of the area $A_p=A_{p1}+A_{p2}$. If this argument is pursued for each of the lines L in p and through P, the demand is that P itself be the centroid of the area A_p. This completes the proof.

In case II the above argument fails for then the function $A_{q1}(\theta)$ may have discontinuities with a saltus possible when the variable plane through P is in coincidence with a portion of the surface of M which is flat. Thus $\partial V_1/\partial\theta$ may fail to exist for the critical plane p which maximizes V, and then this plane need bear no nice relation to the point P. But even in case II a maximizing plane p can sometimes be discovered by noticing that for it the left-hand and right-hand derivatives $\partial V/\partial\theta$ exist, are evaluated by $(A_p\bar{x}_p)^-$ and $(A_p\bar{x}_p)^+$, and are respectively positive and negative, creating the "tent" type of maximum.

In general, the function $V(\theta, L)$ is continuous, bounded by the volume of M, and periodic of period 2π in each of the variables. If $\overline{V}$ indicates the complement of V in M, then $V(\theta+\pi, L)=\overline{V}(\theta, L)$, so that V may be described as skew symmetric of period π in the variable θ.

The condition of the principal theorem was described as necessary, but of course is not sufficient, for a maximum of V; for it characterizes equally well

horizontal points of inflection of the volume function. Minimum points can all be found by virtue of the skew symmetric property already mentioned.

We state next the form which the theorem takes for two-space leaving the proof to the interested reader since he can proceed in a manner analogous to that given above, using that variation of the Theorem of Pappus which gives the plane area swept out by one or more segments of a rotating line.

Given the set M of a finite number of bounded areas in a plane; P, a point in the plane; and L, a variable line through P, separating M into two subsets, on the area of one of which attention is fixed by the label A_L. If this area is the same as that of M, then L determines an absolute maximum for the function A. Otherwise two cases I and II arise according as P is not, or is, on a line, a portion of which serves as part of the boundary of M. Let the line L have in common with M the line segments S_L.

THEOREM. *In case I a necessary condition that the line L serve to maximize the function A is that P be the centroid of the line segments* S_L.

If M be a single area which is convex with respect to P (*i.e.*, every line through P intersects the boundary of M in just two points, separated by P), then the necessary condition is that the line L maximizing A_L have the line segment S_L bisected at P. But this is a known result* which can be established in various ways. One of these, which expresses A_L as an integral containing a parameter, namely—the slope of the line through P, offers an interesting exercise in the use of Leibniz' rule for differentiating such integrals. A posteriori, it follows for this particular type of area, granting the existence of a maximum for A_L, that there *exists* a line segment S_L bisected at P. This is a theorem established elsewhere by other means and for far more general boundary curves.†

In conclusion let me suggest a further line of theorems which can be proved by a slight modification of the method of this paper. For example for the set of coplanar areas M let the problem be to determine that tangent line L of a curve C with continuously turning tangent which will separate M into two subsets, making the area of one of these a maximum. A necessary condition (in case I) is that the point of tangency P shall be the centroid of the segments S_L in common to L and M. As a special case there is the well known theorem that a triangle of minimum area circumscribing an oval must have the points of tangency bisecting the sides. The generalization of these statements to three dimensions is straightforward.

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* For example, see H. Levy, Modern Science, Knopf, 1939, p. 317.

† J. D. Hill, H. E. Vaughan, this MONTHLY, vol. 46, 1939, p. 657.

DISCUSSIONS AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The Department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

NEW FORMS OF CERTAIN INTEGRALS

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It is a remarkable fact, sufficient in itself perhaps to justify such investigations, that often inquiries into relatively advanced or abstract questions throw new light upon topics long regarded as entirely familiar. The present paper is a case in point. In attempting to systematize certain integrations involving Bessel functions the author encountered formulas which proved to have useful and apparently new analogues in the integration of circular and exponential functions. It is the purpose of this note to call attention to these results and to sketch the process by which they may be derived.

We consider first the integral $\int x^n \cos x \, dx$, and guided by a few particular cases we write

$$\int x^n \cos x \, dx = P(x) \sin x + Q(x) \cos x, \tag{1}$$

where P and Q are functions of x to be determined. Differentiation of (1) gives

$$x^n \cos x = P' \sin x + P \cos x + Q' \cos x - Q \sin x. \tag{2}$$

Since the antiderivative of a function is unique to within an additive constant, any pair of functions P, Q satisfying (2) can be used in (1). A sufficient condition that P and Q satisfy (2) is that*

$$P' - Q = 0, \tag{3}$$

$$P + Q' = x^n. \tag{4}$$

Hence we seek a solution of these equations. Eliminating Q from (3) and (4) we have

$$P'' + P = x^n. \tag{5}$$

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* It is an interesting exercise in calculus to show that the more general equations,

$$P' - Q = F(x) \cos x,\ P + Q' = x^n - F(x) \sin x,$$

where $F(x)$ is arbitrary, afford a solution of (1) differing only by a constant from that given by (3) and (4).

We solve this by assuming

$$P = x^c(a_0 + a_1x + a_2x^2 + \cdots).$$

It turns out that we must choose $c=n+2$. Straightforward calculations then yield at once

$$P \equiv x^{n+2}\left\{\frac{1}{(n+1)(n+2)} - \frac{x^2}{(n+1)\cdots(n+4)} + \cdots\right\},$$

or, after multiplying up and down by $\Gamma(n+1)$,

$$P = \Gamma(n+1)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{n+2k}}{\Gamma(n+2k+1)}, \tag{6}$$

and from (3)

$$Q = \Gamma(n+1)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{n+2k-1}}{\Gamma(n+2k)}. \tag{7}$$

Equation (1) with P and Q defined by (6) and (7) respectively is valid for all values of n except $n=-1, -2, -3, \cdots$, and has the useful property of converging more rapidly than the formula obtained by multiplying the Maclaurin expansion of $\cos x$ by x^n and integrating termwise.

Particularly useful and elegant results are obtained when n is a positive integer. If, for instance, n is even, we have

$$\begin{aligned} P &= n!\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{n+2k}}{(n+2k)!} = n!(-1)^{(n+2)/2}\sum_{j=(n+2)/2}^{\infty}(-1)^j\frac{x^{2j}}{(2j)!} \\ &= (-1)^{(n+2)/2}n![\cos x - S_{(n+2)/2}(\cos x)], \end{aligned}$$

$$\begin{aligned} Q &= n!\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{n+2k-1}}{(n+2k-1)!} = n!(-1)^{n/2}\sum_{j=n/2}^{\infty}(-1)^j\frac{x^{2j+1}}{(2j+1)!} \\ &= (-1)^{n/2}n![\sin x - S_{n/2}(\sin x)], \end{aligned}$$

where $S_r(\cos x)$ denotes the rth partial sum of the Maclaurin series for $\cos x$. Finally, then,

$$\int x^n \cos x\, dx = (-1)^{n/2}n![S_{(n+2)/2}(\cos x)\sin x - S_{n/2}(\sin x)\cos x], \quad n \text{ even}, n > 0. \tag{8}$$

To this are to be added these companion formulas, all obtained in the same fashion:

$$\int x^n \cos x\, dx = (-1)^{(n-1)/2}n![S_{(n+1)/2}(\sin x)\sin x + S_{(n+1)/2}(\cos x)\cos x], \quad n \text{ odd}, n > 0, \tag{9}$$

$$(10)\qquad \int x^n \sin x\,dx = (-1)^{(n+2)/2} n!\left[S_{n/2}(\sin x)\sin x + S_{(n+2)/2}(\cos x)\cos x\right],$$

$$n \text{ even, } n > 0,$$

$$(11)\qquad \int x^n \sin x\,dx = (-1)^{(n-1)/2} n!\left[S_{(n+1)/2}(\cos x)\sin x - S_{(n+1)/2}(\sin x)\cos x\right],$$

$$n \text{ odd, } n > 0,$$

$$\left.\begin{aligned} &(12)\qquad \int x^n e^x dx = (-1)^n n! S_{n+1}(e^{-x})e^x \\ &(13)\qquad \int x^n e^{-x} dx = -\,n! S_{n+1}(e^x)e^{-x} \end{aligned}\right\}, \qquad n \text{ integer, } n > 0.$$

These seem to the writer to be superior to the usual one-step reduction formulas to be found in most handbooks.

The application to Bessel functions which first suggested the above results to the author involved the integral $\int x^n J_m(x)dx$. If we write

$$(14)\qquad \int x^n J_m(x)dx = xP(x)J_m(x) + xQ(x)J_{m-1}(x)$$

and differentiate we have

$$(15)\qquad \begin{aligned} x^n J_m = PJ_m + xP'J_m + xP\left(J_{m-1} - \frac{m}{x}J_m\right) + QJ'_{m-1} + xQ'J'_{m-1} \\ + xQ\left(\frac{m-1}{x}J_{m-1} - J_m\right), \end{aligned}$$

where the expressions in parentheses are well-known equivalents of J'_m and J'_{m-1} respectively. A sufficient condition that (15) subsist is that simultaneously

$$(16)\qquad xP' + (1-m)P - xQ = x^n,$$

$$(17)\qquad xQ' + mQ + xP = 0.$$

If P and Q are eliminated from these equations we find

$$(18)\qquad P'' + \frac{P'}{x} + \left[1 - \frac{(m-1)^2}{x^2}\right]P = (n-m-1)x^{n-2},$$

$$(19)\qquad Q'' + \frac{Q'}{x} + \left[1 - \frac{m^2}{x^2}\right]Q = -\,x^{n-1}.$$

P and Q are thus to be determined as particular solutions of non-homogeneous Bessel equations of orders $(m-1)$ and m respectively. In general these solutions will be what are known as Lommell functions. We are interested, however, in

the following rather common special cases in which it turns out that P and Q can be expressed in terms of more familiar and accessible functions:

$$\left.\begin{array}{ll} A.1 & n = m + 2k + 1 \\ A.2 & n = -m + 2k + 1 \end{array}\right\} \qquad k = 0, 1, 2, \cdots,$$

$$\left.\begin{array}{ll} B.1 & n = m + 2k \\ B.2 & n = -m + 2k \end{array}\right\} \qquad k = 0, \pm 1, \pm 2, \cdots.$$

In each of these cases we assume a solution of the form $Q = x^c(a_0 + a_1 x + \cdots)$. In $A.1$ we take $c = m + 2k + 2$ and find at once from (19)

$$Q = -x^{m+2k+2}\left\{\frac{1}{2^2(k+1)(m+k+1)} - \frac{x^2}{2^4(k+1)(k+2)(m+k+1)(m+k+2)} + \cdots\right\}.$$

If we multiply up and down by $2^{m+2k}k!\,\Gamma(m+k+1)$ we have

$$(20) \qquad Q = (-1)^k 2^{m+2k} k!\Gamma(m+k+1)[J_m - S_{k+1}(J_m)].$$

Then from (17) we find

$$(21) \qquad P = (-1)^{k+1} 2^{m+2k} k!\Gamma(m+k+1)[J_{m-1} - S_{k+1}(J_{m-1})].$$

Substituting (20) and (21) into (14) and observing that products of the Bessel functions cancel, we obtain the useful result

$$(22) \quad \int x^{m+2k+1} J_m dx = (-1)^k 2^{m+2k} k!\Gamma(m+k+1)[S_{k+1}(J_{m-1})J_m - S_{k+1}(J_m)J_{m-1}]x.$$

With proper care (22) can be used for all values of m. The apparently troublesome cases arising when m is a negative integer and $m+k+1 \leq 0$ are best handled however by writing $J_{-m} = (-1)^m J_m$ in the original integral and considering only $m = 1, 2, 3, \cdots$.

In A.2 we must have $c = -m + 2k + 2$. P and Q are then

$$Q = (-1)^k 2^{-m+2k} k!\Gamma(-m+k+1)[J_{-m} - S_{k+1}(J_{-m})],$$
$$P = (-1)^k 2^{-m+2k} k!\Gamma(-m+k+1)[J_{-m+1} - S_k(J_{-m+1})].$$

Substituting these values into (14) and observing that products of the Bessel functions combine to give precisely the Wronskian of the solution-pair $[J_m, J_{-m}]$, that is C/x, and hence contribute only to the integration constant we find

$$(23) \quad \int x^{-m+2k+1} J_m dx = (-1)^{k+1} 2^{-m+2k} k!\Gamma(-m+k+1)x[S_k(J_{-m+1})J_m + S_{k+1}(J_{-m})J_{m-1}].$$

With the proper interpretation (23) can be applied without restriction on m. However the apparently troublesome cases when $m=1, 2, 3, \cdots$ and $-m+k+1\leqq 0$ are best handled by writing $J_m=(-1)^m J_{-m}$ in the original integral and considering only $m=-1, -2, \cdots$.

In B.1 let $c=m+2k+1$. We then find that

$$Q = (-1)^{k+1}\Gamma(k+\tfrac{1}{2})\Gamma(m+k+\tfrac{1}{2})2^{m+2k-1}[H_m \mp S_k(H_m)],$$

$$P = (-1)^{k}\Gamma(k+\tfrac{1}{2})\Gamma(m+k+\tfrac{1}{2})2^{m+2k-1}[H_{m-1} \mp S_k(H_{m-1})],$$

and finally

$$(24)\quad \int x^{m+2k}J_m dx = (-1)^{k+1}\Gamma(k+\tfrac{1}{2})\Gamma(m+k+\tfrac{1}{2})x2^{m+2k-1} \cdot[-J_m\{H_{m-1}\mp S_k(H_{m-1})\} + J_{m-1}\{H_m \mp S_k(H_m)\}],$$

where H_m is Struve's function of order m,* and the plus or minus sign is to be used according as the subscript of S_k is minus or plus.† This result is valid for all m except when m is half an odd integer and simultaneously $m+k+\frac{1}{2}\leqq 0$. In these instances a logarithmic term is inevitable.

In B.2 we take $c=m+2k+1$ and obtain finally

$$(25)\quad \int x^{-m+2k}J_m dx = (-1)^{k+1}2^{-m+2k-1}\Gamma(k+\tfrac{1}{2})\Gamma(-m+k+\tfrac{1}{2})x[J_m\{H_{-m+1} \mp S_{k-1}(H_{-m+1})\} + J_{-m+1}\{H_{-m}\mp S_k(H_{-m})\}],$$

where again the plus or minus sign is to be used according as the subscript of S_k is minus or plus. This result is valid for all m although inconvenient to use when m is half an odd integer and $-m+k+\frac{1}{2}\leqq 0$ because of the indeterminate forms which occur.

The present analysis may of course be used to obtain formulas expressing the integral in (14) in terms of J_m and J_{m+1}. In cases A.1 and A.2 these are perhaps of sufficient interest to be included here.

$$(22.1)\quad \int x^{m+2k+1}J_m dx = (-1)^k 2^{m+2k}k!\Gamma(m+k+1)[S_{k+1}(J_m)J_{m+1} - S_k(J_{m+1})J_m]x.$$

$$(23.1)\quad \int x^{-m+2k+1}J_m dx = (-1)^k 2^{-m+2k}k!\Gamma(-m+k+1)[S_{k+1}(J_{-m})J_{m+1} + S_{k+1}(J_{-m-1})J_m]x.$$

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* Watson, H. N., Theory of Bessel Functions, p. 328.

† By the symbol $S_{-k}(\sum_0^\infty a_i x^i)$ we mean $\sum_{-1}^{-k} a_i x^i$.

NOTE ON WHITTAKER'S METHOD FOR THE ROOTS OF A POWER SERIES

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Several years ago, in running through back issues of the *Proceedings of the Edinburgh Mathematical Society*, the writer found in a paper of the English mathematician E. T. Whittaker a certain, very elegant theorem that deserves more attention than it has evidently received.*

The root of the equation

$$0 = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \tag{1}$$

which is the smallest in absolute value is given by

$$x = -\frac{a_0}{a_1} - \frac{a_0^2a_2}{a_1\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}} - \frac{a_0^3\begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}} - \frac{a_0^4\begin{vmatrix} a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \end{vmatrix}} - \cdots . \tag{2}$$

This theorem is of interest both for the neat manner in which the root is displayed as a function of the coefficients of the equation and as a means of determining the roots of the algebraic and transcendental equations encountered in the solution of technical problems. In this latter respect, the writer has found use of it usually preferable to the other methods available: for no differentiations, cumbersome divisions, or tables of transcendental functions are required; by depressing the roots by one synthetic division each root in turn can be made smallest and by a slide-rule or calculating machine be computed rapidly to the desired degree of exactness. As, often, in the preliminary design of electrical networks or coupled mechanical systems, or in the analysis of their subsequent operation, numerous formidable algebraic or transcendental equations must be solved, the time saved by use of a calculating machine may be considerable. If a ten or twenty inch slide-rule provides sufficient accuracy, the saving is yet greater.

* E. T. Whittaker, A formula for the solution of algebraic or transcendental equations, Proceedings of the Edinburgh Mathematical Society, vol. 36, 1917–18, pp. 103–106.

The following examples serve to indicate the scope and the flexibility of (2).

I. In the calculation of the transient field current of a synchronous machine by Heaviside's expansion theorem,* it was necessary to find the roots of

$$p^5 + 0.6193\,p^4 + 0.1273\,p^3 + 1.044 \times 10^{-2}\,p^2 + 2.966 \times 10^{-4}\,p + 5.739 \times 10^{-7} = 0.$$

By (2) the root smallest in absolute value is

$$p = -\frac{5.739 \times 10^{-7}}{2.966 \times 10^{-4}} - \frac{(5.739 \times 10^{-7})^2\, 1.044 \times 10^{-2}}{2.966 \times 10^{-4} \begin{vmatrix} 2.966 \times 10^{-4} & 1.044 \times 10^{-2} \\ 5.739 \times 10^{-7} & 2.966 \times 10^{-4} \end{vmatrix}} - \cdots$$

$$= -1.936 \times 10^{-3} - 0.1415 \times 10^{-3} = -0.002078.$$

The indicated computation, carried out on a ten inch slide-rule in less than three minutes, yields a value sufficiently accurate for engineering purposes. The known, more exact value (-0.002084) could be obtained by computing a third term.

II. Determine the least root of

$$J_0(x) = 1 - x^2/2^2 + x^4/2^2\cdot 4^2 - x^6/2^2\cdot 4^2\cdot 6^2 + \cdots$$
$$= 1 - z/2^2 + z^2/2^2\cdot 4^2 - z^3/2^2\cdot 4^2\cdot 6^2 + \cdots .$$

As the form of the coefficients enables one to manipulate the determinants with ease, (2) quickly yields

$$z = 4 + 4/3 + 20/57 + 316/4009 = 5.763; \quad x = 2.40+.$$

The value given in standard tables is 2.405.

Note by the Editor. Whittaker gives a proof of his theorem only for the case in which $f(x)$ is a polynomial. The combined efforts of R. P. Agnew, Barkley Rosser, and myself have sufficed to prove the general theorem in the following more useful, though less spectacular, form.

THEOREM. *Let $f(z)$ be an analytic function, and r a complex number, such that*

(i) $f(0) = -1$,

(ii) *r is a simple root of $f(z) = 0$,*

(iii) *If s is a root of $f(z) = 0$ distinct from r then $|s| > |r|$,*

(iv) *$f(z)$ has no singular points, except possibly poles, in or on the circle $|z| = |r|$.*

Let $-1 + a_1z + a_2z^2 + \cdots$ be the Maclaurin expansion for $f(z)$, and define $A_0, A_1, \cdots$ by

(3) $$A_0 = 1, \qquad A_n = a_1A_{n-1} + a_2A_{n-2} + \cdots + a_nA_0, \qquad n > 0.$$

Then $r = \lim_{n\to\infty} A_n/A_{n+1}$.

* Doherty and Keller, Mathematics of Modern Engineering, p. 129.

Proof. We first prove a purely formal identity. Namely, if α is any complex number and if $g(z)=(1-z/\alpha)/f(z)=b_0+b_1z+b_2z^2+\cdots$, then $A_n=\alpha^{-n}(b_0+b_1\alpha+b_2\alpha^2+\cdots+b_n\alpha^n)$. Define $F(z)$ by $F(z)=A_0+A_1z+A_2z^2+\cdots$. Then from (3) we have $f(z)F(z)=-1$. Hence

$$F(z) = -1/f(z) = g(z)/(1-z/\alpha),$$

that is

$$\begin{aligned} A_0 + A_1z + A_2z^2 + \cdots &= (b_0 + b_1z + b_2z^2 + \cdots)(1 + z/\alpha + z^2/\alpha^2 + \cdots) \\ &= b_0 + (b_0/\alpha + b_1)z + (b_0/\alpha^2 + b_1/\alpha + b_2)z^2 + \cdots . \end{aligned}$$

The desired expression for A_n follows on equating corresponding coefficients. (The above relations between power series merely express finite algebraic relations between the coefficients, and so questions of convergence need not be considered.)

Now take $\alpha=r$. By conditions (ii), (iii), and (iv) there is a circle $|z|=R$ inside of which $f(z)$ has the one and only one root r and no singularities except poles. It follows that $g(z)=-(1-z/r)/f(z)$ is regular for $|z|<R$ and so $g(r)$ exists, is not zero, and equals $b_0+b_1r+b_2r^2+\cdots$. We have therefore

$$A_n/A_{n+1} = r\,\frac{b_0 + b_1r + \cdots + b_nr^n}{b_0 + b_1r + \cdots + b_{n+1}r^{n+1}},$$

so that

$$\lim_{n\to\infty} A_n/A_{n+1} = r\,\frac{g(r)}{g(r)} = r.$$

This statement of the theorem is convenient for calculational purposes. That it is equivalent to Whittaker's formulation follows from the fact that for $a_0=-1$ we have

$$A_n = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_1 \end{vmatrix},$$

and A_n/A_{n+1} is the sum of the first $n+1$ terms of (2).

The following examples indicate how much easier it is to use the new form of the theorem than the old.

III. To solve $\log(1+z)=1/2$ we find the least root of

$$-1 + 2z - \tfrac{2}{2}z^2 + \tfrac{2}{3}z^3 - \tfrac{2}{4}z^4 + \cdots = 0.$$

Thus

$$A_n = 2(A_{n-1} - \tfrac{1}{2}A_{n-2} + \tfrac{1}{3}A_{n-3} + \cdots).$$

We find successively

$$A_0 = 1, A_1 = 2, A_2 = 3, A_3 = 14/3, A_4 = 43/6, A_5 = 166/15, A_6 = 767/45,$$

and so

$$A_0/A_1 = .5, A_1/A_2 = .667, A_2/A_3 = .643, A_3/A_4 = .651, A_4/A_5 = .648, A_5/A_6 = .6493.$$

The root is $e^{1/2} - 1 = .64872$.

IV. To find the least root of $J_0(\sqrt{z}) = 0$ (see example II) it is convenient to put $B_n = 4^n(n!)^2 A_n$ so as to avoid fractions. Then

$$B_n = \binom{n}{1}^2 B_{n-1} - \binom{n}{2}^2 B_{n-2} + \cdots,$$

$$A_n/A_{n+1} = 4(n+1)^2 B_n/B_{n+1},$$

and we obtain for $B_0, B_1, \cdots, B_7$, the values 1, 1, 3, 19, 211, 3651, 90921, 3081513. These give for $A_0/A_1, \cdots, A_6/A_7$ the values 1, 5.333, 5.684, 5.763, 5.779, 5.782, 5.783. The root of $J_0(x)$ is then $\sqrt{5.783} = 2.405-$.

V. The function

$$-\frac{\tan\sqrt{z}}{\sqrt{z}} = -1 - \frac{1}{3}z - \frac{2}{15}z^2 - \frac{17}{315}z^3 - \frac{62}{2835}z^4 - \frac{1382}{155925}z^5 - \frac{21844}{6081075}z^6 + \cdots$$

has $\pi^2 = 9.870$ as its smallest root. We obtain for $A_0, A_1, \cdots, A_6$, the values 1, $-1/3$, $-1/45$, $-2/945$, $-1/4725$, $-2/93555$, $-1382/648750375$; and for $A_0/A_1, \cdots, A_5/A_6$, the values -3, 15, 10.5, 10, 9.9, 9.877.

R. J. W.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Allegheny College, Meadville, Pa.

WAR TOPICS IN CLUB REPORTS

The opening of another year of club activities finds this country in the throes of a great war. With it the war has brought a great increase of interest in mathematics, and a greater realization of its importance. Among mathematicians and mathematics clubs there has been considerable emphasis on the applications of mathematics to the war effort. About half of the clubs whose reports have been received to date list in their programs topics related to this subject. The *Math X*, publication of the Washington Square College Mathematics Club, New York University, opens with the following paragraph. "The war effort has greatly

increased the need for training in mathematics with the emphasis on applied mathematics. There is a greater demand than can at present be supplied for young men and women who are prepared in the fields of mathematics, physics, engineering, and statistics."

UNDERGRADUATE PUBLICATIONS

The *Math X*, just quoted, sold out its complete edition of 400 copies. The papers included were as follows: *Königsberg's seven bridges* by Jack Moshman, *Mathematics and physical theories* by Frank Grace, *Non-dimensional variables* by Melvin Lax (including an interesting derivation of Van der Waal's equation connecting the pressure and volume of a gas at constant temperature), *A trick with numbers* by Marvin Forray and *An introduction to the calculus of finite differences* by Marvin Forray, *An extension of the max- and min- problem* by Harold Lewis, *Infinity and transfinite numbers* by S. M. Forman.

Other publications received were the Chapter News Letters of the *Kappa Mu Epsilon* chapters at Illinois State Normal University, Kansas State Teachers College at Emporia, and Mt. St. Scholastica College at Atchison, Kansas. Printed programs were received from Brown University, Central Michigan College, and the University of Kansas.

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CLUB REPORTS 1941–42

Mathematics Society, University of Wisconsin at Milwaukee

Of the six papers presented, three were offered in competition for the annual Euler Prize in Mathematics. *Pi, its derivation and transcendence* by D. E. Mereen (the prize winner), *Chance-probability* by Herbert Reider, and *"e"* by Virginia Kubacki. The other three papers submitted dealt with subjects particularly suited to a nation at war. They were *Mathematics in artillery*

warfare by Professor W. E. Roth, *Navigation through the U. S. and Canada* by Miss Iverson (a discussion of methods of mapping with particular stress upon the navigation from an aeronautical standpoint), and *Simple mathematics in the present war* by Professor Morris Marden (a presentation of warfare problems set up for solution by the group). President, D. E. Mereen; Vice-President, R. E. Barr, Jr.; Secretary-Treasurer, Jean Petran; Faculty Adviser, Professor Marden.

Mathematics Club, Connecticut College

Members of our Mathematics Club gave short speeches on various subjects as follows: *Euclid* by Claire Peterson, *Quadrature of the circle* by Doris Kaske, *Trisecting an angle* by Dorothy Green, and *Squaring the cube* by Mary Powers. Our only outside speaker was Professor J. S. Frame of Brown University who spoke on *Mathematical problems in national defense.* At another meeting the Sophomores presented the play *The Flatlanders* taken from this MONTHLY. President, Doris Kaske; Vice-President, Alyce Watson; Secretary, Barbara Pilling; Treasurer, Mary Powers; Faculty Adviser, Professor Julia W. Bower.

Pi Mu Epsilon, Lehigh University

The Chapter held regular monthly meetings with mathematical programs, and sponsored one open lecture, given by Professor Otto Neugebauer, Brown University, on the subject *Origins of ancient mathematics.* The lecture was well attended and enjoyed by all who heard it. Officers for 1941–42 were: President, A. B. Brown, '42; Secretary, W. A. Eisele, '42; Treasurer, R. M. Maiden, '42; Faculty Adviser, Tomlinson Fort. Elected for 1942–43 were: President, M. G. Arsove, '43; Secretary, Stanley Caplan, '43; Treasurer, C. S. Bennett, '43.

Pi Mu Epsilon, University of Nebraska

Demonstrations of mathematical machines were held at a number of the meetings. A model of the Rigge curve tracing machine was explained by Joseph Martin, the slide rule by Vernon Vrana, a poor man's planetarium—an astronomical projector constructed and discussed by O. C. Collins, the harmonic analyzer by L. A. Rife, At a joint meeting with the local chapter of the American Institute of Electrical Engineers, Herbert Gaba spoke on the *Multiplication of complex numbers.* An initiation banquet, a quiz program and a spring initiation picnic completed the year's calendar. Director, D. L. Christensen; Vice-Director, Theodore Roesler; Secretary, Jean Hakanson; Treasurer, Dayle D. Rippe.

Kappa Mu Epsilon, Mount St. Scholastica College

At the opening initiation meeting of the Kansas Gamma Chapter ingenious tests were given to determine the candidates' fitness. The November meeting was a round table discussion on *Mathematics and national defense* with special emphasis on the role of the American college woman. A Christmas party was held in December, and in January a pop-corn sale, the proceeds of which were contributed for the support and continuance of the *Boletin Mathematico.* Two plays were presented by the chapter at an all school assembly: *Modern mathematics looks up its ancestors* by Marion Stark, and *A trip to infinity* by Tena Anders, a pledge. A meeting was held in February in honor of Lincoln, with emphasis upon his accomplishments in mathematics and upon those of his contemporaries. A "St. Patrick's" party was the second social event of the year, featuring an original skit *Irish medley*, with songs, recitations and dances. Pledges demonstrated their mathematical fitness for membership into *Kappa Mu Epsilon* on pledge day in April, and four new members were initiated at the May meeting. Officers for 1941–42 were: President Tartaglia, Bobbe Powers; Vice-President Cauchy, Mary Margaret Downs; Secretary Galileo, Margaret Mary Kennedy; Treasurer Napier, Mary Hughes; Faculty Sponsor, Sister Helen Sullivan, OSB. Elected for 1942–43 were President Proclus, Margaret Molley; Vice-President Cauchy, Mary Margaret Downs; Secretary Leibnitz, Virginia Meyers; Treasurer Bernouilli, Jane Schweizer.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Engineering Drawing. By D. E. Hobart. New York, D. C. Heath & Company, 1941. 7+430 pages. $2.75.

The text is one which will find a useful and welcomed place in the field. The book is well done with careful attention to conventions and accepted indications in standard practice. Particular commendation should be given the treatment of limit dimensioning and welding. Isometric drawing is clearly illustrated to expedite its understanding and use.

The chapter on fasteners is complete with ample tables to cover student needs. Much has been made of that portion on charts and graphs.

In construction drawing, architectural drawing perhaps purposely was neglected but the treatment of structural steel is highly commended.

In intersections and developments there is given the needed descriptive geometry for a full understanding of the subject and the timely treatment of surfaces of double curvature will prove of value to those interested in lofting.

In general the book is concisely expressed and clearly written. The theory is ably presented and consideration given to detail shows the author completely familiar with the problems which arise in teaching the subject. For greater choice or needed variety for those using the book in successive years the problem content could be expanded.

R. W. BOCKHORST

An Introduction to Analytical Geometry and Calculus. By T. K. Raghavachari. Mount Road, Madras, Oxford University Press, 1941. 20+192 pages. Rs 2.

This little book was written specifically as a textbook in the intermediate courses of South Indian Universities and without doubt adequately serves the purpose for which it was intended. There seems, however, to be little place in the American colleges or universities for such a treatment of analytic geometry and calculus as it is too long for a course of three hours and too short for six. It could possibly be used in a one-year cultural course by giving a very short treatment of trigonometry between the analytic geometry and calculus. The author is an excellent expositor and if he could maintain his clarity throughout a longer book covering what Americans consider standard he would have a book that would undoubtedly find its place among the textbooks of this country.

The book is pocket size and contains 192 pages about equally divided be-

tween geometry and calculus. Only the straight line and circle are included in analytic geometry but functions, graphs, and tangents to curves are discussed under calculus. The part on calculus, though short, can by no means be considered a "calculus made easy" in the bad sense of the phrase, as more than usual care is taken to tell the truth. For example, the area under a curve is defined (not proved) to be the limit of the sum of areas of rectangles and moreover the existence of this limit is made quite plausible.

There are of course omissions from standard courses. Conic sections are not mentioned and the definite integral is applied only to areas under or between curves and to volumes of revolution. On the other hand, the treatment of the straight line is most exhaustive. A somewhat startling inclusion is an extension of directed distances to directed areas; "Areas will therefore be positive if the perimeter is described in the counter-clockwise direction and negative if clockwise." The author should probably not be blamed for his "proof" that $dx=\Delta x$.

It is with some regret that such an attractive book both in appearance and exposition cannot be recommended for use in America,

J. F. Randolph

Tables of Natural Logarithm. Volume IV. Contains the Decimal Numbers from 5.0001 to 10.0000. (Prepared by the Federal Works Agency Projects Administration for the City of New York. Conducted under the Sponsorship of the National Bureau of Standards.) New York, Work Projects Administration, 1941. 22+506 pages. $2.00.

This is Volume IV of a series of four volumes of natural logarithms. It contains the natural logarithms of decimal numbers from 5 to 10 at intervals of 0.0001.

The corrections of errors in the Wolfram Tables, and the explanations of the procedure for direct and inverse interpolation are repeated from the earlier volumes. The arrangement of the page and the safeguards for a high degree of accuracy are the same as in the preceding volumes of the series.

Virgil Snyder

TechniData Hand Book. By Edward L. Page. New York, The Norman W. Henley Publishing Company, 1942. 64 pages. Spiral binding, $1.00. Cloth binding, $1.50.

This book contains a well-selected group of formulas, definitions, laws, figures, tables, etc., from mathematics, mechanics, physics, chemistry, and engineering, in a form convenient for quick reference. In many places the typography is not clear, exponents being particularly hard to read. Quite a few obvious errors were noted, the most serious of which occurred in the solution of the general cubic equation. It seems especially unfortunate that so many errors should occur in a handbook.

N. G. Gunderson

Brief Course in Analytics. By M. A. Hill and J. B. Linker. New York, Henry Holt and Company, 1940. 9+204 pages. $1.36.

This elementary text in Analytic Geometry, though designated brief, covers all the topics usually given in a first course in analytics. It includes the following chapters: Basic Definitions and Theorems, Equations and Loci, Straight Lines, Special Equations of the Second Degree, General Equation of the Second Degree, Equations in Other Forms, Elements of Solid Analytic Geometry.

The explanations are clear but concise. The arrangement of topics is good. "Many problems have been included in the text, some of which are quite easy, others difficult. By solving the former, the student will gain confidence in himself and will be able to apply his knowledge in solving the latter."

Answers are given for the odd numbered problems. The text should satisfy both the student and the instructor.

R. P. STEPHENS

College Algebra. By C. H. Sisam. New York, Henry Holt and Company, 1940. 12+395 pages. $1.90.

The author of an earlier work on Analytic Geometry has here produced a new College Algebra. The outstanding feature seems to be the flexibility permitted in using the text to meet college algebra classes of varying ability and different high-school background. Sufficient material is presented on the ordinary elementary algebraic processes to make the text suitable for the high-school graduate who has been exposed to but one year of algebra. The insertion of a section on business mathematics and some mathematical tables is a unique feature. The exercises seem to be sufficiently numerous to permit instructors to select those best suited to the needs of the individual students.

A. V. KOZAK

Intermediate Algebra for College Students. By T. S. Peterson. New York, Harper and Brothers, 1942. 8+358 pages. $1.85.

This well written book covers those topics usually found in intermediate algebra textbooks; to quote, "the author has purposely neglected to stress certain principles which are more effectively studied in college algebra and which have little value for the student who does not intend to continue work in mathematics." In accordance with this purpose, about three-fourths of the book is devoted to algebra through simultaneous quadratic equations, and about fifty pages are devoted to ratio, variation, the binomial theorem, logarithms, and progressions. A number of sections are marked "optional." The rigor is sufficient for the purposes intended.

There are a number of features which may appeal to teachers. Word statements of rules, usually approached intuitionally, are very prominentlydisplayed; symbolic statements receive less emphasis. Common errors, and points which the student might overlook otherwise, are brought to his attention through

"notes." A great deal of emphasis is placed upon the checking of solutions; there is a good note on page 295 on estimating the reasonableness of logarithmic solutions.

The "stated problem" is stressed throughout the book, the recommended procedure consisting of analysis, solution, statement of the answer in words, and check. The familiar tabulated forms of the secondary school textbooks are utilized in the analysis of problems, and certain problems are diagrammed as well.

There are many examples of illustration; many students who wish to acquire facility in the use of elementary algebraic operations could use this book without additional instructional assistance. The exercises, and the review lists at the end of each chapter, contain an ample number of problems for classwork as well as for outside preparation; all of the problems are of familiar types and they are well chosen. Answers are given to most of the odd numbered problems; in the sample selected for solution by this reviewer no errors were detected.

Some teachers may think that too much attention is given to "transposition" and that too little attention is given to the operation of adding the same quantity to each member of an equation. A suggestion, perhaps a very minor one, for future editions of the book, is that some principle of order be adopted for the entries in the tables printed on pages 216, 217, 255 and 256; most teachers probably would prefer the order used in the tables printed on pages 219 and 223 to that used in the table on page 220.

The general appearance of the book is attractive; the only detected printing error is an obvious and harmless one on page 31.

C. W. MUNSHOWER

An Introduction to the Theory of Newtonian Attraction. By A. S. Ramsey. Cambridge, The University Press; New York, The Macmillan Company, 1941. 9+184 pages. $2.50.

This is a textbook setting forth the elementary groundwork of the classical theory of gravitational attraction and potential. In this purposely limited respect it is a welcome addition to the textbook literature in English. It is the last of a series of the well-known books on Mechanics which Mr. Ramsey has written in recent years, and he is to be congratulated on his perseverance to so complete and successful a conclusion of his task.

The first chapter is concerned with some "Preliminary Mathematics" including the Laplacian in curvilinear coordinates and the convergence of volume integrals. There are some remarks on "vectorial methods," but the vector analysis is not employed in this book. In the remaining six chapters one finds in order: the consideration of the usual elementary theorems and problems on the gravitational attraction and potential at points outside and inside attracting continuous matter (II, III); the theorems of Laplace, Poisson, and Gauss (IV) and of Green (V); harmonic functions of integral degree (VI); and the attraction of

ellipsoids, with a few brief final remarks on the equilibrium of a rotating fluid mass, (VII).

The author's style is pleasingly direct; he writes with an admirable economy of language. The physical ideas are everywhere carefully presented, but in certain places important questions should have received, in the reviewer's opinion, more attention to mathematical rigor. The book is well illustrated with over fifty diagrams, and the usual long sets of valuable College examination problems have been selected with care. There is an occasional and discriminating use of geometrical argument. Although there are a few incidental references scattered throughout the book there is no bibliography, which is most unfortunate. Moreover, this subject has a long and glorious history and the author's bald biographical footnotes are hardly the way of meeting the responsibility of referring to it.

Since formal courses in this subject at the undergraduate level for which this book is expressly written are apt to be rare, it should be stated that serious students in mathematics and physics will profit from reading this book independently during or immediately following the first course in analytic mechanics.

S. G. HACKER

Tools. A Mathematical Sketch and Model Book. By R. C. Yates. Baton Rouge, Louisiana State University, R. C. Yates, 1941. 194 pages. $1.60.

This interesting and different book "has been designed especially for college students who are prospective students of mathematics" and treats of geometrical instruments, their use and their equivalence. Freshman work in algebra, trigonometry, and analytics is the only mathematics presupposed.

The nature of the material presented is best seen from the titles of the eleven sections of the book. These are: The Straightedge and Modern Compasses (Modern Geometry), Dissection of Plane Polygons, The Compasses (Geometry of Mascheroni), Folds and Creases (Geometry of Paper Folding), The Straightedge (Synthetic Projective Geometry), Line Motion Linkages (How to Draw a Straight Line), The Straightedge with Immovable Figure (Geometry of Poncelet-Steiner), The Assisted Straightedge, Parallel and Angle Rulers, Higher Tools and Quartic Systems, and General Plane Linkages.

Not presuming to be complete, each section is preceded by a bibliography for further reading and consists of explanatory text, together with many plates which are to be worked out by the student in laboratory periods.

The present book is a loose-leaf affair, printed by the author, and hence not of a very durable nature. However, much of unusual interest is to be found here, which will be valuable to any teacher of mathematics and will make excellent supplementary class material.

R. A. HARRISON

NEW BOOKS RECEIVED

General Trade Mathematics. By E. P. Van Leuven. New York, McGraw-Hill Book Co., 1942. 10+575 pages.

Logarithms, Trigonometry, Statistics. First Year College Mathematics. By H. R. Cooley, P. H. Graham, F. W. John, and A. Tilley. New York, McGraw-Hill Book Co., 1942. 12+280 pages. $2.00.

A Short Course in Tensor Analysis for Electrical Engineers. By G. Kron. New York, John Wiley and Sons, Inc., 1942. 15+250 pages. $4.50.

The Fundamental Principles of Mathematical Statistics with Special Reference to the Requirements of Actuaries and Vital Statisticians and an Outline of a Course in Graduation. By H. H. Wolfenden. New York, Actuarial Society of America, and the Macmillan Company of Canada, 1942. 15+379 pages.

Introduction to the Theory of Relativity. By P. G. Bergmann. New York, Prentice-Hall, Inc., 1942. 16+287 pages. $4.50.

Tables of Natural Logarithms, Volume IV. Logarithms of the decimal numbers from 5.0000 to 10.0000. Prepared by the Federal Works Agency, Work Projects Administration for the City of New York. Conducted under the sponsorship of the National Bureau of Standards, 1941. 22+510 pages. $2.00.

Elements of Spherical Trigonometry. By J. E. Thompson. New York, D. Van Nostrand Company, 1942. 12+144 pages. $1.65.

An Outline of College Algebra. By G. E. Moore. New York, Barnes and Noble, Inc., 1942. 224 pages. $1.00.

Essentials of Astronomy. By J. C. Duncan. New York, Harper and Bros., 1942. 181 pages. $1.85.

Spherical Trigonometry with Naval and Military Applications. By L. M. Kells, W. F. Kern, and J. R. Bland. New York, McGraw-Hill Book Co., Inc., 1942. 13+163 pages. $1.50.

Mathematical Dictionary. By G. James and R. C. James. Van Nuys, California, The Digest Press, 1942, 5+259+22 pages. $3.00.

First Year College Mathematics. By C. C. Richtmeyer and J. W. Foust. New York, F. S. Crofts and Company, 1942. 11+461 pages. $3.25.

Industrial Statistics. Statistical Technique Applied to Problems in Industrial Research and Quality Control. By H. A. Freeman. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1942. 9+178 pages. $2.50.

Contributions to the Calculus of Variations, 1938–1941. Theses submitted to the Department of Mathematics of the University of Chicago and Bibliography. Chicago, University of Chicago Press, 1942. 7+527 pages.

Mathematics of Business and Finance. By W. B. Dyees and R. C. Gilmore. (Including Compound Interest and Annuity Tables by F. C. Kent and M. E. Kent.) New York, McGraw-Hill Book Co., 1942. 10+221+8+214 pages. $3.50.

Basic College Mathematics. (A General Introduction.) By C. W. Munshower and J. F. Wardwell. New York, Henry Holt and Co., 1942. 11+612 pages. $3.20.

PROBLEMS AND SOLUTIONS

Edited by Otto Dunkel, Orrin Frink, Jr., and H. S. M. Coxeter

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 531. *Proposed by P. R. Hill, University of Georgia*

Suppose six students be standing an examination in a row of seats with an aisle at each end. If they finish in random order, what is the probability that a student will have to pass over one or more other students in order to reach an aisle?

E 532. *Proposed by V. Thébault, San Sebastián, Spain*

Find a perfect square whose digits form one of the permutations of five consecutive digits.

E 533. *Proposed by N. A. Court, University of Oklahoma*

Prove that, if an orthocentric group of points occurs as a section of an orthocentric group of lines, then the plane of section is perpendicular to one of the lines.

E 534. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that 4, 5, 7 are the only values of n for which $n!+1$ is a perfect square.

E 535. *Proposed by A. H. Stone, Institute for Advanced Study*

Let A', B', C' be three points on the circumcircle of a triangle ABC, whose Simson lines with respect to ABC all meet in a point, O. Prove that the Simson lines of A, B, C, with respect to the triangle $A'B'C'$, concur at the same point O.

E 511 [1942, 195]. *Proposer's Correction*

After "Doppler effect," the problem should read as follows:

The train of sound waves of maximum apparent pitch *leaves* the bomb at the elevation at which the component of the bomb's velocity in the direction of the observer is a maximum. Find this angular elevation, and show that it approaches two-thirds of the initial elevation if the initial elevation is small.

SOLUTIONS

A Magic Square of Triangular Numbers

E 496 [1941, 699]. *Proposed by R. V. Heath, Wall St., New York City*

What is the smallest value of n for which the n^2 triangular numbers 0, 1, 3, 6, 10, $\cdots$, $\frac{1}{2}n^2(n^2-1)$ can be arranged to form a magic square?

Partial solution by the proposer

Such an arrangement can certainly be found for $n=8$.

A magic square is said to be doubly-magic if its sums remain uniform when all its numbers are squared. Clearly, the magic property will still be retained if each of the original numbers is subtracted from its square. The resulting numbers are all even, and their halves are the triangular numbers; in fact each original number r leads to $\frac{1}{2}r(r-1)$. Here, for instance, is a doubly-magic pandiagonal square (cf. Rouse Ball, *Mathematical Recreations and Essays*, 11th edition, p. 212, Fig. xxviii) along with the corresponding magic square of triangular numbers:

16	41	36	5	27	62	55	18
26	63	54	19	13	44	33	8
1	40	45	12	22	51	58	31
23	50	59	30	4	37	48	9
38	3	10	47	49	24	29	60
52	21	32	57	39	2	11	46
43	14	7	34	64	25	20	53
61	28	17	56	42	15	6	35

120	820	630	10	351	1891	1485	153
325	1953	1431	171	78	946	528	28
0	780	990	66	231	1275	1653	465
253	1225	1711	435	6	666	1128	36
703	3	45	1081	1176	276	406	1770
1326	210	496	1596	741	1	55	1035
903	91	21	561	2016	300	190	1378
1830	378	136	1540	861	105	15	595

(In this last square, each of the eight rows, eight columns, and two main diagonals has the sum 5460.)

But it remains possible that a smaller set of triangular numbers might form a magic square without the corresponding natural numbers forming a magic square. Moreover, it has never been satisfactorily proved that there is no doubly-magic square of order 7.

Partition of a Triangle

E 497 [1941, 699]. *Proposed by V. Thébault, San Sebastián, Spain*

The sides of a triangle $A'B'C'$, of constant size, remain parallel to those of a fixed triangle ABC, and form with it three more triangles and three pentagons. Show that the position of $A'B'C'$ which minimizes the sum of the areas of these three triangles makes the areas of the three pentagons all equal.

Solution by Howard Eves, Chattanooga, Tenn.

Let us designate the three triangles (all similar to ABC) by $T(A)$, $T(B)$, $T(C)$, where $T(A)$ corresponds to the vertex A in the obvious manner. First, let $A'B'C'$ move so that $T(A)$ remains constant in area. This means that the side $B'C'$ slides along a fixed line parallel to BC. In order for the area $T(B)+T(C)$ to be a minimum, we must have $T(B)=T(C)$. Treating $T(B)$ and $T(C)$ similarly, we see that, in order for the area $T(A)+T(B)+T(C)$ to be a minimum, we must have $T(A)=T(B)=T(C)$. This guarantees that the parallelograms AA', BB', CC' be all equal in area (since in pairs they have equal bases and altitudes); and this in turn implies equality of area for the three pentagons.

Note. Triangles ABC and $A'B'C'$ have a common centroid, since the lines AA', BB', CC' are medians of both.

Also solved by the proposer.

A Convergent Sequence

E 498 [1941, 699]. *Proposed by E. C. Kennedy, Texas College of Arts and Industries*

Consider the relation

$$T_{n+1} = \sqrt{\frac{k + T_n}{2 - T_n}}.$$

What is the largest value of k such that the sequence $\{T_n\}$, for a suitable range of values of T_0, converges to a positive number? What is this number?

Solution by P. D. Thomas, Southeastern State College, Durant, Okla.

Assume that $\{T_n\}$ has a positive limit, T. Then

$$T^2 = (k + T)/(2 - T),$$

or

$$T^3 - 2T^2 + T + k = 0. \tag{1}$$

This equation has no positive root if k is positive, but at least one otherwise. (If $-4/27<k<0$, it has three.) Thus the required largest value of k is 0, and then $T=1$.

(Cf. No. 379 on p. 380 of the *National Mathematics Magazine*, April, 1941.)

Also solved by the proposer.

Editorial note. If $-2<k\leqq 0$, the sequence $\{T_n\}$ converges to the largest root of (1), provided T_0 lies between this root and 2.

Centers of Similitude

E 499 [1941, 699]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Two intersecting circles (A) and (B) have centers mutually external. Two other circles (C) and (D), orthogonal to (A) and (B) respectively, are drawn through the points of intersection. Show that the two common tangents of (C) and (D) are concurrent with the two common tangents of (A) and (B).

Solution by W. F. Clarke, San Jose, Calif.

Let E and F be the intersections of (A) and (B), and P the point where their common tangents meet AB. By the requirements of the problem, (C) and (D) must pass through E and F, with EC perpendicular to EA, and ED to EB. Clearly, angles AEB and CED have a common bisector, which meets AB at K, the harmonic conjugate of P with respect to A and B. Since P and K are the external and internal centers of similitude of (A) and (B), it follows that EP is the external bisector of both angles AEB and CED. Thus P is the harmonic conjugate of K with respect to C and D, whence the common tangents of (C) and (D) must also pass through P.

Also solved by Howard Eves, P. D. Thomas, and the proposer.

Editorial Note. The result can be obtained more rapidly as follows. With respect to the circle through E with center P, (A) inverts into (B). Therefore (C) inverts into (D), and the tangents from P to (C) touch (D) also.

Differences of Factorials

E 488 [1942, 377]. *Generalization by H. W. Becker, Vallejo, Calif.*

Let $\{u_k\}$ be any sequence, and $v_k=(E-x)^k u_0$. Then

$$(E+x)^n v_0 = u_n.$$

Proof. Since $E^k v_0=(E-x)^k u_0$, we have

$$(E+x)^n v_0 = \sum_{r=0}^{n}\binom{n}{r}x^r E^{n-r}v_0 = \sum_{r=0}^{n}\binom{n}{r}x^r(E-x)^{n-r}u_0$$
$$= (E-x+x)^n u_0 = u_n.$$

The case when $v_k=k!$ suggests the notation

$$k!_x = (E+x)^k 0! = k!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^k}{k!}\right),$$

whence

$$0!_x = 1, \qquad k!_x = k(k-1)!_x + x^k,$$

and $k!_0$ is the ordinary factorial. The above theorem then gives

$$(E-x)^n 0!_x = n!.$$

Problem E 488 itself is included, $k!_1$ being the "super-factorial," and $k!_{-1}$ the "sub-factorial."

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4045. *Proposed by A. M. Glicksman, The Bronx High School of Science*

Show that γ, Euler's constant, is given by

$$\gamma = \sum_{r=2}^{\infty} (-1)^r \frac{g_r}{r}, \qquad g_r = \sum_{k=1}^{\infty} \frac{1}{k^r}, \qquad \gamma = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \log n \right).$$

4046. *Proposed by Otto Dunkel, Washington University*

Show that Euler's constant γ is given by

$$\gamma \doteq 2\left[1 - \log 2 - \frac{\tau_3}{3} - \frac{\tau_5}{5} - \cdots \right], \qquad \tau_r = \sum_{i=1}^{\infty} \frac{1}{(2i+1)^r}.$$

4047. *Proposed by Theodore R. Running, Ann Arbor, Mich.*

Triangles have the sides $x-1$, x, $x+1$, the altitude h with x as base, an area A, where x, h, A are whole numbers. The first six possible triangles are given by the table

n	h	x	A
0	0	2	0
1	3	4	6
2	12	14	84
3	45	52	1170
4	168	194	16296
5	627	724	226974
.	.	.	.

Do the relations

$$h_{n+2} = 4h_{n+1} - h_n, \qquad x_{n+2} = 4x_{n+1} - x_n, \qquad A_{n+2} = 14A_{n+1} - A_n,$$

hold for all the triangles fulfilling the given conditions?

4048. *Proposed by V. Thébault, San Sebastián, Spain*

Find a number of six digits $N = abcdef$ such that the product NN' is a perfect square, where $N' = defabc$.

4049. *Proposed by V. Thébault, San Sebastián, Spain*

In an orthocentric tetrahedron $ABCD$ the straight lines joining the centroid with the circumcenter of the triangles of the faces cut the respective radical planes of the circumsphere and the spheres with the medians of $ABCD$ as diameters in four points of the same plane.

SOLUTIONS

Inequalities for Intervals

3949 [1940, 182]. *Corrected. Proposed by P. Turán, Budapest, Hungary*

Given the angles $0 \leqq \phi_1 < \phi_2 < \cdots < \phi_n < 2\pi$ with the common initial line Ox, show that there exists an angle β with the properties: $\beta \geqq \pi/2^{n(n-1)/2+1}$, and there exist no integers k and ν such that $\phi_\nu + \beta < \phi_k < \phi_\nu + 2\beta$, or $\phi_\nu - 2\beta < \phi_k < \phi_\nu - \beta$.

Solution by the Proposer

We consider in turn three cases:

I. If min $|\phi_\mu - \phi_\nu| \geqq \pi/2^\lambda$, $\lambda = n(n-1)/2$, we may take $\beta = \pi/2^{\lambda+1}$.
II. If max $|\phi_\mu - \phi_\nu| \leqq \pi/2$, we may take, for example, $\beta = 5\pi/8$.
III. If min $|\phi_\mu - \phi_\nu| < \pi/2^\lambda$, and max $|\phi_\mu - \phi_\nu| > \pi/2$, let $\phi_\mu - \phi_\nu = \Delta_{\mu\nu}$, $\mu > \nu$.

We now arrange these differences in monotonic increasing order, and for simplicity we use simple subscripts $0 < \Delta_1, \Delta_2, \cdots, \Delta_\lambda < 2\pi$. The conditions here evidently mean that

$$\Delta_1 < \pi/2^\lambda = 2\pi/2^{\lambda+1}, \qquad \Delta_\lambda > 2^\lambda\pi/2^{\lambda+1};$$

and it is obvious that there exists an integer r such that

$$1 \leqq r < \lambda, \qquad \Delta_r \leqq 2^r\pi/2^{\lambda+1}, \qquad \Delta_{r+1} \geqq 2^{r+1}\pi/2^{\lambda+1}.$$

This means that for *no* k and ν is

$$2^r\pi/2^{\lambda+1} < |\phi_k - \phi_\nu| < 2^{r+1}\pi/2^{\lambda+1},$$

and hence a suitable choice is

$$\beta = 2^r\pi/2^{\lambda+1}.$$

Tetrahedral Polars

3993 [1941, 273]. *Proposed by N. A. Court, University of Oklahoma*

A variable plane passing through a fixed point of the face ABC of the tetrahedron $DABC$ meets the edges DA, DB, DC in the points P, Q, R. Show that the locus of the point U common to the three planes PBC, QCA, RAB is a cone of the second degree.

I. *Solution by H. S. M. Coxeter, University of Toronto*

Let (x, y, z, t) be the barycentric coordinates of U, referred to the tetrahedron $ABCD$, so that the points P, Q, R are

$$(x, 0, 0, t), \qquad (0, y, 0, t), \qquad (0, 0, z, t), \qquad (t \neq 0).$$

The condition for these to be coplanar with the fixed point $(f, g, h, 0)$ is

$$\begin{vmatrix} x & 0 & 0 & 1 \\ 0 & y & 0 & 1 \\ 0 & 0 & z & 1 \\ f & g & h & 0 \end{vmatrix} = 0,$$

or $fyz+gzx+hxy=0$. Thus the locus of U is a quadric cone with vertex D.

II. *Solution by the Proposer.*

Let s be the trace of the variable plane PQR in the face ABC, and S the harmonic pole of the line s for the triangle ABC.

The harmonic pole M of the plane PQR for the tetrahedron $DABC$ lies on the line DS, and also in the plane BCP' projecting from BC the harmonic conjugate P' of P to the vertices D, A. Hence the plane BCP meets the line DS in the harmonic conjugate U of M for the pair of points D, S. Similarly the planes CAQ and ABR pass through U.

If the trace s remains fixed and the variable plane PQR revolves about s, the point U describes the fixed line DS.

The line s passes through the fixed point L given in the plane ABC. As the line s revolves about L, the point S describes a conic (L) circumscribed about the triangle ABC (J. J. Mathieu, *Nouvelles Annales de Mathématiques*, 1865, p. 407). Consequently the locus of the point U is the cone obtained by projecting the conic (L) from the vertex D.

Remark. We have incidentally proved the proposition:

If a plane revolves about a fixed point in a face of a tetrahedron, the harmonic pole of the variable plane for that tetrahedron describes a cone of the second degree having for vertex the vertex of the tetrahedron opposite the face containing the fixed point.

Indeed, the cone $D(L)$ is also the locus of the point M.

Solved also by G. B. Huff, W. T. Short, C. E. Springer and P. D. Thomas.

Editorial Note. The remaining solutions are analytic, and Huff stated that the same analysis applies to any number of dimensions greater than one. The theorem by Mathieu used by the proposer is easily proved. For, if the plane of P, Q, R cuts the sides BC, CA, AB in A_1, B_1, C_1 on the straight line s, let A_1', B_1', C_1' be the respective harmonic conjugates of the last three points with respect to the corresponding pair of vertices of triangle ABC. Then AA_1', BB_1', CC_1' intersect in S, the pole of s with respect to triangle ABC. If s rotates in the plane of ABC about the fixed point L, the points A_1, B_1, C_1 describe projective ranges on their respective bases; and we have the projective ranges $(A_1')\barwedge(A_1)\barwedge(B_1)\barwedge(B_1')$. It follows that $A(A_1')$, $B(B_1')$, $C(C_1')$ are projective pencils, and that S describes a conic (L) passing through the vertices of ABC. If AL cuts BC in A_0, then AA_0' is the tangent at A, where A_0, B, A_0', C is a harmonic set of points. If B_0', C_0' are similarly defined, then the three tangents

AA_0', BB_0', CC_0' are such that A_0', B_0', C_0' lie on a straight line l which is the polar of L with respect to ABC. If L lies on a side, say AB, then C_1 is the fixed point L for all positions of s, and C_1' is also a fixed point, and S describes the straight line CC_1'. The conic degenerates into the pair of straight lines of AB and CC_1' which is of the hyperbolic type, except when L is at the midpoint of BC, and then we have the parabolic type with AB and CC_1' as parallel straight lines.

Conversely, if any conic passes through the vertices of triangle ABC, there exists a point L by means of which the conic is generated in the above manner. For, let the tangents at the vertices cut the opposite sides in A_0', B_0', C_0'. Then the triangle and the three tangents form a degenerate hexagon inscribed in the conic whose opposite sides meet respectively in A_0', B_0', C_0', and these three points lie on a straight line l by Pascal's theorem. The polar of A_0' with respect to the conic is the straight line through A cutting BC in A_0 so that B, A_0, C, A_0' is a harmonic set; hence AA_0, BB_0, CC_0 must meet in a point L which is the pole of l with respect to the conic and also with respect to the triangle ABC; and L is the desired point.

We now determine the regions of the plane for L which give the different types of the conic (L). If the conic (L) has a point at infinity S_∞, the polar of this point with respect to ABC is the straight line s_∞ passing through L, and if it has two distinct points at infinity, the intersection of their polars is L. We consider therefore the envelope γ of the one-parameter family of polars with respect to ABC of points on the line at infinity. The envelope γ must be a conic; for, if L is any point of the plane, the conic (L) cuts the line at infinity in two points, and hence there are precisely two tangents from L to γ, which may coincide and then L is on γ. We show next that the conic γ is tangent to the sides of ABC at their midpoints, and this determines it as an ellipse whose center is at G, the centroid of ABC. The polar with respect to ABC of the point at infinity on AB is AB, and hence this side is tangent to γ. At any point C_1 of AB other than the midpoint M_c the conic (C_1) is of the hyperbolic type and there are two distinct tangents from C_1 to γ, one of which is AB; but for the midpoint M_c the conic (M_c) is parabolic and the two tangents from M_c to γ coincide in AB, with the limit point M_c as the point of contact of AB. We now have the result:

If L is inside the ellipse γ the conic (L) is an ellipse; if L is on γ the conic (L) is a parabola; if L is inside ABC but outside of γ the conic (L) is a hyperbola with one branch passing through the vertices of ABC; if L is outside of ABC then (L) is a hyperbola with one branch through one vertex of ABC and the other branch through the remaining two vertices.

If L is at the centroid G of ABC the conic (G) is an ellipse with center G tangent to the sides of the triangle complementary to ABC at their midpoints A, B, C. The two ellipses γ and (G) have other important properties.

This completes the determination of the form of the cone for the various positions of L.

There are three polars of a given point $P(y_1, y_2, y_3, y_4)$ with respect to a given tetrahedron $A_1A_2A_3A_4$, whose equations are

$$(1)\ \sum \frac{y_i}{x_i} = 0, \qquad (2)\ \sum_{i \neq j} \frac{y_i y_j}{x_i x_j} = 0, \qquad (3)\ \sum \frac{x_i}{y_i} = 0, \qquad i, j = 1, 2, 3, 4.$$

The third polar, or polar plane of P, has the following property: If P' is a point on the polar plane of P, and the straight line joining P and P' cuts the faces of the tetrahedron, the surface $x_1x_2x_3x_4 = 0$, in the points Q_1, Q_2, Q_3, Q_4, then we have the harmonic relation

$$\frac{4}{PP'} = \sum \frac{1}{PQ_i}.$$

This result and also corresponding relations for the first and second polars are easily obtained by requiring the sum of the coordinates of each finite point to be a constant, say unity.

Partial Differential Equation

3995 [1941, 273]. *Proposed by Cezar Coşniţă, Focşani, Roumania*

Integrate the partial differential equation

$$y(x + y)z_{xx} - (x^2 - y^2)z_{xy} - x(x + y)z_{yy} + (x - y)(z_x + z_y) = 0.$$

Solution by H. W. Bailey, University of Illinois

This equation is readily solved by a Laplace transformation (e.g., Morris and Brown, *Differential Equations*, Prentice-Hall, New York, 1935, p. 340 ff.). The Monge auxiliary equation

$$(x + y)[ydy^2 + (x - y)dydx - xdx^2] = 0$$

has solutions $x^2+y^2=a$, $x-y=b$. On making a transformation of variables $u=x^2+y^2$, $v=x-y$, the differential equation becomes $z_{uv}=0$ which has the general solution $z=\phi(u)+\psi(v)$. Hence, in terms of x and y, the general solution is

$$z = \phi(x^2 + y^2) + \psi(x - y).$$

Solved also by G. W. Petrie, P. D. Thomas, A. K. Waltz, and C. L. Weaver. The proposer stated that the solution could be obtained by the above change of variables to u and v.

Summation of Series

3996. [1941, 341] *Proposed by Elbert H. Clarke, Hiram College*

Sum the series

$$\sum_{n=1}^{\infty} [(n - 1)k]!/(nk)!,$$

where k is any integer greater than unity.

Solution by the Proposer

The general term, $1/(nk+1)(nk+2)\cdots(nk+k)$ can be separated into its partial fractions by the usual methods, and it is easily discovered that we have a common factor $1/(k-1)!$ and that the successive fractions have numerators which are the binomial coefficients in the expansion of $(1-1)^{k-1}$. This suggests that the original sum can be expressed as $f(1)$ where

$$\begin{aligned} f(x) = \frac{1}{(k-1)!}\Bigg[&\frac{x}{1} - \binom{k-1}{1}\frac{x^2}{2} + \binom{k-1}{2}\frac{x^3}{3} - \cdots \\ &+ (-1)^{k-1}\binom{k-1}{k-1}\frac{x^k}{k} + \frac{x^{k+1}}{k+1} - \binom{k-1}{1}\frac{x^{k+2}}{k+2} \\ &+ \binom{k-1}{2}\frac{x^{k+3}}{k+3} - \cdots + (-1)^{k-1}\binom{k-1}{k-1}\frac{x^{2k}}{2k} + \cdots\Bigg] \end{aligned} \tag{1}$$

each cycle of terms carrying the same binomial coefficients while the powers of x rise steadily in step with the regular progression of the denominators. It is seen at once that the derivative $f'(x)$ has a simple form in which the factor $(1-k)^{k-1}$ occurs multiplied by a geometric series of ratio x^k. In symbols,

$$f'(x) = \frac{1}{(k-1)!}\,\frac{(1-x)^{k-1}}{1-x^k}\,.$$

Hence the original series may be written

$$\sum_{n=1}^{\infty}\frac{[(n-1)k]!}{[nk]!} = \frac{1}{(k-1)!}\int_0^1 \frac{(1-x)^{k-2}}{1+x+x^2+\cdots+x^{k-1}}\,dx. \tag{2}$$

Again using partial fractions, the integrand may be written $B_1/(x-\alpha_1) + B_2/(x-\alpha_2) + \cdots + B_j/(x-\alpha_j) + \cdots + B_{k-1}/(x-\alpha_{k-1})$ where α_j is one of the complex kth roots of 1. By the usual reductions it is found that $B_j = -\alpha_j(1-\alpha_j)^{k-1}/k$.

Hence the original infinite series is reduced to a finite sum

$$\frac{1}{k!}\sum_{j=1}^{k-1}(-\alpha_j)(1-\alpha_j)^{k-1}\log\left(\frac{1-\alpha_j}{-\alpha_j}\right).$$

When this is translated in the ordinary manner into a sum of real and imaginary terms it is easy to show that the imaginary terms cancel out and the real terms double up, all of them if k is odd, and all but one if k is even. If we use $\phi(k,j)/2^k$ to stand for the expression to be summed

$$\begin{aligned}\phi(k,j) = \left(\sin\frac{\pi j}{k}\right)^{k-1}\Bigg[&\cos\frac{\pi(k-2j)(k+1)}{2k}\log\left(2\sin\frac{\pi j}{k}\right) \\ &+ \frac{\pi}{2k}(k-2j)\sin\frac{\pi(k-2j)(k+1)}{2k}\Bigg],\end{aligned}$$

and the sum of the infinite series with which we started is

$$\sum_{n=1}^{\infty} \frac{[(n-1)k]!}{[nk]!} = \frac{2^k}{k!} \sum_{j=1}^{(k-1)/2} \phi(k, j),$$

if k is odd, and

$$\frac{2^{k-1}\log 2}{k!} + \frac{2^k}{k!} \sum_{j=1}^{(k-1)/2} \phi(k, j),$$

if k is even.

Series for $k=2, 3, 4, 5$, are given for reference.

$$k = 2: \qquad \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots = \log 2,$$

$$k = 3: \qquad \frac{1}{1\cdot 2\cdot 3} + \frac{1}{4\cdot 5\cdot 6} + \cdots = \frac{1}{4}\left[\frac{\pi\sqrt{3}}{3} - \log 3\right],$$

$$k = 4: \qquad \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7\cdot 8} + \cdots = \frac{1}{4}\left[\log 2 - \frac{\pi}{6}\right],$$

$$k = 5: \quad \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{1}{6\cdot 7\cdot 8\cdot 9\cdot 10} + \cdots = \frac{1}{48}\left[\log 2 - \frac{1+\sqrt{5}}{2}\log 20 \right.$$

$$\left. + \sqrt{5}\log(5-\sqrt{5}) + \frac{\pi\sqrt{2}}{10}\left(\sqrt{25+11\sqrt{5}} - 3\sqrt{25-11\sqrt{5}}\right)\right].$$

Editorial Note. The equality (2) results from the Abel theorem given in Goursat-Hedrick's *Mathematical Analysis*, vol. 1, p. 378. The summation of series of a similar form is given in the solution of 2907 [1923, 206].

Slopes and Curvatures

3999 [1941, 409]. *Proposed by G. B. Van Schaack, Michigan State College*

Let $f(x)$ be a polynomial of degree n with n distinct real roots x_i, $(i=1, 2, \cdots, n)$. Let λ_i be the reciprocal of the slope of the curve $y=f(x)$ at $x=x_i$. Let ρ_j, $(j=1, 2, \cdots, n-1)$, be the algebraic radius of curvature of the curve at the critical point of the curve which lies between x_j and x_{j+1}. (a) Show that if $n>1$, then $\lambda_1+\lambda_2+\cdots+\lambda_n=0$. (b) Show that if $n>2$, then $\rho_1+\rho_2+\cdots+\rho_{n-1}=0$.

Solution by Paul Brock, Student, Brooklyn College

We may consider the coefficient of the highest power of x in $f(x)$ to be unity, and then

$$f'(x_1) = (x_1 - x_2)(x_1 - x_3) \cdots (x_1 - x_n),$$

where $n>1$ and no two x_i's are equal. We shall prove that the following sum is zero

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)} = \frac{1}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4) \cdots (x_1 - x_n)}$$
$$+ \frac{1}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4) \cdots (x_2 - x_n)}$$
$$+ \frac{1}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4) \cdots (x_3 - x_n)} + \cdots$$
$$+ \frac{1}{(x_n - x_1)(x_n - x_2)(x_n - x_3) \cdots (x_n - x_{n-1})}.$$

The first term on the right is equal to the sum of the partial fractions

$$\frac{1}{x_1 - x_2}\left[\frac{1}{(x_2 - x_3)(x_2 - x_4) \cdots (x_2 - x_n)}\right]$$
$$+ \frac{1}{x_1 - x_3}\left[\frac{1}{(x_3 - x_2)(x_3 - x_4) \cdots (x_3 - x_n)}\right] + \cdots$$
$$+ \frac{1}{x_1 - x_n}\left[\frac{1}{(x_n - x_2)(x_n - x_3) \cdots (x_n - x_{n-1})}\right],$$

and hence the original sum is zero. This proves the first part. Since $\rho_j = 1/f''(\bar{x}_j)$, where $f'(\bar{x}_j) = 0$, and the $\bar{x}_j$'s are distinct and real, the second part follows from the above on replacing $f(x)$ by $f'(x)$ with $n>2$.

Solved also by P. Chairulli, H. W. Eves, F. A. Ficken, P. M. Hummel, D. C. Lewis, R. K. Morley, and the proposer.

Editorial Note. Chiarulli's solution is similar to the above. The proposer's solution denotes by V the Vandermonde determinant of order n with the ith row $x_i^{n-1}, x_i^{n-2}, \cdots, x_i, 1$. In this determinant the cofactor of x_i^{n-1} is $(-1)^{i-1}V_i$, where V_i is a similar determinant obtained from V by omitting the first column and ith row. Then, taking unity as the leading coefficient of $f(x)$, $f'(x_i) = V/(-1)^{i-1}V_i$; and it follows that the sum of the reciprocals of $f'(x_i)$ is $1/V$ times the determinant obtained from V by replacing the elements of its first column by unity. Hence if $n \geqq 2$, this sum is zero. The solutions of Lewis and Morley consider the residues of $1/f(z)$, where $f(z)$ is a polynomial of the nth degree with distinct zeros z_i in the complex plane. The proofs are essentially as follows: The sum of the residues for the whole z-plane is zero, and the residue for the point at infinity is zero since $n \geqq 2$. It is obvious that the residue for the pole z_i is $1/f'(z_i)$, and this completes the proof of (a).

Hummel stated that the theorem of the problem results from the long known relations

$$(1)\quad \sum_{i=1}^{n} \frac{x_i^k}{f'(x_i)} = 0, \qquad k = 0, 1, \cdots, n-2; \qquad (2)\quad \sum_{i=1}^{n} \frac{x_i^{n-1}}{f'(x_i)} = 1/c_0,$$

where $f(x)$ is a polynomial with complex coefficients, of which c_0 is the leading one, of degree $n \geqq 2$, and with distinct zeros x_i (see Burnside and Panton's *Theory of Equations*, vol. 1, p. 172, problem 4). The proof in this reference is different from his own which is briefly as follows: Assume that (1) is true for any chosen set of n distinct x_i's, let x_{n+1} be different from each of these, and set $F(x) = (x - x_{n+1})f(x)$. Then $F'(x_i) = (x_i - x_{n+1})f'(x_i)$, $i = 1, 2, \cdots, n$, and

$$0 = \sum_{i=1}^{n} \frac{x_i^k}{f'(x_i)} = \sum_{i=1}^{n} \frac{(x_i - x_{n+1})}{F'(x_i)}, \qquad k \leqq n - 2.$$

We may then write

$$\sum_{i=1}^{n+1} \frac{x_i^{k+1}}{F'(x_i)} = x_{n+1} \sum_{i=1}^{n+1} \frac{x_i^k}{F'(x_i)},$$

and, since x_{n+1} may be any one of the $n+1$ distinct zeros of $F(x)$, we see at once that in this last equation the sum on the right and the one on the left must each be zero. Since (1) is easily verified for $n = 2$, the induction proof of it is now complete. Set $1 = nc_0x_i^{n-1}/f'(x_i) + (n-1)c_1x_i^{n-2}/f'(x_i) + \cdots$, sum each side for $i = 1, 2, \cdots, n$, and then (2) follows after using (1). The induction proof by Eves for the case $k = 0$ is essentially the same.

Ficken used Lagrange's interpolation formula on an auxiliary function and differentiation of the result gives

$$\sum_{i=1}^{n} \frac{\lambda_i}{x_i - x_0} + \frac{1}{f(x_0)} = 0, \qquad f(x_0) \neq 0, \qquad n \geqq 1.$$

Let $F(x) = (x - x_0)f(x)$, then $\Lambda_0 = 1/f(x_0)$, $\Lambda_i = \lambda_i/(x_i - x_0)$; and it follows that the sum of the Λ_i's is zero for $i = 0, 1, \cdots, n$, thus proving (a). The above formula results directly by the use of the partial fraction expansion of $1/f(x)$, then the proof is closely related to the one by Brock.

After the completion of his solution Lewis discovered that a generalization of the problem is given in Pólya und Szegö's *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 87, problem 67. He stated that his proof can be adapted to this generalization. The proof in this reference is essentially the same as the one in the first reference above.

A solution by Han Ming-tê, Peiping, China, was received after the preparation of the above. In the same cover with it were solutions of 3968 and 3969 by Li Ou using inversive geometry.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to C. O. Oakley, Haverford College, Haverford, Pennsylvania.

The members of the Mathematical Association are reminded that, as previously announced, the Annals of Mathematics and the Duke Journal of Mathematics are continuing the generous plan of allowing the half rate for those journals to those whose membership in the Association and whose subscription to the respective journals are unbroken from 1942 to the year in question.

Since the beginning of 1942, Professor G. D. Birkhoff has been acting as exchange professor with Latin-American universities. He has given series of lectures in the universities at Mexico City; Lima and Arequipa, Peru; and Santiago, Chile. At a special session he was made an honorary member of the faculty of the Universidad Nacional Mayor de San Marcos de Lima, which is the oldest institution of higher learning in the western hemisphere.

The University of Michigan announces the following promotions: Dr. C. C. Craig and Dr. R. V. Churchill have been promoted to professorships; Dr. P. S. Dwyer has been promoted to an associate professorship; Dr. R. M. Thrall has been promoted to an assistant professorship.

The University of Oregon has granted leaves of absence to Assistant Professors K. S. Ghent and T. S. Peterson for service with the Naval Ordnance Department.

Purdue University announces the following promotions and appointments: Dr. M. W. Keller and Dr. J. W. T. Youngs have been promoted to assistant professorships; Dr. Leonidas Alaoglu, B. H. Arnold, N. J. Fine, Dr. J. H. Giese, Dr. Michael Golomb, Dr. Ivan Niven, Dr. Maxwell Reade, and Dr. G. S. Young have been appointed as instructors.

At Washington and Jefferson College Associate Professor H. C. Shaub has been promoted to a professorship and made head of the department. D. T. Finkbeiner has been appointed to an instructorship.

Professor C. R. Adams has been made chairman of the department of mathematics in Brown University.

Professor C. B. Allendoerfer is on leave from Haverford College to become associate physicist in the Bureau of Ordnance, Navy Department.

Assistant Professor Frances E. Baker of Mount Holyoke College has been appointed to an associate professorship at Vassar College.

Assistant Professor P. N. Carpenter of Grove City College has been promoted to an associate professorship.

Cornell College, Iowa, has appointed W. M. Davis of the Illinois Institute of Technology to an assistant professorship.

Dr. D. M. Dribin of the University of Nebraska was appointed in June for work in cryptanalysis in the Intelligence Service, Office of the Chief Signal Officer in Washington.

Assistant Professor H. A. Giddings of the Illinois Institute of Technology has been appointed to an associate professorship at the University of New Hampshire.

Professor H. V. Gummere of Haverford College retired in June, at which time Haverford College conferred upon him the doctorate of science.

Dr. E. R. Hedrick retired on July 1 as vice-president of the University of California and provost at the Los Angeles campus.

Dr. P. R. Halmos has been appointed associate at the University of Illinois.

Dr. T. J. Higgins of Tulane University has been appointed an associate professor of electrical engineering at the Illinois Institute of Technology.

Professor M. H. Ingraham, head of the department of mathematics at the University of Wisconsin, has been appointed dean of the College of Letters and Science, and Professor R. E. Langer succeeds him as head of the department.

Professor S. C. Kleene of Amherst College is now a lieutenant (j.g.) in the U. S. Naval Reserve.

G. J. Neupert, head of the department of mathematics at Lewiston, Idaho, State Normal School has retired. He is succeeded by A. J. Boosinger who has been a member of the staff since last September.

Assistant Professor C. R. North of Rutgers University has been granted leave of absence to teach mathematics at the U. S. Naval Academy.

Dr. H. L. Olson has been appointed assistant professor at Southwestern University, Georgetown, Texas.

Professor H. A. Robinson of Agnes Scott College is teaching mathematics at the U. S. Military Academy, West Point, with the rank of major, F.A.

Assistant Professor E. A. Saibel of Carnegie Institute of Technology has been promoted to an associate professorship.

L. W. Swanson of the University of Minnesota has been appointed assistant professor at Coe College, Iowa.

At Fenn College, Dr. W. R. Van Voorhis has been promoted from instructor to assistant professor.

Dr. R. K. Wakerling of Texas Technological College has been appointed assistant professor at Fresno State College.

THE SIXTH CORPS AREA CONFERENCE ON PRE-INDUCTION TRAINING

On June 26 and 27, 1942, a conference representing colleges and universities in the Sixth Corps Area was held at Evanston, Illinois, to consider programs for pre-induction training of students who enlist for military service on a deferred basis. This conference was attended by over one hundred delegates from forty institutions, and by fifteen officers of the Army and the Navy. After a session devoted chiefly to a preliminary discussion by these officers of the problems before the conference, divisional meetings for mathematics and the sciences occupied the afternoon of the 26th, and most of the following morning. The various divisions then reported their recommendations to the conference as a whole.

The meetings of the mathematics division were attended by about fifty men representing more than twenty-five institutions. Lieutenant Commander E. P. Wilson was present and gave valuable advice and information.

At the afternoon session, Professor L. R. Ford presiding, discussion centered about the necessity or advisability of radical changes in the usual beginning college courses in mathematics. Accounts were given of programs that have been followed, or are planned for next year, at various institutions. A motion was passed expressing the sense of the meeting that it is feasible to meet the requirements of military pre-training programs, so far as Algebra and Plane Trigonometry are concerned, by suitable modification of existing courses, but it is desirable to organize a new course in Solid Geometry and Spherical Trigonometry. It was also recognized that something must be done to improve the technique of students in arithmetical computation.

Two committees were then appointed to draw up recommendations which might be presented to the conference as a whole. The first committee, on the program for a course in Solid Geometry and Spherical Trigonometry, consisted of Professors H. P. Evans (chairman), E. J. Moulton, and E. W. Schreiber. The second committee, whose members were Professors G. E. Moore (chairman), D. R. Curtiss, and E. B. Miller, prepared outlines of mathematical programs for each of several kinds of pre-induction training. These reports were presented and adopted by the division at the next morning's session, Professor C. N. Mills presiding, and were later adopted by the conference as a whole. The substance of these reports, as approved by the conference, follows.

The mathematics division recommends that courses be uniformly planned in the Sixth Army Corps Area as shown in the following chart. Since many institutions divide the academic year into semesters, while others use the quarter plan, this chart lists class hours in terms of both semester-hours (class hours per week for one semester) and quarter-hours (class hours per week for one quarter). Thus in the table below, Algebra (3s or 4q) indicates a course of 3 semester-hours, or one of 4 quarter-hours.

MINIMUM COURSE RECOMMENDATIONS

	Courses	Program Involved	
I	(1) Algebra (3s or 4q) (2) Plane Trigonometry (3s or 4q) (3) Solid Geometry and Spherical Trigonometry (3s or 4q)	Pre-Aviation Cadets as Bombardiers, Navigators, or Pilots	Flight
II	(1) Algebra (3s or 4q) (2) Plane Trigonometry (3s or 4q) (4) Analytic Geometry (3s or 4q) Calculus (8s or 12q in addition to above)	Communications Meteorology	Ground Forces
III	(1) Algebra (3s or 4q) (2) Plane Trigonometry (3s or 4q) (4) Analytic Geometry (3s or 4q)	Geology Option	Photography
	Calculus (6s or 8q in addition to above)	Physics and Chemistry Option	Photography

Note 1. For course (1), students entering college with only one year of high school algebra should take a course in intermediate algebra; students entering with one and one half years should take a course in ordinary college algebra.

Note 2. In courses (1), (2), and (3), special emphasis should be placed on the use of tables, on interpolation, and on computation.

Note 3. Course (3), being of a combined character, was given special consideration by a separate committee. Details are found later in this report.

Note 4. Special emphasis should be placed on graphical methods in course (4), Analytic Geometry.

Note 5. Students who enter college with credit in College Algebra or Plane Trigonometry, or both, should be encouraged to take continuation work in mathematics beginning with their first semester in college. One suggestion is that, if they enter with two courses, they should take Analytic Geometry. If they enter with credit in College Algebra but not Trigonometry, under proper circumstances, they might continue with both Trigonometry and Analytic Geometry for groups II and III above.

Note 6. It is recommended to the supervising military authorities that the comprehensive examinations given in the various military programs cover in algebra just those topics which are common to the two courses: Intermediate Algebra and College Algebra.

COMPUTATIONAL ARITHMETIC

In view of the reiterated plea from representatives of the armed forces that we strive for greater accuracy and speed in computational arithmetic, we recommend that colleges give special drill work to improve computation. This may be done by definite drill in connection with courses in Algebra, Trigonometry, and Solid Geometry—or, schools may wish to offer special course work designed to remedy the serious lack of computational ability.

In the report of the committee, topical outlines of the courses were presented. Since these followed traditional subject outlines of such courses, only the following for the new course on Solid Geometry and Spherical Trigonometry is here given.

SOLID GEOMETRY AND SPHERICAL TRIGONOMETRY

Credit, 3 semester-hours or 4 quarter-hours. About 43 class periods.

Solid Geometry (intuitive development).

Straight Lines and Planes. 5 lessons.

Drawing of three-dimensional figures. Relative positions of lines and planes in space. Projections. Dihedral and polyhedral angles.

Polyhedrons, Prisms, Cylinders, Cones. 10 lessons.

In the presentation of these topics emphasis should be placed on the making and interpretation of drawings of three-dimensional figures and the development of space intuition. Statements of theorems should be clarified by construction and use of models. When possible, the drawings of figures should be motivated by word problems involving mensuration formulas (the more important of these formulas should be memorized). The problems assigned should include many designed to give practice in computation; some should require the use of trigonometric functions and of logarithms. Problems concerning loci, frustums, and prismatoids might also be included.

The Sphere and Trihedral Angles. 10 lessons.

Definitions. Circles on a sphere. Area and volume of a sphere. Spherical angles. Spherical triangles and trihedral angles. Area of a spherical triangle.

Spherical Trigonometry and Applications.

Spherical Trigonometry. 10 lessons.

Polar triangles. Formulas for right spherical triangles. Napier's rules. Rules regarding the relative magnitude of parts. Solution of right spherical triangles, isosceles triangles, quadrantal triangles. The law of sines. The law of cosines. Formulas for half-angles and half-sides. Napier's analogies. Solution of oblique spherical triangles, six cases.

Applications of Plane and Spherical Trigonometry to Navigation. 8 lessons.

Plane sailing. Parallel sailing. Middle latitude sailing. Dead reckoning. Great circle sailing. Positions on the celestial sphere. The astronomical triangle, with applications.

A limited number of reports of the conference is available on application to the President of Northwestern University. Mimeographed accounts of the complete agenda of the mathematics division can be had, not more than one to an institution, and so long as they last, by writing to the Department of Mathematics, Northwestern University, Evanston, Illinois.

D. R. CURTISS

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fifth Summer Meeting, Poughkeepsie, N. Y., September 7–9, 1942.
Twenty-seventh Annual Meeting, New York, N. Y., December 30–31, 1942.

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, State College, Pa., Oct. 23–24, 1942
ILLINOIS
IOWA
INDIANA, Notre Dame, April 9–10, 1943
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Dec. 5, 1942
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI, fall, 1942
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, Jan. 30, 1943
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 28, 1942
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE, fall, 1942
WISCONSIN, Milwaukee, May 7, 1943

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WHAT IS A CURVE?

G. T. WHYBURN, University of Virginia

1. Introduction. When the searching light of modern mathematical thinking is focused on the classical notion of a curve, this idea is found to involve elements of vagueness which must be clarified by accurate and exact definition. Fortunately this has been made possible and relatively simple by development in the field of set-theoretic topology. We shall endeavor to set forth below, first the need for explicit definition of a curve, then the definition itself, and finally several illustrations of types of simple curves which can be completely characterized by their topological properties and which more nearly approach the classical notion of a curve.

2. The classical notion. The concept of a curve as the "path (or locus) of a continuously moving point" usually is accompanied by intuitive notions of

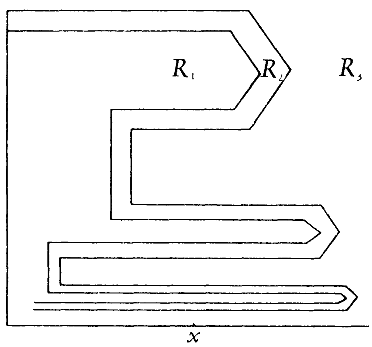


FIG. 1.

thinness and *two-sidedness*. When the curve is in a plane, these were thought to be consequences of the rather vaguely formulated definition of a curve as just given.

That the path of a continuously moving point is not necessarily a thin or curve-like set was shown by Peano and somewhat later by E. H. Moore, who demonstrated the remarkable fact that a square plus its interior can be exhibited as the continuous image of the interval. In other words, if S denotes a square plus its interior, we can define continuous functions $x(t)$ and $y(t)$ on the interval $0 \leqq t \leqq 1$ so that as t varies from 0 to 1, the point $P[x(t), y(t)]$ moves continuously through all the points of S.

A still more striking result in this direction is the remarkable theorem proved independently by Hahn and Mazurkiewicz about 1913. This theorem asserts that in order for a point set M (in euclidean space of any number of dimensions) to be representable as the continuous image of the interval $0 \leqq t \leqq 1$, it is necessary and sufficient that M be a locally connected continuum. (A *continuum* in euclidean space is a closed, bounded, and connected set; and a con-

tinuum M is *locally connected* provided that for any $\epsilon > 0$ a $\delta > 0$ exists such that any two points x and y of M at a distance apart $< \delta$ can be joined by a subcontinuum of M of diameter $< \epsilon$). Thus since obviously not only a square but also a cube, an n-dimensional interval, an n-dimensional sphere and a multitude of other sets are locally connected continua, any such set M can be represented as the path of a continuously moving point in the sense that we can define continuous functions

$$x_i = x_i(t) \qquad 0 \leqq t \leqq 1,\ i = 1, 2, \cdots, n,$$

such that as t varies from 0 to 1 the point P with coordinates $(x_1, x_2, \cdots, x_n)$ moves continuously through all the points of M.

Even when a set is sufficiently "thin" or "1-dimensional" that we would probably call it a curve it may be in a plane and still not be two-sided. To illustrate we note that in Figure 1 any point on the base of the continuum, such as x, is a boundary point of each of the three regions R_1, R_2, R_3 into which the continuum divides the plane. Hence there are *three sides* of the base of this continuum. (Clearly we could add extra oscillating curves to the figure so as to make an arbitrarily large number or even an infinite number of regions each having all base points x on their boundaries). Nevertheless our continuum is a thin 1-dimensional set made up of an infinite number of line segments. Now it is possible to construct in a plane a continuum which is thin in the sense that it will not contain the interior of any circle and yet is so unusual that it will divide the plane into any finite number or an infinite number of regions and, further, it will be the boundary of each one of these regions. Also a plane continuum can be constructed which not only itself cuts the plane into infinitely many regions but has the remarkable property that every subcontinuum of it (any "piece" of it) also cuts the plane into infinitely many regions.

3. Dimensionality. General definitions of curve, surface, solid. Undoubtedly sufficient evidence has been given of the necessity of being precise in our definitions and statements concerning curves, surfaces, etc., and of the unreliability of our intuition concerning these concepts.

We leave aside the continuous traversibility of the set as a criterion characterizing or distinguishing between curves, surfaces, solids, etc., since we have seen how it fails in this respect, and concentrate on content or dimensionality of the set as a guide.

Hence it seems natural and adequate to define a *curve* as a 1-dimensional continuum, a *surface* as a 2-dimensional continuum and a *solid body* as a 3-dimensional continuum.

These definitions are satisfactory provided we give an adequate definition of dimensionality of a set. To this end let us concentrate our attention on compact sets, *i.e.*, sets K which have the property that any infinite subset has a limit point belonging to K, sets which are closed and bounded if they lie in a euclidean space.

We then define the dimensionality of the empty set to be -1 and agree the dimensionality of any other set is to be $\geqq 0$. Assuming, then, that we have defined the dimensionality concept for dimensions $\leqq n-1$, by induction we define a set K to be of dimensionality n provided (1) every pair of distinct points p and q of K can be separated in K by some set X of dimensionality $\leqq n-1$, *i.e.*, $K-X$ falls into two separated sets K_p and K_q containing p and q respectively; and (2) some pair of points of K cannot be separated in K by a subset of K of dimensionality $<n-1$. Thus for $n \geqq 0$, a set K is of dimension n provided n is the least integer such that every pair of distinct points of K can be separated in K by the removal of a subset of dimension not greater than $n-1$.

According to this definition, then, a compact set K is of dimension 0 provided every two points of K can be separated in K by omitting the empty set, *i.e.*, provided they are already separated in K. Hence a 0-dimensional set is one which is non-empty but is totally disconnected in the sense that its only connected subsets are single points. A compact set K is 1-dimensional provided any two points can be separated in K by omitting from K a 0-dimensional or totally disconnected set but some two points cannot be separated without omitting some points from K. A compact set K is 2-dimensional provided each pair of points of K can be separated in K by omitting a 1-dimensional set but not every pair can be separated by omitting a 0-dimensional set, and so on.

Stated in other terms, if we accept our definition that a curve is a 1-dimensional continuum, a surface is a 2-dimensional continuum, and a solid body is a 3-dimensional continuum, we see that a non-empty compact set K is 0-dimensional if every pair of its points are separated in K. The set is 1-dimensional at most provided we can (with shears if you like) separate any two of its points by cutting the set along a 0-dimensional set, *i.e.*, by cutting out only single points as connected pieces. The set is 2-dimensional at most provided we can separate any two points by cutting the set along a 1-dimensional set, *i.e.*, by cutting out only curves as connected sets. The set is 3-dimensional at most if we can separate any two points by cutting (with a saw perhaps) the set along a 2-dimensional set, *i.e.*, by cutting out only surfaces as connected sets.

4. Some simple types of curves. Having defined exactly the notions of curve, surface, and solid in terms of their topological properties in such a way that they correspond roughly to the geometrical notions of line, plane, and space, we consider now some interesting particular kinds of curves which may be similarly characterized.

Take first a straight line interval ab joining two points a and b and ask the question "What properties of a set make it essentially like an interval?" or "When are the points in a set associated together like those in the interval ab?" For example, if ab is a taut string and we release the tension and let it go slack but do not allow it to loop over onto itself, it is no longer straight but it retains its same essential structure. It can still be severed by cutting out any one of its points other than a or b; and it is this property in particular which char-

acterizes the interval completely from the topological point of view. In other words, if we understand by a *simple arc* any set of points which is topologically equivalent to an interval in the sense that its points can be put into one-to-one and continuous correspondence with the points of an interval, then *in order that a continuum T be a simple arc it is necessary and sufficient that T contain two points a and b such that the removal of any point of T other than a or b will disconnect T.* Thus in Fig. 2, (a) is a simple arc, but (b) is not a simple arc because the removal of neither a, b, nor x will separate the set (*i.e.*, will make it fall apart).

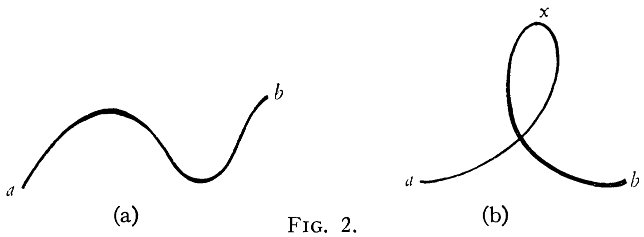


FIG. 2.

Consider next a circle C and let us ask similar questions. If C is distorted, as was our interval, by letting it slacken and bend but not fold onto itself or be broken violently, it is seen to retain its essential set structure. It retains the property, for example, of being severed by the removal of any two of its points whatever. Here again the property mentioned is characteristic for the type of curves which are topologically equivalent to the circle. In other words, if we define a *simple closed curve* as a set which can be put in one-to-one and continuous correspondence with a circle, then *in order that a continuum C be a simple closed curve it is necessary and sufficient that C be disconnected by the omission of any two of its points.* Thus in Figure 2, (a) is not a simple closed curve since the removal of both a and b leaves the set connected. In Figure 3, (a) and (b) are simple closed curves but (c) is not a simple closed curve because the removal of x and y leaves the set connected.

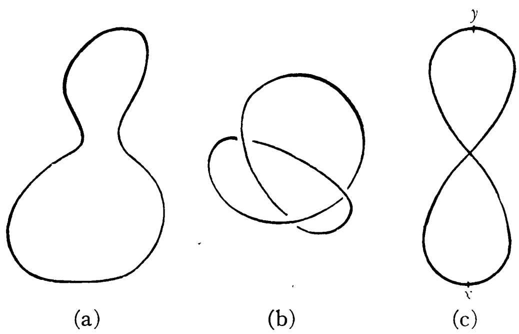
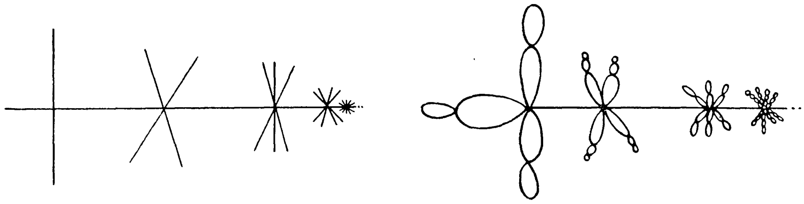


FIG. 3.

A curve which is made up of a finite number of simple arcs which overlap with each other only at end points of themselves is called a *graph* or a *linear*

graph. A graph, then could be regarded as being constructed by putting together in any one of numerous ways a finite number of simple arcs so that no two of the arcs will overlap anywhere except possibly at an end point of both. All of the curves illustrated in Figs. 2 and 3 are graphs; and of course many more complicated structures could be made which would still be graphs. However, if a graph is in a plane it, like the simpler curves previously discussed, will have the classical property of 2-sidedness which does not belong to all curves.



(a) (b)

FIG. 4.

Finally, we mention two further types of curves which in general are not graphs and yet whose structure is interesting and simple, namely the *dendrite* or *acyclic curve* and the *boundary curve*. A *dendrite* is a locally connected continuum which contains no simple closed curve. It may contain infinitely many simple arcs [See Fig. 4 (a)]. In fact it may be impossible to express it as the sum even of countably many arcs, and yet it has the property that any two of it's points are end points of one and only one arc in the curve. A *boundary curve* is a locally connected continuum which can be so imbedded in a plane that it will be the boundary of a connected region of the plane. Although it is true that every dendrite is a boundary curve, in general a boundary curve will contain one and may contain infinitely many simple closed curves [See Fig. 4 (b)]. However, it is interesting to note that no such curve could contain a cross bar on a simple closed curve. In other words, the most that any two simple closed curves can overlap is in a single point (point of "tangency"). Thus any boundary curve breaks up into so called cyclic elements which are either single points or simple closed curves, no two of these have more than one common point, and these fit together to make up the curve and give it a structure relative to these elements which is very similar to that of a dendrite. [Compare Fig. 4 (a) with Fig. 4 (b)].

5. Conclusion. We have touched but a few of the many interesting aspects of the fundamental theory of curves. The subject has an extensive literature, particularly from the topological point of view, which the explorative reader will find fascinating as well as instructive. The field is a live one and it is currently receiving important contributions. Interesting and difficult problems remain unsolved. There is much to attract and repay the student who will expend the effort necessary to acquire a knowledge of these problems and to master the methods which have been devised for attacking them.

ON THE TEACHING OF ELEMENTARY MATHEMATICS*

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The following remarks are addressed to those who are preparing themselves to be teachers of elementary mathematics. They begin with some horrible examples of what I think a good teacher should not do, and some general principles that I think a good teacher would do well to follow; they conclude with a serious thesis concerning the proper emphasis of various aspects of elementary mathematics.

Most of the examples come out of my own first-hand experience at various institutions (although some do not) and could be, but will not be, documented. If this be undiplomatic, it is at least not appeasement. I shall not discuss what should be taught to freshmen because this depends to some extent on what they are being prepared for, and because I have expressed my views fully on this subject elsewhere. Rather, I shall confine myself to remarks which I consider to be valid regardless of what topics the curriculum contains.

There are at least three different French languages: Parisian French, Gascon French, and a language spoken by American tourists called High School French. In our subject, we have Pure Mathematics, Applied Mathematics, and a peculiar subject sometimes called "Freshman Mathematics." Every instructor, in the normal exercise of his duties, accumulates his own collection of specimens of "Freshman Mathematics." An innocuous example is that of the student who divided a large number by 1 by the process of long division. I call this innocuous because at least he was using a general algorithm which could be applied, however inadvisedly, to his problem. A more common and harmful sample is the following:

$$\frac{2+x}{2x} = \frac{\overset{1}{\cancel{2}} + \overset{1}{\cancel{x}}}{\cancel{2}\cancel{x}} = 1 + 1 = 2. \tag{1}$$

This illustrates the "law of universal cancellation" which says that whenever a symbol occurs twice on the same paper or blackboard it may be crossed out in both places. A well-known "proof" of it proceeds as follows:

$$\frac{1\cancel{6}}{\cancel{6}4} = \frac{1\cancel{6}\cancel{6}}{\cancel{6}\cancel{6}4} = \frac{1\cancel{6}\cancel{6}\cancel{6}}{\cancel{6}\cancel{6}\cancel{6}4} = \cdots = \frac{1}{4}.$$

Very common specimens of "Freshman Mathematics" are the following:

$$\sin(x+y) = \sin x + \sin y, \qquad \log(x+y) = \log x + \log y, \tag{2}$$
$$\sqrt{x+y} = \sqrt{x} + \sqrt{y}.$$

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* Based on an address delivered at the triennial convention of Pi Mu Epsilon held at Lehigh University, January 1, 1942.

Other common errors are "transposing" $2x=1$ to obtain

$$x = \frac{1}{-2} \tag{3}$$

(presumably because whenever a symbol is "brought" from one side of an equation to the other its sign must be changed); and adding fractions by adding numerators and denominators separately as in

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}\,. \tag{4}$$

A favorite specimen of mine is the following "proof" of the remainder theorem:

$$\begin{array}{r|l|l}
x - a & \overline{f(x) \qquad\quad} & f \\
\hline
 & f(x) - f(a) & \\
\cline{2-2}
 & \qquad\quad f(a)\,. &
\end{array}$$

A young teacher may well feel at times as though he has somehow blundered into a psychopathic ward. But successful psychoanalytic technique requires sympathy, understanding of the difficulties, and probing into the past of the patient to find the events leading up to the tragedy. I can speak at first hand of this latter aspect since I have taught both high school and college freshman classes for many years.

It is essential for the teacher to understand both the student's preparation and difficulties, and, if I may be so bold as to say so, the subject he is teaching. I emphasize this because I once heard a man high in educational circles assert that one who knows too much can't be a good teacher. This is unquestionably true, but only because one who knows too much cannot exist. Of course, there are distinguished scholars who do not teach well, and excellent teachers who are not distinguished scholars. But if one has the patience and sympathy needed for good teaching and is willing to devote time and thought to the problems of teaching, then increased scholarship can only enrich one's teaching, not spoil it. On the other hand it is impossible for any one to teach well something he doesn't himself understand. There are, I am afraid, some people teaching the third reader who have never progressed beyond the fourth. By this I do not mean at all that it is necessary for a teacher of elementary subjects to be well versed in partial differential equations or the calculus of variations in the large, or similar advanced technical subjects; but I do think it essential that he have a broad, deep, and clear understanding of the fundamental concepts underlying elementary mathematics. That such understanding is not always present becomes obvious upon contact with some teachers and some textbooks.

For example, I was once asked to settle a bet between two ex-colleagues, one of whom contended that $0/0=0$ because 0 divided by anything is 0, while the other held that $0/0=1$ because anything divided by itself is 1. Neither was

willing to listen to an explanation of why both were wrong; they wanted only a verdict as to which was right. This would make a better story if there were a third who claimed that $0/0=\infty$ because anything divided by 0 is ∞, but I shall not yield to the temptation to embellish the facts.

Then there is the teacher who seriously claimed that it was literally correct to write $1/0=\infty$ in a trigonometry class on the grounds that Veblen and Young write this equation in their Projective Geometry. Needless to say, Veblen and Young are considering a particular coordinate system on a line in the projective plane and division means to them not numerical division, but a certain quadrilateral construction.

Another instructor "derived" an equation for vertical lines as follows:

$$y - b = \infty\ (x - a),$$

$$x - a = \frac{1}{\infty}(y - b) = 0.$$

In fact, so many people and textbooks have the courage of their confusion on the subject of infinity that it is small wonder that some of our brightest students come away with the impression that the cardinal number of the set of all natural numbers is ∞ which is obtained by dividing 1 by 0 and represents the place where parallel lines meet. I have often wondered why no one has ever "proved" the statement $1/0=\infty$ by appealing to the fact that if one tries to divide 1 by 0 on a calculating machine it will run until worn out.

Then there are many books which carelessly ask the student to write the nth term of a sequence whose first few terms are given, as though this were a reasonable mathematical question. To answer it, as you know, requires not mathematics but clairvoyance. For example, the nth term of a sequence beginning with 2, 4, 6, 8, $\cdots$ may be $2n$ but it may also be $2n+(n-1)(n-2)(n-3)(n-4)f(n)$ where $f(n)$ is almost completely arbitrary.

Perhaps more serious than these troublesome confusions is the failure of some teachers and some textbooks to pay due attention to converse propositions. For example, one of my ex-colleagues once remarked to me that he liked trigonometric identities because nothing could be more satisfyingly certain than to come out with the same expression on both sides. I answered, "Yes, as long as all the steps are reversible." The man's jaw dropped. He had never heard of that, and it took me 45 minutes to persuade him that the trouble lay with the converse proposition. It turned out that, although he liked trigonometric identities and had taught them for years, he had never understood them. Similarly, many texts on analytic geometry leave the student without the slightest suspicion that a converse proposition is needed in discussing equations of loci.

I say that this is perhaps a more serious matter because it is symptomatic of a more or less widespread tendency that I regard as vicious: namely, the

tendency to *say* that one of the main objectives of teaching mathematics is to train the student to think logically and then to *teach* the subject as though it had nothing to do with logical reasoning at all. The man just referred to paid careful attention to converses in plane geometry because that was traditional but had little suspicion that algebra or trigonometry had anything to do with reasoning. In fact, I recently had the distressing experience of hearing someone high in educational circles say that after all plane geometry is mostly a matter of memory. Is it any wonder that this opinion of mathematics is so widespread when it is all too often taught as a matter of memorized formulas, routine meaningless manipulations, and undigested proofs? It is my opinion that many of the ills of mathematics come from such excessive formalization and neglect of fundamental principles and common sense. If mathematics is presented as a collection of arbitrary rules of thumb for performing peculiar manipulations which somehow solve problems in which the student is not interested anyway, it is not surprising that many intelligent students are repelled from the subject for life. Such a collection of tricks is merely a branch of parlor magic, while the ideas of mathematics constitute an important current in the history of human thought. When these disgruntled individuals grow up some of them lead movements to abolish mathematics as a required subject. I am convinced that this often happens because they have never had contact with genuine mathematics at all, but only with formalized, memorized regurgitated techniques.

This souring of public opinion among non-technical people is by no means the only ill effect of excessive formalization. I am convinced that many purely technical difficulties arise from the same source. Thus, the errors given in (2) arise from thoughtless, superficial, formal analogy with the distributive law $a(x+y)=ax+ay$. The troubles with cancellation, transposition, and fractions ((1), (3), (4)) are due to thoughtless, unrationalized formalization. A memorized rule that has not been understood is difficult to recapture when the smallest part of it is forgotten.

I do not know whether one can attribute the tendency toward excessive formalization to a unique source. Perhaps it is partly due to the fact that teachers are sometimes judged by the results of their students on stereotyped examinations which can usually be passed with only formal techniques and without understanding. This practice may put pressure on the teacher to forget about the difficult (though worth while) task of teaching mathematics and cause him to give instead a coaching course for the examination. This makes the teacher's life easier, but harder to justify. He can then "cover" all the topics required for the examination easily, by the simple process of cramming memorized techniques into his students and ignoring the opportunity to give real insight into mathematics. He can teach "box" methods for solving problems since he is sure that the examination will not contain a strange "type" of problem that will not fit into his boxes. I have actually seen a secondary teacher who taught the logarithmic solution of triangles at the outset of the term before the very

definitions of the trigonometric functions. He "justified" this procedure on the grounds that he found students lost most credits on those questions on the Regents' examinations and consequently should get as much drill as possible on them. Beginning on the first day of the term surely gives the maximum drill. I met another secondary teacher who disliked teaching plane geometry because he was unable to formalize it as he did algebra. He did, however, like to teach some "calculus" in his advanced algebra section. By "teaching calculus" he seems to have meant training his students to write nx^{n-1} whenever they saw x^n, much as a mouse is trained to run a maze when a gong rings.

By now, I think I have complained enough about some of the extreme cases that I have had the misfortune to encounter. I hope they are uncommon. Let me now say a few words about some much less rare situations. There is the teacher who does "take up" the fundamental ideas and the reasoning behind his processes. But he does it once and rapidly, hopes that the best students will get it, and races on. This practice arises from a point of view with which I am not in sympathy. I do not consider the fundamentals important only for the few best students and the techniques important for everybody. I think that the fundamentals are important for all and will justify, in the long run, the time and effort needed to get them over to most of the class. I think it is part of the teacher's job to disseminate the ideas of mathematics as widely as possible, not to keep them within an esoteric circle of witch-doctors. The "devil take the hindmost" theory of teaching, if pushed to extremes, leads the teacher to give lectures which only he understands.

Another practice of which I disapprove and which is prevalent among college teachers, is that of blaming the student's incompetence on his high school training and letting it go at that. In fact, the high school teachers proceed similarly to blame the elementary school teachers, who blame the parents, who presumably can blame nothing but heredity, so that the original fault seems to lie with Adam. I can see no sense whatever in grumbling because a freshman has not mastered the routine techniques of high school algebra and then rushing him on into further ill-understood memorized techniques. We must accept our students as incontrovertible data; if they never learned to add fractions sensibly let us not consider it beneath our dignity to teach it to them. Even where they have had prerequisite material, they often have it no longer. Unless one is willing to face the facts as they are and do the best job possible under the circumstances, one might as well lapse into one's anecdotage and go about mumbling about the good old days.

I have said a great deal about what I think a good teacher of freshman mathematics should not do. It is harder to say what he should do since that depends on his own point of view, his temperament, training, and interests. However I shall undertake to enunciate a few generalities.

I think it is important for a teacher of elementary mathematics to be thoroughly familiar with the foundations of the subjects he teaches and to have as

broad a mathematical background as possible. A teacher who has only superficial knowledge himself, slips comfortably into the habit of regarding things as "easy" because he is "familiar" with them. On the other hand, genuine familiarity with the deep-lying difficulties underlying elementary mathematics will, if nothing else, prevent false emphases and tend to make one more tolerant towards the students' difficulties. This was brought home to me in the first high school class I ever taught when a student bisected an angle thus:

An angle had been defined as a figure consisting of two rays with a common vertex. But insufficient stress had been placed on a careful definition of what is meant by "bisecting" an angle. For example, one may be more inclined to sympathize with a student's troubles with minus signs if one realizes oneself that $-(-2)$ and $(-1)\ (-2)$ represent different situations. In any case, you will not teach them that "two minuses make a plus" because you can lay one minus sign vertically across the other thus: $+$. Many an advanced student, well trained in advanced techniques, remains unfamiliar with fundamental concepts because he has been expected to acquire familiarity with them by osmosis. This process does not always take place, perhaps because it proceeds only from the more dense to the less dense.

I think it is important that you try not to teach falsehoods that have to be corrected later, unless, as Forder remarks,* it be contended that an unsound proof has an educational value not possessed by a sound one. Needless to say, an unsound proof may well have educational value not possessed by a sound one provided it is used as an example of faulty reasoning. By this I do not mean at all that elementary teaching should be rigorous. I mean only that where gaps occur they should be pointed out, and that theorems which cannot be treated soundly at the student's level should be assumed or discussed informally rather than be given a bad proof.

Try to keep alive your interest in mathematics and its teaching. It is easier to give your students enthusiasm for mathematics if you have it yourself. Despite heavy schedules and other adverse conditions, it is always possible to come yet a little closer to attaining your ideal objectives, provided you know what you want them to be.† Try to teach in such a way that the students will acquire:

* H. G. Forder, The Foundations of Euclidean Geometry, Cambridge Press, 1927, p. viii.

† For an excellent discussion of the objectives of teaching elementary mathematics, see the Fifteenth Yearbook of the National Council of Teachers of Mathematics, which was drawn up by a joint committee of the Council and the Mathematical Association of America.

(1) an appreciation of the natural origin and evolutionary growth of the basic mathematical ideas from antiquity to the present; (2) a critical logical attitude and a wholesome respect for correct reasoning, precise definitions, and clear grasp of underlying assumptions; (3) an understanding of the role of mathematics as one of the major branches of human endeavor and its relations with other branches. Try to emphasize the distinction between familiarity and understanding, between proof and routine manipulation, between a critical attitude of mind and habitual unquestioning belief, between scientifically organized knowledge and both encyclopedic collections of facts and mere opinion and conjecture. Try to give them not only formulas but a wholesome appreciation of the nature and importance of mathematics.

Let me try to summarize my point of view in another way. There are three main aspects that must be brought out to some extent in the teaching of elementary mathematics, namely:

A. The routine techniques which are basic for future work. This aspect will be designated by *"techniques."*

B. The concrete applications of mathematics, the concrete settings in which mathematics originated, the interrelations among the "branches" of mathematics, and it relevance to the real world in general. For lack of a better term, this aspect will be referred to as the *"relevance"* of mathematics.

C. The reasonable justification of the techniques, and the logical nature of mathematics in general. This aspect will be designated by *"reasonableness."*

There seems to be some fear in certain quarters that overemphasis of C leads to excessive abstraction which will repel the general public from mathematics. This fear seems to be founded on the undeniable fact that some abstractions which appear in research papers lack a rich concrete background. This is true, but it is also true that few mathematicians are seriously interested in such an abstraction per se except possibly for the purpose of acquiring a Ph.D. or a promotion. Nevertheless, it must be granted that even such an empty abstract science is pure mathematics even though it is not important pure mathematics. It does not seem to me that anything is to be gained by hiding this innocent fact from the general public as though it were an important trade secret, much as the Pythagoreans are said to have tried to hide the irrationality of the square root of two. The truth will gain more for our subject than suppression of allegedly harmful ideas, which are, in my opinion, not harmful at all as long as one emphasizes that mathematicians are almost always interested only in abstractions which have rich concrete interpretations and which serve to unify or clarify concrete subjects which are interesting in themselves. As for this tendency to abstraction seeping harmfully into the teaching of elementary mathematics, such fear strikes me as being not only groundless but actually pointed in precisely the wrong direction.

In the actual practice of elementary teaching, as I have observed it, "techniques," "relevance," and "reasonablesness" are usually given attention in precisely that order, with "techniques" far in the lead almost to the total exclu-

sion of the other two, with "relevance" running a very bad second, and "reasonableness" a much worse third. It is my contention that this order should be reversed; in any case "reasonableness" and "relevance" deserve at least as much importance as "techniques." For students whose major interest is in the sciences, all three aspects should be virtually tied for first place. By this, I do not mean at all to belittle the importance of "techniques" for technical students, but I am convinced that when their reasonable justification is properly stressed, the techniques themselves require far less attention than is ordinarily given them. Understanding really does improve technical facility.

For students with little or no interest in the sciences, who really have little need for such facility, "techniques" should trail considerably behind. It is definitely my experience that these students can become very interested in the logical nature of mathematics, whereas they have no personal interest whatever in solving problems of a practical (to a technician) nature, no matter how interesting such problems may seem to a mathematician. It may seem surprising to many teachers, but it is nevertheless a fact that to many college freshmen the idea that algebra is a reasonable subject comes as a complete revelation. If there is danger to the status of mathematics, it does not arise from overemphasis of its "reasonableness." It comes from the deadly overemphasis on routine "techniques," and the unwholesome neglect of its "reasonableness" and of its "relevance" to the real world.

THE FIRST ANNUAL MEETING OF THE METROPOLITAN NEW YORK SECTION

The first annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Hunter College, New York City, on Saturday, April 18, 1942. Professor F. H. Miller presided at the morning session and Professor T. F. Cope, chairman of the Section, presided at the afternoon meeting.

The attendance was about one hundred and thirty-six, including the following seventy members of the Association: R. Lucile Anderson, R. G. Archibald, L. A. Aroian, Aaron Bakst, Brother Bernard Alfred (Welch), Frank Boehm, A. B. Brown, Jewell Hughes Bushey, J. H. Bushey, S. S. Cairns, H. R. Cooley, Elizabeth M. Cooper, T. F. Cope, W. H. H. Cowles, Marguerite D. Darkow, D. R. Davis, Carolyn Eisele, W. H. Fagerstrom, J. M. Feld, Edward Fleisher, R. M. Foster, B. P. Gill, Marion C. Gray, Etta Greenberg, Harriet M. Griffin, J. I. Griffin, C. C. Grove, N. A. Hall, L. S. Hill, J. H. Hlavaty, R. A. Johnson, L. S. Kennison, E. H. Koch, Jr., Edna E. Kramer-Lasser, Helen L. Kutman, A. W. Landers, Mary K. Landers, Nathan Lazar, C. H. Lehmann, C. C. MacDuffee, H. F. Mac Neish, J. J. McCarthy, P. H. McGrath, Mary McKenna, D. May Hickey Maria, A. E. Meder, F. H. Miller, E. C. Molina, Philip Newman, M. A. Nordgaard, Walter Penney, Mina S. Rees, Selby Robinson, S. G. Roth, H. D. Ruderman, Arthur Sard, Edna C. Schnefel, L. P. Siceloff, Lao G. Simons,

James Singer, C. S. Stuckey, J. A. Swenson, J. J. Tanzola, H. E. Wahlert, Louis Weisner, Mary E. Wells, A. Marie Whelan, D. E. Whitford, John Williamson, Jack Wolfe.

At the beginning of the morning session President G. N. Shuster of Hunter College welcomed the Section to Hunter College. At the beginning of the afternoon meeting the following officers were elected for the coming year: Chairman, H. F. Mac Neish, Brooklyn College; Vice-Chairman, Edna E. Kramer-Lasser, Thomas Jefferson High School; Secretary, H. E. Wahlert, New York University; Treasurer, F. H. Miller, Cooper Union.

The following four papers were presented at the morning session:

1. "An application of matrix theory to cryptography" by Professor L. S. Hill, Hunter College.

2. "The teaching of mathematics at the defense training institute" by C. H. Lehmann, Cooper Union.

3. "On the principles of statistical inference" by Dr. Abraham Wald, Columbia University and Queens College, introduced by Professor Cope.

4. "Mathematical training for aeronautical engineers" by Dr. N. A. Hall, Vought-Sikorsky Aircraft Company.

At the afternoon session a symposium on "Integrated mathematics in high school" was held. Dr. J. A. Swenson of Andrew Jackson High School presided at this symposium and the following four papers were presented:

5. "The significance of Δx in secondary mathematics" by Agnes Morley, Andrew Jackson High School, introduced by Dr. Swenson.

6. "Integrated mathematics with special application to the tenth year (geometry)" by Harry Sitomer, New Utrecht High School, Brooklyn, introduced by Dr. Swenson.

7. "Spatial and probable relationships in secondary mathematics" by Dr. Edna E. Kramer-Lasser, Thomas Jefferson High School, Brooklyn.

8. "Integrated mathematics in Catholic high schools" by Brother Anselm. St. Joseph's Normal Institute, introduced by Brother Bernard Alfred.

Abstracts of the papers follow.

1. Professor Hill's presentation was based upon two articles published in this MONTHLY: *Cryptography in an algebraic alphabet*, July, 1929, and *Certain linear transformation apparatus of cryptography*, March, 1931. The articles attracted wholly unanticipated attention in the United States and abroad. Their interest was strengthened by the circumstance that the author's learned, ingenious, and mechanically-minded colleague of some fifteen years, Professor Louis Weisner, was able to plan a machine for the speedy and effortless operation of those simpler types of transformation which were considered in the 1929 article. These transformations include, as special cases several of the more prominent military cipher systems, and afford extensions which seem to be quite beyond cryptanalytic approach.

2. Mr. Lehmann described the origin, organization, and scope of the Defense Training Institute of the Engineering Colleges of Greater New York. This new

school, in actual operation since February 10, 1941, is the only one in the country having a full daytime coordinated curriculum set up primarily to train engineering personnel for war industries. The address featured the place of mathematics in the Institute's curriculum and described some of the teaching problems in mathematics encountered in what is probably one of the most highly concentrated and intensive technical programs ever undertaken.

3. Dr. Wald discussed some modern developments of the theory of statistical estimation. Suppose x is a random variable and its probability distribution function $f(x, \theta)$ involves an unknown parameter θ. In the theory of estimation the value of the unknown parameter θ has to be estimated on the basis of N observations $x_1, \cdots, x_N$ on the variable x, *i.e.*, we have to construct a function $t(x_1, \cdots, x_N)$ of the observations which can be considered as a "good statistical estimate" of the parameter θ. In the modern theory definitions for a "good" or "best" statistical estimate are formulated and solutions are given for certain classes of cases. If the number of observations N is large, the so-called maximum likelihood estimate provides a satisfactory solution of the problem of estimation under fairly general conditions.

4. Dr. Hall discussed the mathematical training required by aeronautical engineers from the viewpoint of the engineer, whose primary consideration is the production of the most efficient airplane. He presented a brief picture of the program of development of airplane design from the original conception in terms of function and specifications to the final detail design for production. The various places where mathematics entered the program were pointed out to indicate the type of mathematical training required. It was observed that two distinct types of ability were demanded. First, the average engineer was called upon to use mathematics of a difficulty up to the grade of the calculus with absolute accuracy and dependability. The need for training in thoroughly reliable work as a basic requirement was stressed. The second demand was for a preparation in advanced fields of mathematical analysis with emphasis on numerical methods and the possession of the viewpoint of mathematical service to the engineering program. "When we are training aeronautical engineers, we must train men to develop airplanes and not mathematics."

6. Mr. Sitomer pointed out that, whether in the tenth year or not, integrated mathematics must not be confused with simultaneous, correlated, comprehensive, or general mathematics, which refer to the scope of instructional materials but not to the methods of presentation and solution. Integrated mathematics may be defined as a program, not a syllabus, which aims to enlist any branch of mathematics which is best fitted to solve a given problem, whether it be computation or proof. The more branches in use, the better the integration. The simpler the computation or proof, the better the integration. In the tenth year we may select methods from algebra, plane and solid geometry, trigonometry, calculus, function theory, number theory, vector analysis and topology. Courageous experimentation in the classroom by teachers, well-read in mathematics, will determine possible syllabi. In the tenth year special emphasis on the

nature of proof and on a postulational system demands a simple synthetic introduction with a gradual transition to analytic geometry and other methods of proofs. Integrated mathematics results in a rich set of concepts arranged spirally in a syllabus.

7. Dr. Kramer-Lasser showed how integrated mathematics is used to develop general space concepts. Even as early as the seventh year, elementary procedures form a foundation for more advanced work. For example, in working with paper patterns, proper numbering of vertices and edges eventually leads pupils to think of the surfaces they construct as sets of triangles with suitable identification of vertices and edges. In the ninth and tenth years, space geometry is studied along with plane. By rotating and translating $3, 4, \cdots, n-1$ dimensional forms, pupils obtain an abstract realization of $4, 5, \cdots, n$ dimensional figures. Combinatorial methods are also used to obtain the concept of higher dimensions. In the eleventh and twelfth years, elementary analytic geometry of three dimensions is studied and analytic generalization to higher dimensions is made.

8. Brother Anselm stated that, in general, the Catholic High Schools in and around New York City adhere to the traditional courses in mathematics. They do this because they think that their present curriculum is the best for their particular needs. They are reluctant to change to the integrated course because most of the leading educators in this field are not certain that integrated mathematics courses are better than the traditional courses. Moreover, they fear that lack of drill and understanding of the reasons for manipulative processes may be brought about by the integrated program. They are convinced also that the much desired purpose of an integrated program can be achieved with the traditional subjects. Hence the Catholic schools look, not to a change of courses to improve secondary mathematics, but rather to better teaching of the present courses.

H. E. WAHLERT, *Secretary*

THE ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The twenty-sixth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the Colorado School of Mines, Golden, Colorado, April 17 and 18, 1942. There were three sessions. Professor J. C. Fitterer presided at the first two and at the business session of the third. The Saturday morning session was a joint meeting with the mathematics section of the Eastern Division of the Colorado Education Association. Mr. H. W. Charlesworth of East Denver High School presided at this meeting.

The attendance was forty-four, including the following fifteen members of the Association: C. F. Barr, J. R. Everett, J. C. Fitterer, I. L. Hebel, A. J. Kempner, Claribel Kendall, A. J. Lewis, S. L. Macdonald, A. E. Mallory, W. K.

Nelson, Greta Neubauer, M. G. Pawley, A. W. Recht, C. H. Sisam, and W. E. Wilson.

At the business meeting the following officers were elected for the coming year: Chairman, Professor A. E. Mallory, Colorado State College of Education; Vice-Chairman, Professor A. W. Recht, University of Denver.

The following papers were presented:

1. "An improved cosine-law slide rule" by Professor I. L. Hebel, Colorado School of Mines.

2. "On determining the 'best' critical region for testing the null hypothesis when the parent populations follow the Poisson law" by H. T. Guard, Colorado State College, introduced by Professor Macdonald.

3. "The use of Cauchy's integral formula in evaluating certain improper integrals" by Professor A. J. Lewis, University of Denver.

4. "A study of roulettes with the aid of the cathode-ray oscillograph" by Professor M. G. Pawley, Colorado School of Mines.

5. "Geometrical demonstration of a theorem on envelopes and its application to solve a maximum and minimum problem" by G. E. Uhrich, University of Colorado, introduced by Professor Kempner.

6. "The mathematical approach to the fundamentals of hydraulics" by C. P. Vetter, Senior Engineer, Bureau of Reclamation, introduced by the Secretary.

7. "Remarks on repeating decimal fractions" by Professor Emeritus I. M. De Long, University of Colorado. Read by Professor A. J. Kempner.

8. "Periodic decimal fractions, primitive roots and quadratic residues" by Professor A. J. Kempner, University of Colorado.

9. "A report of instruction and learning activities as observed in geometry class rooms" by Professor A. E. Mallory, Colorado State College of Education.

10. "The handmaiden spurned" by Professor O. H. Rechard and Professor C. F. Barr, the University of Wyoming.

11. "Some simple proofs of the addition law in trigonometry" by G. E. Uhrich, University of Colorado, introduced by Professor Kempner.

12. "The use of mathematics in industry" by L. A. McElroy, Denver public Schools, introduced by the Secretary.

13. "Constructions by means of a marked ruler and other instruments" by Professor Claribel Kendall, University of Colorado.

Abstracts of the papers follow.

1. Professor Hebel gave an extension of an earlier paper concerning a slide rule for the solution of the cosine law of spherical trigonometry, particularly as applied to distances on the earth. By properly manipulating the equation, an improved slide rule has been devised which gives a direct solution by purely mechanical means, as contrasted to the original model where it was necessary to "take out" intermediate answers to reach the final results. The operation was demonstrated on a specially constructed two meter long classroom model slide rule.

2. This problem was first considered by Przyborowski and Wilenski in *Biometrica*, 1939.

Given the random variables x_i where $p(x_i|\lambda_i)=\lambda_i^{x_i}e^{-x_i}/x_i!$, $(i=1, 2)$. To determine the best critical region for testing the composite hypothesis, H_0, that $\lambda_1=\lambda_2$ with respect to the alternative hypothesis $\lambda_1<\lambda_2$. The test consists in finding a region, w_0, in the sample space, W, such that, if E is the observed sample point,

1. $P\{E\epsilon w_0 \mid H_0\}\leqq\epsilon$, the desired level of significance.
2. $P\{E\epsilon w_0 \mid H_1\}>P\{E\epsilon w \mid H_1\}$

where H_1 is any alternative hypothesis and w is any other region that satifies 1.

The test reduces to rejection of H_0 when $x_1\leqq x_\epsilon$ where x_ϵ is chosen so that

$$\sum_{x_1=0}^{x_\epsilon}\binom{s}{x_1}\left(\frac{1}{1+A}\right)^{x_1}\left(\frac{A}{1+A}\right)^{s-x_1}\leqq\epsilon$$

is satisfied. A is the ratio of the sample sizes and $s=x_1+x_2$.

The power function is

$$B(\theta\mid s)=\sum_{x_1=0}^{x_\epsilon}\binom{s}{x_1}\left(\frac{1}{1+A\theta}\right)^{x_1}\left(\frac{A\theta}{1+A\theta}\right)^{s-x}\quad\text{where}\quad\theta>1.$$

The calculations of tables given the critical region and values of the power function for specified values of s and θ and also with s unspecified were demonstrated by Mr. Guard.

3. Professor Lewis outlined methods of using Cauchy's integral formula and Cauchy's residue formula in evaluation of certain improper integrals with real integrands. He illustrated the method by applying it to two examples.

4. Professor Pawley made a study of the parametric equation

$$x=a_1\cos n_1t+a_2\cos n_2t+a_3\cos n_3t,$$
$$y=a_1\sin n_1t\pm a_2\sin n_2t+a_3\sin n_3t.$$

The equation was shown to represent a group of roulettes, including epicycloids, hypocycloids, epitrochoids, and hypotrochoids, depending upon the relations between the a's and n's. The various curves were pictured on the flourescent screen of a cathode-ray oscillograph. Simple rules were given for anticipating the form of roulette obtained when the a's and n's in the equation are varied. An equation was given for a curve approximating the N-sided regular polygon and these curves were displayed on the screen of the oscillograph.

5. Mr. Uhrich dealt mainly with a geometrical proof of the theorem: The envelope of the family of chords which cut segments of equal area from a given curve is tangent to each chord at its midpoint. The converse of this theorem is proved in Salmon's *Analytische Geometrie der Kegelschnitte*, Kap. 13, with the purpose of application to conic sections; however the method seems to be general.

Besides some special examples, Mr. Uhrich used this theorem to prove: Of all the chords which can be drawn through a fixed point within a closed convex curve, that chord which is bisected by the point cuts off a segment of minimum area from the curve. This theorem was given as a problem in the *Analyst*, vol. 5.

6. Mr. Vetter outlined the various methods of mathematical approach to the solution of problems connected with fluid flow. These methods permit of practical solutions and of solutions that may be verified experimentally only in comparatively few and comparatively simple cases. He demonstrated further, that as an alternative it is possible to attack the problems from purely dimensional considerations without the necessity of simplifying assumptions. The latter method, without leading to final answers, nevertheless gives precise information as to the mathematical form of the functional relationship between the physical quantities involved and, in many instances, permits experimental verification or experimental evaluation of specific functional constants. He analyzed the relationship of the method to the theory of models.

7. Professor Emeritus Ira M. De Long of the University of Colorado is eighty-seven years old, and has been for four months in a Boulder hospital. He has maintained his life-long interest in the properties of repeating decimals, and never tires of telling me interesting and amusing relations which he has discovered for himself. Without claims of priority I present two examples.

(1) If we expand a fraction a/p, p a prime, say $3/7$, we have $10\cdot 3=4\cdot 7+2$, $10\cdot 2=2\cdot 7+6$, $10\cdot 6=8\cdot 7+4$, $10\cdot 4=5\cdot 7+5$, $10\cdot 5=7\cdot 7+1$, $10\cdot 1=1\cdot 7+3$, with the digit period $428571'$ and the remainder period 264513. The $7(4+2+8+5+7+1)=9(2+6+4+5+1+3)$. For a/p, $p\cdot\sum(\text{digits})=9\cdot\sum(\text{remainder})$. For a base g instead of 10, $g\neq 1$, $\not\equiv 0 \bmod p$, 9 is replaced by $g-1$.

(2) To obtain the period of 1/49 from the period 142857 or 1/7, proceed as follows: Write

.020408	020408	020408	020408	020408	020408	020408
	142857	285714	428571	571428	714285	857142
020408	163265	306122	448979	591836	734693	877550

which is (except for the last 0 which has to be replaced by 1) the correct period.

8. The examination of the period length and digit properties, etc., of an ordinary repeating decimal have a particularly special character, on account of the orientation with respect to the base 10. By considering simultaneously the representation of all fractions a/p, $(a/p)=1$, for a given p, with respect to *all* bases $g>1$, and by fixing attention on the periodic *remainder* sequences rather than upon the periodic *digit* sequences, the general pattern or relations appears clearly. For a/p, $a=1, 2, \cdots, p-1$ we have for each base $g>1$, $(p/g)=1$, the following theorems:

I. Let $d_1=1, \cdots, d_\nu, \cdots, d_\mu=p-1$ be the divisors of $p-1$. Then, denoting

by $L(a/p)_g$ the period length in the remainder sequence, there are exactly $\phi(d_\nu)$ remainder sequences for which $L(a/p)_g=d_\nu$. (For $p=7$, $a=4$, $g=2, 3, \cdots, 6, 8$ we have remainder sequences 124, 513264, 214, 623154, 34, 4, respectively.) The same holds of course for the digit periods.

II. For a given g the remainders $1, 2, \cdots, p-1$, of a given a/p will either form exactly a remainder period, or the $p-1$ remainders break up into α sets of β each, $\alpha\beta=p-1$. In each of these $L(a/p)_g$ one of the three cases occur: Either each number of the period is quadratic residue of p, or each number is a non-residue; or the numbers are alternately residues or non-residues.

III. If $(g/p)=1$, $L(a/p)_g$, is a divisor of $(p-1)/2$, 1 and $(p-1)/2$ inclusive. But if $(g/p)=-1$, $L(a/p)_g$, which must divide $p-1$, does not divide $(p-1)/2$.

9. Professor Mallory gave a report of the type of work being done in certain geometry classes. The classes visited showed a high degree of understanding of the subject matter as revealed by the procedures.

10. The paper by Professor Rechard and Professor Barr grew out of the rejection by a geologist of a simple formula involving a few trigonometric functions on the basis that geologists would not use it. Instead, in his published paper, he described and illustrated a graphic method for solving a chain of right triangles. Where the fault lies for such a situation is the question raised by this paper, in the hope that out of it might come some clue to the old problem of how to make usable mathematics used.

11. In this paper Mr. Uhrich gave a presentation of three of the four proofs with slight variations for the addition theorem for the sine of the sum or difference of two angles which are given in Professor Gerhard Hessenberg's *Ebene und sphärische Trigonometrie*, Kap. IV.

12. The successful person in industry is the one who can separate essential and non-essential factors, get the job done and show a profit. Mathematical training is important to the extent that it applies to the job in hand. Pupils should be offered basic knowledge of mathematics with the problems that are *real to them*. This should help them to develop the habit of applying mathematical knowledge to practical situations. Industry specializes; mathematics essential to one industry has no value to another. But, since no one can predict with 100% accuracy the potentialities of an individual, it would seem wise to recommend training in mathematics for each pupil, at least until a choice of occupation has been made.

13. We are all familiar with constructions by means of an unmarked ruler and compasses, the euclidean instruments. Some or all of these constructions can be carried out by means of other instruments such as an angle ruler, a marked ruler, an unmarked ruler without compasses or by compasses alone. Professor Kendall paid particular attention to constructions by means of a marked ruler, including the trisection of the angle and the finding of a length which is the cube root of a given line segment.

A. J. LEWIS, *Secretary*

MODIFIED CONTINUED FRACTIONS

J. W. BRADSHAW, University of Michigan

In an article* in this MONTHLY the "modification" of slowly convergent continued fractions was suggested by the author as a means of obtaining forms that would converge more rapidly and thus be more useful in computation. In the present article the general considerations underlying such "modification" are developed and the procedure is illustrated with some concrete examples. The notion of a modified continued fraction is not new; Sylvester† in 1869 called attention to a first modification of Euler's continued fraction for $\pi/2$, and Glaisher‡ in 1874 compared this result with a similarly modified Wallis product. But its usefulness seems not to have been appreciated.

1. Definition. Given the convergent infinite continued fraction $b_0+a_1/b_1+a_2/b_2+\cdots$, by $K_\nu=A_\nu/B_\nu$ we denote the convergent of order ν, and by $L_\nu=C_\nu/D_\nu$, the same finite continued fraction with the last partial quotient, a_ν/b_ν, replaced by c_ν/d_ν. The problem is to determine c_ν/d_ν so that

(a) $\lim_{\nu\to\infty} L_\nu=\lim_{\nu\to\infty} K_\nu$,

(b) L_ν will converge more rapidly than K_ν.

We call the function of ν thus obtained the convergent of order ν of the "modified continued fraction."

2. Formulation of conditions. The variation with ν of the convergents of the continued fraction and the modified continued fraction is governed by the two pairs of equations:

$$(1)\quad \begin{cases} A_\nu = b_\nu A_{\nu-1} + a_\nu A_{\nu-2}, \\ B_\nu = b_\nu B_{\nu-1} + a_\nu B_{\nu-2}, \end{cases} \qquad (2)\quad \begin{cases} C_\nu = d_\nu A_{\nu-1} + c_\nu A_{\nu-2}, \\ D_\nu = d_\nu B_{\nu-1} + c_\nu B_{\nu-2}, \end{cases}$$

$$(\nu = 1, 2, 3, \cdots; \text{ it being assumed as usual that } A_{-1} = 1,\ B_{-1} = 0).$$

From the relation

$$(3)\qquad A_\nu B_{\nu-1} - A_{\nu-1}B_\nu = (-1)^{\nu-1}a_1a_2\cdots a_\nu, \quad (\nu = 1, 2, 3, \cdots),$$

it follows that

$$(4)\qquad \begin{aligned} A_\nu/B_\nu - C_\nu/D_\nu = {} & (-1)^{\nu-2}a_1a_2\cdots a_{\nu-1}(b_\nu c_\nu - a_\nu d_\nu) \\ & \div \left[b_\nu d_\nu B_{\nu-1}^2 + (b_\nu c_\nu + a_\nu d_\nu)B_{\nu-1}B_{\nu-2} + a_\nu c_\nu B_{\nu-2}^2\right], \\ & (\nu = 2, 3, \cdots). \end{aligned}$$

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* J. W. Bradshaw, Modified series, this MONTHLY, vol. 46, 1939, p. 486.

† J. J. Sylvester, Note on a new continued fraction applicable to the quadrature of the circle, Philosophical Magazine, Series 4, vol. 37, 1869, pp. 373–375.

‡ J. W. L. Glaisher, On the transformation of continued products into continued fractions, Proceedings of the London Mathematical Society, vol. 5, 1874, pp. 78–88.

If L_ν is to approach the same limit as K_ν, the limit of this expression must be zero. Since the continued fraction is assumed to be convergent,

$$\lim_{\nu\to\infty} (A_\nu/B_\nu - A_{\nu-1}/B_{\nu-1}) = \lim_{\nu\to\infty} (-1)^{\nu-1}a_1a_2\cdots a_\nu/B_\nu B_{\nu-1}$$
$$= \lim_{\nu\to\infty} (-1)^{\nu-1}a_1a_2\cdots a_\nu/(b_\nu B_{\nu-1}^2 + a_\nu B_{\nu-1}B_{\nu-2}) = 0.$$

The condition $\lim_{\nu\to\infty} L_\nu = \lim_{\nu\to\infty} K_\nu$ will then certainly be satisfied, if

$$\begin{aligned} &| a_\nu d_\nu - b_\nu c_\nu | \leqq | a_\nu d_\nu |, \\ (5)\qquad &| b_\nu d_\nu B_{\nu-1}^2 + (b_\nu c_\nu + a_\nu d_\nu)B_{\nu-1}B_{\nu-2} + a_\nu c_\nu B_{\nu-2}^2 | \qquad (\nu = 2, 3, \cdots) \\ &\qquad \geqq | b_\nu d_\nu B_{\nu-1}^2 + a_\nu d_\nu B_{\nu-1}B_{\nu-2} |. \end{aligned}$$

Now further, we wish $C_\nu/D_\nu - C_{\nu-1}/D_{\nu-1}$ to approach zero faster than $A_\nu/B_\nu - A_{\nu-1}/B_{\nu-1}$. Use of the reduction formulas (1) and (2) yields

$$C_\nu D_{\nu-1} - C_{\nu-1}D_\nu = (-1)^{\nu-2}a_1a_2\cdots a_{\nu-2}(d_\nu d_{\nu-1}a_{\nu-1} - d_\nu c_{\nu-1}b_{\nu-1} - c_\nu c_{\nu-1}),$$
$$(\nu = 2, 3, \cdots, \text{ if for } \nu = 2 \text{ we put } a_1a_2\cdots a_{\nu-2} = 1).$$

The comparative rate of approach to zero is then given by

$$\begin{aligned} (6)\qquad &(C_\nu/D_\nu - C_{\nu-1}/D_{\nu-1})/(A_\nu/B_\nu - A_{\nu-1}/B_{\nu-1}) \\ &= - B_\nu B_{\nu-1}(d_\nu d_{\nu-1}a_{\nu-1} - d_\nu c_{\nu-1}b_{\nu-1} - c_\nu c_{\nu-1})/D_\nu D_{\nu-1}a_\nu a_{\nu-1}, \ (\nu = 2, 3, \cdots). \end{aligned}$$

The limit of this multiplied by ν^α should equal a finite quantity G for some positive α.

3. Application to a continued fraction for $\pi/2$. As an illustration we consider the continued fraction, $1+K_{\nu=1}^{\infty} 1/(1/\nu)$, which is equivalent to Euler's continued fraction for $\pi/2$. Here $b_0=1$, $a_\nu=1$, $b_\nu=1/\nu$; we assume $c_\nu=1$, and conditions (5) and (6) take the simpler form

$$(7)\qquad d_\nu > 0, \quad | d_\nu B_{\nu-1}^2 + (1+\nu d_\nu)B_{\nu-1}B_{\nu-2} + \nu B_{\nu-2}^2 | \geqq | d_\nu B_{\nu-1}^2 + \nu d_\nu B_{\nu-1}B_{\nu-2} |,$$

$$(8)\qquad - B_\nu B_{\nu-1}(d_\nu d_{\nu-1} - d_\nu b_{\nu-1} - 1)\nu^\alpha/D_\nu D_{\nu-1} = G \neq 0 \text{ for a positive } \alpha.$$

For d_ν we choose a rational fraction in ν, seeking to determine coefficients so that

$$(9)\qquad d_\nu(b_{\nu-1} - d_{\nu-1}) + 1$$

shall be of low degree in ν. It is clear that numerator and denominator of d_ν should be of the same degree and have the same leading coefficient. Assume

$$(10)\qquad d_\nu = (q_0\nu^t + q_1\nu^{t-1} + \cdots + q_t)/(p_0\nu^t + p_1\nu^{t-1} + \cdots + p_t),$$
$$q_0 = p_0, \ t \text{ an integer}.$$

Then the numerator of the expression (9) takes the form

$$(11)\quad \begin{aligned} &(q_0\nu^t + q_1\nu^{t-1} + \cdots + q_t) \\ &[-q_0(\nu-1)^{t+1} + (p_0 - q_1)(\nu-1)^t + \cdots + (p_{t-1} - q_t)(\nu-1) + p_t] \\ &+ (p_0\nu^t + p_1\nu^{t-1} + \cdots + p_t)[p_0(\nu-1)^{t+1} + p_1(\nu-1)^t + \cdots + p_t(\nu-1)], \end{aligned}$$

while the denominator is

$$(12)\quad (p_0\nu^t + p_1\nu^{t-1} + \cdots + p_t)[p_0(\nu-1)^{t+1} + p_1(\nu-1)^t + \cdots + p_t(\nu-1)].$$

The number of terms, powers of ν including the zeroth, in (11) is $2t+2$, while the number of independent coefficients at our disposal is $2t+1$. It is therefore possible to make this numerator independent of ν. Equating to zero coefficients of successive powers of ν we obtain the sequence of equations:

$$(13)\quad \begin{aligned} p_0 - q_0 &= 0, \\ p_1 - q_1 &= -\frac{1}{2}p_0, \\ p_2 - q_2 &= -\frac{1}{2}p_1 - \frac{3}{8}p_0, \\ p_3 - q_3 &= -\frac{1}{2}p_2 - \frac{3}{8}p_1, \\ p_4 - q_4 &= -\frac{1}{2}p_3 - \frac{3}{8}p_2 + \frac{45}{128}p_0, \\ p_5 - q_5 &= -\frac{1}{2}p_4 - \frac{3}{8}p_3 + \frac{45}{128}p_1, \\ p_6 - q_6 &= -\frac{1}{2}p_5 - \frac{3}{8}p_4 + \frac{45}{128}p_2 - \frac{1125}{1024}p_0, \\ p_7 - q_7 &= -\frac{1}{2}p_6 - \frac{3}{8}p_5 + \frac{45}{128}p_3 - \frac{1125}{1024}p_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots . \end{aligned}$$

These yield the following sequence of possibilities for d_ν:

$$(14)\quad \begin{aligned} \frac{4\nu - 1}{4\nu - 3} &= 1 + \frac{2}{|4\nu - 3}, \\ \frac{16\nu^2 - 12\nu + 11}{16\nu^2 - 20\nu + 15} &= 1 + \frac{2\ |}{|4\nu - 3} + \frac{9\ |}{|4\nu - 2}, \\ \frac{64\nu^3 - 80\nu^2 + 168\nu - 47}{64\nu^3 - 112\nu^2 + 200\nu - 105} &= 1 + \frac{2\ |}{|4\nu - 3} + \frac{9\ |}{|4\nu - 2} + \frac{25\ |}{|4\nu - 2}. \end{aligned}$$

This suggests that the successive convergents of a certain infinite continued fraction will furnish the modification we desire.

4. Continued fraction form of modification. We now define a new continued fraction, using ρ as index and accents on the quantities connected with this to distinguish them from those of the continued fraction with which we started:

$$4\nu - 3 + \mathop{K}_{\rho=1}^{\infty} \frac{(2\rho + 1)^2}{4\nu - 2} \cdot \tag{15}$$

The convergents of order 0, 1, and 2 have entered into the values of d_ν given above. We now seek to show that if $A'_\rho(\nu)/B'_\rho(\nu)$ denotes the convergent of order ρ of this fraction,

$$d_\nu = 1 + 2 \div A'_\rho(\nu)/B'_\rho(\nu) = [A'_\rho(\nu) + 2B'_\rho(\nu)]/A'_\rho(\nu) \tag{16}$$

furnishes the desired modification for $\rho = 0, 1, 2, \cdots$. The expression (9) now takes the form

$$\frac{A'_\rho(\nu) + 2B'_\rho(\nu)}{A'_\rho(\nu)}\left[\frac{1}{\nu - 1} - \frac{A'_\rho(\nu - 1) + 2B'_\rho(\nu - 1)}{A'_\rho(\nu - 1)}\right] + 1$$

$$= \frac{1}{(\nu - 1)A'_\rho(\nu)A'_\rho(\nu - 1)}[A'_\rho(\nu)A'_\rho(\nu - 1) - 2(\nu - 1)A'_\rho(\nu)B'_\rho(\nu - 1)$$
$$- 2(\nu - 2)A'_\rho(\nu - 1)B'_\rho(\nu) - 4(\nu - 1)B'_\rho(\nu)B'_\rho(\nu - 1)].$$

The numerator of this fraction we denote by $M_\rho(\nu)$ and proceed to show that it is independent of ν. We have, in fact, $M_0(\nu) = -9$, $M_1(\nu) = 225$, $M_2(\nu) = -11025$, $M_3(\nu) = 893025$. By means of the reduction formulas $M_\rho(\nu)$ can be expressed in terms of $M_{\rho-1}(\nu)$, $M_{\rho-2}(\nu)$, and two new functions that we denote by $U_\rho(\nu)$ and $V_\rho(\nu)$:

$$U_\rho(\nu) = A'_\rho(\nu)A'_{\rho-1}(\nu - 1) - 2(\nu - 1)A'_\rho(\nu)B'_{\rho-1}(\nu - 1)$$
$$- 2(\nu - 2)B'_\rho(\nu)A'_{\rho-1}(\nu - 1) - 4(\nu - 1)B'_\rho(\nu)B'_{\rho-1}(\nu - 1),$$
$$V_\rho(\nu) = A'_\rho(\nu - 1)A'_{\rho-1}(\nu) - 2(\nu - 1)B'_\rho(\nu - 1)A'_{\rho-1}(\nu)$$
$$- 2(\nu - 2)A'_\rho(\nu - 1)B'_{\rho-1}(\nu) - 4(\nu - 1)B'_\rho(\nu - 1)B'_{\rho-1}(\nu).$$

We then have

$$M_\rho(\nu) = (4\nu - 2)(4\nu - 6)M_{\rho-1}(\nu) + (4\nu - 2)(2\rho + 1)^2U_{\rho-1}(\nu)$$
$$+ (4\nu - 6)(2\rho + 1)^2V_{\rho-1}(\nu) + (2\rho + 1)^4M_{\rho-2}(\nu).$$

Further application of the reduction formulas yields

$$U_{\rho-1}(\nu) = (4\nu - 2)M_{\rho-2}(\nu) + (2\rho - 1)^2V_{\rho-2}(\nu),$$
$$V_{\rho-1}(\nu) = (4\nu - 6)M_{\rho-2}(\nu) + (2\rho - 1)^2U_{\rho-2}(\nu),$$

and then from $M_\rho(\nu)$ and $M_{\rho-2}(\nu)$ the quantities $U_{\rho-3}(\nu)$ and $V_{\rho-3}(\nu)$ can be eliminated, giving finally

$$
\begin{aligned}
M_\rho(\nu) = {} & (4\nu - 2)(4\nu - 6)M_{\rho-1}(\nu) \\
& + (2\rho + 1)^2(32\nu^2 - 64\nu + 40 + 8\rho^2 + 2)M_{\rho-2}(\nu) \\
& + (2\rho + 1)^2(2\rho - 1)^2(4\nu - 2)(4\nu - 6)M_{\rho-3}(\nu) \\
& - (2\rho + 1)^2(2\rho - 1)^2(2\rho - 3)^4 M_{\rho-4}(\nu).
\end{aligned} \tag{17}
$$

By mathematical induction it can then be shown that

$$M_\rho(\nu) = (-1)^{\rho-1}3^2\cdot 5^2\cdot \cdots \cdot(2\rho+3)^2, \quad (\rho = 0, 1, 2, \cdots). \tag{18}$$

The original continued fraction converges very slowly; the value of $\pi/2$ obtained by its use shows an error of 15 in the second decimal place when ν is taken equal to 4, and even when $\nu=8$, the error is still more than 8 in this second decimal place. This is to be contrasted with an error of 3 in the seventh decimal place obtained by using the modified continued fraction with $\nu=4$, $\rho=4$.

5. Other solutions. Other solutions of the problem proposed can be obtained by imposing different conditions on d_ν. If instead of requiring that (11) shall be independent of ν, we now require that it shall be the product of $\nu-1$ and a quantity independent of ν, we obtain a second solution. This will enable us to cancel the factor $\nu-1$ from numerator and denominator of d_ν. To this end we put $p_t=0$ and proceed to determine coefficients as before. The sequence of values of d_ν are now convergents of a modified continued fraction, namely, the same continued fraction as before, except that the ρth partial quotient $(2\rho+1)^2/(4\nu-2)$ is to be replaced by $(2\rho+1)^2/(4\nu+2\rho+1)$. A suggested notation for this modified continued fraction is the following:

$$4\nu - 3 + \mathop{K}_{1}^{\rho} \frac{(2\rho+1)^2}{4\nu - 2} (m) \frac{(2\rho+1)^2}{4\nu + 2\rho + 1} \cdot \tag{19}$$

The rate of convergence is somewhat improved as soon as $\rho \geqq 2\nu$.

A third solution may be obtained by arranging for further cancellation. Since we have taken $p_t=0$, the denominator (12) is now divisible by $(\nu-1)^2$. The numerator will also contain this factor and it may be cancelled, if we take $p_{t-1}-q_t=0$. If the remaining coefficients are then determined, the result is the same continued fraction modified in a different way; the ρth partial quotient is $(2\rho+1)^2/(2\nu+2\rho+1)$:

$$4\nu - 3 + \mathop{K}_{1}^{\rho} \frac{(2\rho+1)^2}{4\nu - 2} (m) \frac{(2\rho+1)^2}{2\nu + 2\rho + 1} \cdot \tag{20}$$

This modified continued fraction converges still more rapidly, if $\rho \geqq 2\nu$.

6. Double modification. The two modified continued fractions met above suggest the application of the general method directly to the continued fraction $4\nu-3+K_{\rho=1}^{\infty}(2\rho+1)^2/(4\nu-2)$. For this purpose we take $c_\rho'=a_\rho'=(2\rho+1)^2$ and

$$d_\rho' = (q_0'\rho^{t+1} + q_1'\rho^t + \cdots + q_{t+1}')/(p_0'\rho^t + p_1'\rho^{t-1} + \cdots + p_t').$$

Condition (6) now leads us to try to make $d'_\rho(d'_{\rho-1}-b'_{\rho-1})-c'_\rho$ of low degree in ρ. The coefficients of powers of ρ in the numerator of this expression we equate to zero, obtaining the sequence of equations:

$$
\begin{aligned}
q_0' - 2p_0' &= 0,\\
q_1' - 2p_1' &= (2\nu+1)p_0',\\
q_2' - 2p_2' &= (2\nu+1)p_1' + \nu^2 p_0',\\
q_3' - 2p_3' &= (2\nu+1)p_2' + \nu^2 p_1' - \nu^2 p_0',\\
q_4' - 2p_4' &= (2\nu+1)p_3' + \nu^2 p_2' - \nu^2 p_1' - \tfrac{1}{4}(\nu^4+2\nu^3-3\nu^2)p_0',\\
q_5' - 2p_5' &= (2\nu+1)p_4' + \nu^2 p_3' - \nu^2 p_2' - \tfrac{1}{4}(\nu^4+2\nu^3-3\nu^2)p_1'\\
&\quad + \tfrac{1}{4}(3\nu^4+6\nu^3-\nu^2)p_0',\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\tag{21}
$$

From these we obtain for d'_ρ a sequence of rational fractions in ρ, which may be thrown into the form of 4 times successive convergents of a new continued fraction, for which we take the new parameter σ:

$$\tfrac{1}{4}(2\nu+2\rho+1)+K_{\sigma=1}^{\infty}\ \tfrac{1}{4}(\nu-1+\sigma)^2/(\rho+1).$$

We shall distinguish the elements and convergents of this continued fraction by double accents:

$$b_0'' = \tfrac{1}{4}(2\nu+2\rho+1),\qquad a_\sigma'' = \tfrac{1}{4}(\nu-1+\sigma)^2,\qquad b_\sigma'' = \rho+1.$$

7. The function $f(\nu,\rho,\sigma)$. Our approximations to the value of the original continued fraction, viz., $\pi/2$, are now functions of three indices; we proceed as far as we like with the original continued fraction, $1+K_\nu 1/(1/\nu)$, replace $1/\nu$ by $1+2$ divided by the second continued fraction, $4\nu-3+K_\rho(2\rho+1)^2/(4\nu-2)$ carried as far as we wish, and finally replace the denominator of $(2\rho+1)^2/(4\nu-2)$ by four times the third continued fraction, again continued as far as may be desired. The result of these steps may be put in a single formula:

$$
f(\nu,\rho,\sigma) = \frac{\begin{aligned}&[4A_\sigma''A_{\rho-1}' + (2\rho+1)^2B_\sigma''A_{\rho-2}'][A_{\nu-1}+A_{\nu-2}]\\ &\qquad + 2[4A_\sigma''B_{\rho-1}' + (2\rho+1)^2B_\sigma'\ B_{\rho-2}']A_{\nu-1}\end{aligned}}{\begin{aligned}&[4A_\sigma''A_{\rho-1}' + (2\rho+1)^2B_\sigma''A_{\rho-2}'][B_{\nu-1}+B_{\nu-2}]\\ &\qquad + 2[4A_\sigma''B_{\rho-1}' + (2\rho+1)^2B_\sigma''\cdot B_{\rho-2}']B_{\nu-1}\end{aligned}}.
\tag{22}
$$

In this we have for

ν: $\quad A_0 = 1,\qquad B_0 = 1;$

$$A_\nu = \frac{1}{\nu}A_{\nu-1} + A_{\nu-2},\qquad B_\nu = \frac{1}{\nu}B_{\nu-1} + B_{\nu-2},\qquad (\nu = 1, 2, 3, \cdots).$$

$$\begin{aligned}
\rho:\quad & A_0' = 4\nu - 3, \qquad B_0' = 1;\\
& A_\rho' = (4\nu - 2)A_{\rho-1}' + (2\rho + 1)^2 A_{\rho-2}',\\
& B_\rho' = (4\nu - 2)B_{\rho-1}' + (2\rho + 1)^2 B_{\rho-2}', && (\rho = 1, 2, 3, \cdots).\\
\sigma:\quad & A_0'' = \tfrac{1}{4}(2\nu + 2\rho + 1), \qquad B_0'' = 1;\\
& A_\sigma'' = (\rho + 1)A_{\sigma-1}'' + \tfrac{1}{4}(\nu - 1 + \sigma)^2 A_{\sigma-2}'',\\
& B_\sigma'' = (\rho + 1)B_{\sigma-1}'' + \tfrac{1}{4}(\nu - 1 + \sigma)^2 B_{\sigma-2}'', && (\sigma = 1, 2, 3, \cdots).
\end{aligned}$$

By substituting in this formula we find, for example, that $f(4, 4, 4)$ yields an approximation with an error of 2 in the ninth decimal place. But we also note that $f(5, 4, 3)$ and $f(8, 4, 0)$ have exactly the same value. Investigation of a considerable number of such cases leads to the surmise that, in general,

$$f(\nu, \rho, \sigma) = f(\nu + 1, \rho, \sigma - 1). \tag{23}$$

It may be argued that if this surmise is correct our last modification has accomplished nothing from a practical point of view; we would have done just as well to use formula (20), replacing ν by $\nu+\sigma$. As an illustration of the method, however, it has some interest.

THE ROOTS OF A QUATERNION

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The theorems relative to the roots of a quaternion given by Niven in this MONTHLY, vol. 49, 1942, pp. 386–388, may be very simply obtained by writing a quaternion $q=d+ai+bj+ck$ in the form

$$q = k(\cos\theta + \epsilon \sin\theta), \qquad 0 \leqq \theta < 2\pi. \tag{1}$$

Here $k=\sqrt{d^2+a^2+b^2+c^2}$,

$$\cos\theta = \frac{d}{k}, \qquad \sin\theta = \pm\frac{\sqrt{a^2+b^2+c^2}}{k},$$

and in case $a^2+b^2+c^2\neq 0$, ϵ is the unit vector

$$\epsilon = \pm\frac{ai+bj+ck}{\sqrt{a^2+b^2+c^2}}\cdot$$

When q is real $(a=b=c=0)$ ϵ may be chosen at pleasure.

Since $\epsilon^2=-1$, we have by De Moivre's Theorem

$$q^n = k^n(\cos n\theta + \epsilon \sin n\theta). \tag{2}$$

If $q^n=Q$, Q will have the same unit vector as q. With Q given, we can always write

$$Q = K(\cos\phi + \epsilon \sin\phi), \qquad 0 \leqq \phi \leqq \pi. \tag{3}$$

Then, with the same ϵ for q, we have

$$k^n = K, \qquad \cos n\theta = \cos\phi, \qquad \sin n\theta = \sin\phi,$$

and the nth roots of Q are given by (1) provided

$$k = K^{1/n}, \qquad \text{the positive root,} \tag{4}$$

$$\theta = \frac{\phi}{n} + m\frac{2\pi}{n}, \qquad (m = 0, 1, \cdots, n-1). \tag{5}$$

These n values of θ comprise all values in the interval $0 \leqq \theta < 2\pi$ which satisfy the equations above.

When Q is *not real*, q has the same unit vector as Q and the n values of θ in (5) give precisely n quaternion nth roots.

When Q is *real* ϵ is arbitrary.

If $Q>0$, $\phi=0$ and

$$\theta = \frac{2m\pi}{n}, \qquad (m = 0, 1, \cdots, n-1).$$

When $n=2$, the values $\theta=0$, π give just two roots $\pm\sqrt{Q}$, both real. When $n>2$, some values of θ ($\neq 0$ or π) give non-real roots q and with these any unit vector ϵ may be associated.

If $Q<0$, $\phi=\pi$ and

$$\theta = \frac{(2m+1)\pi}{n}, \qquad (m = 0, 1, \cdots, n-1).$$

In every case some values of θ ($\neq\pi$) give non-real roots q and with these any unit vector ϵ may be associated.

We therefore have proved the

THEOREM. *A non-real quaternion has exactly n nth roots. If Q is real and positive it has just two square roots, $\pm\sqrt{Q}$; in all other cases a real quaternion has infinitely many roots.*

In order to *compute* the roots, formulas (4) and (5) suffice in all cases. For example if $Q=1+i+j+k$, we write

$$Q = 2(\cos 60^\circ + \epsilon \sin 60^\circ), \qquad \epsilon = \frac{i+j+k}{\sqrt{3}}\cdot$$

The cube roots of Q are then

$$q = \sqrt[3]{2}\,(\cos\theta + \epsilon\sin\theta), \qquad \theta = 20^\circ, 140^\circ, 260^\circ.$$

THE SPHERICAL PENDULUM AND COMPLEX INTEGRATION

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A century ago, Puiseux [1] published the following theorem: *The increment Φ of the azimuth of a spherical pendulum corresponding to its passage from the lowest level z_1 to the highest level z_2 is greater than $\pi/2$.* Later Halphen [2] found that Φ *is less than* π. Subsequently A. de Saint-Germain [3] and Gerard [4] gave simpler proofs for both inequalities, still involving somewhat laborious computations. Moreover, A. de Saint-Germain [3], following an idea of Hadamard [5], gave a simple and direct proof of Halphen's inequality based on the theory of residues.

In the present note complex integration will be used to give a simple demonstration of both inequalities. In the case of Halphen's inequality the argument reduces to that of Saint-Germain. The method allows a discussion of a new case.

It is well known that Φ is given by the formula

$$\Phi = \int_{z_1}^{z_2} \frac{lA\,dz}{(l^2 - z^2)\sqrt{(z - z_1)(z - z_2)(z - z_3)}} \tag{1}$$

where l is the length of the pendulum and z_1, z_2, z_3, satisfy the following conditions

$$-\,l < z_1 < z_2 < l < z_3, \qquad z_1 + z_2 < 0, \qquad z_3 = -\frac{l^2 + z_1 z_2}{z_1 + z_2},$$

so that the condition $z_3 > l$ is automatically satisfied. We put

$$A^2 = -\frac{(l^2 - z_1^2)(l^2 - z_2^2)}{z_1 + z_2} \tag{2}$$

and denote by A the positive square root of A^2. The polynomial $(z-z_1)(z-z_2)(z-z_3)$ takes for $z = \pm l$ the negative real value $-A^2$. We see that Φ depends only on the two real parameters z_1 and z_2 which are subject to the conditions $-l<z_1<z_2<l$ and $z_1+z_2<0$.

Considering the integrand in (1) as a function of the complex variable $z=x+iy$, we obtain a single valued branch of this function in the domain exterior to the two narrow lanes (or cuts) C_1 and C_2 (see the Figure) by putting

$$z - z_k = r_k e^{i\theta_k}, \qquad 0 \leqq \theta_k < 2\pi, \qquad (k = 1, 2, 3). \tag{3}$$

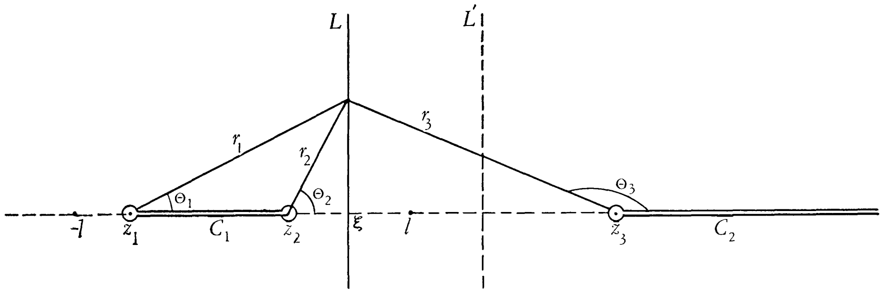
With these conventions Φ is the positive value obtained by integration in the positive sense along the lower border of C_1. Our domain contains the two poles of our integrand $z = \pm l$; the corresponding residues are $i/2$, $i/2$. Applying Cauchy's theorem to our domain, Saint-Germain obtains immediately Halphen's inequality $\Phi<\pi$. It may be noticed that, by Cauchy's theorem, C_2 can be replaced by a *vertical line* L' cutting the real axis between l and z_3.

In order to obtain Puiseux's inequality $\Phi > \pi/2$ we shift L' to a new position L of abscissa $x = \xi$, $0 \leqq \xi < l_1$ and we apply Cauchy's theorem to the domain on the left of this vertical line L. We obtain at once

$$2\Phi = \pi + \int_{\xi - i\infty}^{\xi + i\infty} \tag{4}$$

where the integrand on the right hand side is the same as in (1). Since Φ and π are real the integral in (4) reduces to its real part. We have therefore, putting $\alpha = \alpha(y) = (\theta_1 + \theta_2 + \theta_3)/2$:

$$\int_{\xi - i\infty}^{\xi + i\infty} = \int_{-\infty}^{+\infty} \frac{ilA\,dy}{(l^2 - \xi^2 + y^2 - 2i\xi y)e^{i\alpha}\sqrt{r_1 r_2 r_3}}$$
$$= lA \int_{-\infty}^{+\infty} \left[\frac{(l^2 - \xi^2 + y^2)\sin\alpha - 2\xi y \cos\alpha}{(l^2 - \xi^2 + y^2)^2 + 4\xi^2 y^2} \right] \frac{dy}{\sqrt{r_1 r_2 r_3}}.$$



In order to show that this integral is positive *let us take the non-negative abscissa ξ of L nearer to z_2 than to z_3: $\xi - z_2 < z_3 - \xi$.* Since $l < z_3$ and $-l < z_2 < l$ this condition can be fulfilled by non-negative values of ξ. We have then (see the Figure) for $y > 0$: $\theta_2 + \theta_3 > \pi$ and, *a fortiori*, $\theta_1 + \theta_2 + \theta_3 > \pi$. Moreover, we have obviously $\theta_1 < \pi/2$, $\theta_2 < \pi/2$, $\theta_3 \leqq \pi$ so that $\pi/2 < \alpha < \pi$ and the integrand is positive for $y > 0$. Since $\alpha(-y) = 2\pi + \pi - \alpha(y)$ the integrand is an even function of y and therefore is positive also for $y < 0$. In this way we obtain from (5) and (4), substituting the explicit value for the positive quantity A from (2),

$$\Phi = \pi/2 + l\sqrt{\frac{(l^2 - z_1^2)(l^2 - z^2)}{-(z_1 + z_2)}} \int_0^\infty \frac{[(l^2 - \xi^2 + y^2)\sin\alpha - 2\xi y\cos\alpha]dy}{[(l^2 - \xi^2 + y^2)^2 + 4\xi^2 y^2]\sqrt{r_1 r_2 r_3}} \tag{5}$$

which proves Puiseux's inequality $\Phi > \pi/2$.

By Cauchy's theorem we can shift now the abscissa ξ to any point of the interval $z_2 < \xi < l$. For instance, if z_1 and z_2 are both negative, we can take $\xi = 0$. Then by (6)

$$\Phi = \pi/2 + l\sqrt{\frac{(l^2 - z_1^2)(l^2 - z_2^2)}{-(z_1 + z_2)}} \int_0^\infty \frac{\sin\alpha\,dy}{(l^2 + y^2)\sqrt{r_1 r_2 r_3}}. \tag{7}$$

Consider now the case when z_1 and z_2 tend to $-l$, which implies that z_3 is converging to $l+0$. The integral in (7) remains finite and tends to a value J, so that we obtain $\lim \Phi=\pi/2$, a well known result which is usually obtained by somewhat delicate reasoning. The value J can be easily computed. Putting $z_1=z_2=-l$ we have

$$r_1 = r_2 = r_3,\ \theta_1 = \theta_2 = \pi - \theta_3, \alpha = \frac{\pi + \theta_1}{2}, r_1 \cos\theta_1 = l,\ y = l \tan\theta_1,\ l^2 + y^2 = r_1^2.$$

Therefore we have

$$J = \int_0^\infty \frac{\cos\dfrac{\theta_1}{2}\,dy}{r_1^3\sqrt{r_1}} = \frac{1}{l^2\sqrt{l}}\int_0^{\pi/2} \cos\frac{\theta_1}{2}\cos\theta_1\sqrt{\cos\theta_1}\,d\theta_1$$

or, putting $\zeta=\sqrt{2}\sin\theta_1/2$,

$$J = \frac{\sqrt{2}}{l^2\sqrt{l}}\int_0^1 (1-\zeta^2)\sqrt{1-\zeta^2}\,d\zeta = \frac{\sqrt{2}}{l^2\sqrt{l}}\,\frac{3\pi}{16}\cdot$$

Since $-(z_1+z_2)$ tends to $2l$, we have finally, from (6)[7]

$$\lim\left(\Phi - \frac{\pi}{2}\right)(l^2 - z_1^2)^{-1/2}(l^2 - z_2^2)^{-1/2} = 3\pi/16.$$

Using again (6), we see that Φ converges to $\pi/2$ when z_1 tends to $-l$ and z_2 remains arbitrarily fixed. In this case the motion approximates the oscillations of a simple pendulum of finite amplitude. It seems that this case has not been mentioned in the bibliography of the question. Let us also note that the method of complex integration allows to represent the excess of Φ over $\pi/2$ explicitly in the form of an integral.

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THE HAGGE CIRCLE OF A TRIANGLE

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1. Introduction. If from the vertices A_i, $i=1, 2, 3$, of a triangle, lines are drawn through a fixed point P, meeting the circumcircle again at B_i, and if C_i is the reflection of B_i in A_jA_k, then the points C_i determine the *Hagge Circle* of P with respect to triangle $A_1A_2A_3$ [1, p. 300]. We denote by $H(P, T)$ the Hagge circle of P with respect to triangle T.

In the following discussion we shall use conjugate coördinates,* taking as the vertices of our triangle T, the turns t_i, $i=1, 2, 3$; *i.e.*, complex numbers with unit modulus. Then the origin of coördinates, O, is the circumcenter of T.

Consider a fixed point p not on the circumcircle of T, so that p is not a turn. Then the points C_i which determine $H(p, T)$ are readily found to be

$$c_i = t_j + t_k - \frac{t_jt_k(t_i\bar{p} - 1)}{t_i - p}\cdot \tag{1}$$

Introducing the symmetric functions

$$S_1 = t_1 + t_2 + t_3,$$
$$S_2 = t_1t_2 + t_2t_3 + t_3t_1,$$
$$S_3 = t_1t_2t_3,$$

we may write (1) in the form

$$c_i = \frac{S_2 - S_3\bar{p} - pS_1 + pt_i}{-p + t_i}\cdot \tag{2}$$

If we replace t_i by the variable turn t, we obtain

$$x = \frac{S_2 - S_3\bar{p} - pS_1 + pt}{-p + t} \tag{3}$$

which is the equation of a circle through the points c_i, or precisely $H(p, T)$. If, in particular, we let $t=(S_2-S_3\bar{p})/(S_1-p)$ we have $x=S_1$. But S_1 is the orthocenter of T [4], so that *the Hagge circle of any point with respect to a given triangle passes through the orthocenter of the triangle* [1, p. 300].

Eliminating t from (3) and from the conjugate equation of (3), we have the equation of $H(p, T)$ in the form

$$(x - c)(\bar{x} - \bar{c}) = q\bar{q} \tag{4}$$

where

$$c = \frac{p - S_1p\bar{p} + S_2\bar{p} - S_3\bar{p}^2}{1 - p\bar{p}} \tag{5}$$

* For explanation of this method, see [2], [3] and [4].

and

$$= \frac{S_3\bar{p}^2 - S_2\bar{p} + S_1 - p}{1 - p\bar{p}}; \tag{6}$$

that is, $H(p, T)$ has center at c, and radius $\sqrt{q\bar{q}}$. But from (6), q is the isogonal conjugate [4] of p in T. Hence, *the radius of the Hagge circle of p is the distance from the circumcenter to the isogonal conjugate of p.*

It follows immediately from (5) and (6), that

$$(c + q)/2 = S_1/2. \tag{7}$$

Now $S_1/2$, being midway between the circumcenter and the orthocenter of T, is the nine-point center of T, so that *for any point p, the nine-point center is midway between the Hagge center and the isogonal conjugate of p.*

If we allow p to be on the circumcircle of T, but not at t_i, then $H(p, T)$ is simply the line of images [5] of p with respect to T.

2. Hagge circles in an orthocentric system. Let p be any point not a member of the orthocentric system t_1, t_2, t_3, S_1, and consider $H(p, T_i)$ where T_i is the triangle $t_jt_kS_1$. Let $m = S_1 - p$. Then the isogonal conjugate of p in T_i is found to be

$$q_i = S_1 - \frac{t_i\bar{m}(S_3\bar{m} - S_2 + t_im - t_i^2)}{(t_im\bar{m} - m - t_i^2\bar{m})}.$$

Since the nine-point circle of T is also the nine-point circle of T_i [1, p. 197], it follows from (7) that the center of $H(p, T_i)$ is given by

$$d_i = \frac{t_i\bar{m}(S_3\bar{m} - S_2 + t_im - t_i^2)}{(t_im\bar{m} - m - t_i^2\bar{m})}.$$

Now the circle we are seeking has center at d_i and passes through t_i, the orthocenter of T_i. We find its second intersection with the circumcircle of T to be $E = d_i/(t_i\bar{d}_i) = S_3\bar{m}/m$. And so

$$E = \frac{S_2 - S_3\bar{p}}{S_1 - p}.^* \tag{8}$$

Since this result is symmetric in t_1, t_2, t_3, we have

THEOREM 1. *The Hagge circles of any point distinct from the points of an orthocentric system, with respect to three triangles of the system, are concurrent at a point on the circumcircle of the fourth triangle.*

It follows from (8) that $Ep - S_3\bar{p} = S_1E - S_2$. The set of points p which satisfy this equation is precisely the line of images [5] of the point E with respect to T.

* Other properties of the point E were first stated by J. R. Musselman, [5], [6].

Conversely, for any point p on the line of images of E, but not at the orthocenter S_1, (8) must hold. Hence we have

THEOREM 2. *Let T, T_1, T_2, T_3 be the triangles of an orthocentric system, and let E be any point on the circumcircle of T. Then the circles $H(P, T_i)$, $i=1, 2, 3$, where P is not the orthocenter of T, pass through E if and only if P is on the line of images of E with respect to T.*

3. Hagge circles in the Brocard configuration. Let the circles tangent at t_i to t_it_j and t_it_k respectively meet again at b_i. Then b_i, $i=1, 2, 3$, are the vertices of the *second Brocard triangle* of T [1, p. 279]. We easily obtain

$$b_i = \frac{S_2 - S_1t_i}{S_1 - 3t_i}\cdot \tag{9}$$

The circle determined by the b_i is the *Brocard Circle* of T, and its equation may be obtained from (9) in exactly the same manner in which (4) was obtained from (1), (2) and (3):

$$x\bar{x} - \left(\frac{3S_2 - S_1^2}{9S_3 - S_1S_2}\right)x - \left(\frac{3S_1S_3 - S_2^2}{9S_3 - S_1S_2}\right)\bar{x} = 0. \tag{10}$$

The transformation

$$x = S_1 - 2x' \tag{11}$$

sends triangle T into its medial triangle T', and sends (4) into

$$x\bar{x} - \frac{\bar{q}}{2}\,x - \frac{q}{2}\,\bar{x} = 0, \tag{12}$$

a circle through the circumcenter of T, *i.e.*, through the orthocenter of T'; q here is defined as in (6). If, in particular, we take p as the median point of T, and hence, as the median point of T', we have $p=S_1/3$, and (12) reduces immediately to (10). Hence, *in any triangle, the Hagge circle of the median point with respect to the medial triangle is the Brocard circle of the given triangle* [1, p. 300].

We consider the points c_i' which determine the Brocard circle of T as $H(S_1/3, T')$. Putting $p=S_1/3$ in (2), we have

$$c_i = \frac{2S_2 - S_1^2 + S_1t_i}{3t_i - S_1}\cdot \tag{13}$$

Applying (11), and noticing that this leaves $S_1/3$ invariant, we find that the desired points are

$$c_i' = \frac{S_2 - S_1t_i}{S_1 - 3t_i}\cdot \tag{14}$$

Comparing (9) and (14), we have

THEOREM 3. *The points which determine the Brocard circle of any triangle as a Hagge circle of its medial triangle are the vertices of the second Brocard triangle of the given triangle.*

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CONVEX SETS*

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The systematic study of convex sets was begun by Brunn in 1887 and carried on by Minkowski and others whose works are listed in the bibliography of Bonnesen and Fenchel.† The elementary plane of support properties of convex sets are very well known. However, about the only readily accessible discussion of these properties is that of L. L. Dines which appeared in this MONTHLY in April, 1938. The proof there given‡ for the fundamental theorem, that through any given boundary point of a convex set there passes a plane supporting the set, depends on exhibiting a sequence of bounding planes whose limit, in the sense there defined, turns out to be the desired plane of support. In the present note we present two distinct, and, we believe, new proofs of this theorem neither of which involves any such limiting process. One of these proofs is due to Professor McShane,§ and we wish to thank him for permission to include it here.

The last section contains some simple related theorems on the existence and continuous turning of planes tangent to convex surfaces. Though these theorems seem interesting and non-trivial, so far as we know no one has hitherto bothered to record them.||

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* Presented at the Virginia-District of Columbia-Maryland section meeting of the Mathematical Association of America, May, 1940.

† Bonnesen and Fenchel, Theorie der Konvexen Körper, Berlin, 1934.

‡ Due, as the author remarks, to Carathéodory.

§ This proof will also appear shortly in a paper by Professor McShane in the Revista de Ciencias.

|| We have been informed that the two dimensional case is discussed by J. Hjelmslev in Contribution à la géometrié infinitésimale de la courbe réelle, Overs. Danske Vidensk. Selsk. Forh., 1911, p. 433–494.

1. Notation. The space with which we shall deal is an n dimensional euclidean space E_n. In cartesian coördinates the points of E_n are the vectors $\mathbf{x}=(x^1, \cdots, x^n)$ with real components. By the dimension of any point set A in E_n we mean the dimension of the subspace of E_n of lowest dimension containing A. Repetition of the superscript i in any term will require summation over the values $1, \cdots, n$. Thus for the distance of the point $\mathbf{x}$ from the origin we write

$$|\mathbf{x}| = [x^i x^i]^{1/2} = [(x^1)^2 + \cdots + (x^n)^2]^{1/2}.$$

The linear equation $x^i u^i - u^0 = 0$, u^0 being a constant and $\mathbf{u}=(u^1, \cdots, u^n)$ a fixed non-null vector, defines a *hyperplane* or as we shall say simply a *plane* in E_n. The vector $\mathbf{u}$ defines the normal direction of the plane. If $\mathbf{x}_0, \cdots, \mathbf{x}_p$ are $p+1$ points lying in no $p-1$ dimensional subspace of E_n, the set of points of the form

$$\mathbf{x} = \lambda_0 \mathbf{x}_0 + \cdots + \lambda_p \mathbf{x}_p, \quad \lambda_0 + \cdots + \lambda_p = 1,$$
$$\lambda_j \geqq 0 \qquad (j = 0, \cdots, p)$$

is called the p-simplex $S(x_0, \cdots, x_p)$. Clearly the dimension of a p-simplex is p.

If there exists a neighborhood of a point $\mathbf{x}$ which is contained in a set A, the point $\mathbf{x}$ is said to be *interior* to A. If every neighborhood of a point $\mathbf{x}$ contains both points in A and points not in A, then $\mathbf{x}$ is called a *boundary point* of A. It can easily be shown that

(1.1) *a simplex S in E_n has an interior point if and only if S is an n-simplex.*

We now proceed to the definition and immediate properties of convex sets.

2. Convexity. A set K in E_n is said to be *convex* provided for every pair of points $\mathbf{x}=(x^1, \cdots, x^n)$, $\mathbf{y}=(y^1, \cdots, y^n)$ in K all points of the line segment

$$(1-\lambda)\mathbf{x} + \lambda \mathbf{y} \qquad (0 \leqq \lambda \leqq 1)$$

joining $\mathbf{x}$ and $\mathbf{y}$ are in K. A closed* convex set is called a *convex body.*

The following properties of convex sets are easily established.

(2.1) *If a convex set K contains the vertices of a simplex S, then K contains all the points of S.*

(2.2) *The closure† $\bar{K}$ of a convex set K is a convex body.*

(2.3) *The intersection‡ of any collection of convex sets is a convex set.*

If a convex set K is n dimensional, it contains the vertices of an n-simplex S, and hence by (2.1) K contains all of S. By (1.1) S has an interior point. Therefore

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* A set which contains all its limit points is called closed.

† The closure $\bar{A}$ of a set A is the set consisting of all the points of A and all the limit points of A.

‡ The intersection of a collection of sets is the set of all points x such that x is in every set of the collection.

(2.4) *a convex set K in E_n has an interior point if (and only if) K is n dimensional.*

(2.5) *If the point $\mathbf{x}$ is interior to the convex set K and $\mathbf{y}$ is any point of K, then all points of the form*

$$(1-\lambda)\mathbf{x}+\lambda\mathbf{y} \qquad (0 \leqq \lambda < 1)$$

are interior to K.

(2.6) *If $\mathbf{y}$ is a point external to the convex body K, there is a unique point $\mathbf{x}_0$ of K closest to y.*

A plane π: $x^iu^i-u^0=0$ *bounds* a set A provided all points of A lie together in just one of the open half-spaces $x^iu^i-u^0<0$ and $x^iu^i-u^0>0$ defined by π. A plane π is said to be a *plane of support* of the set A provided the intersection $\pi\cdot A$ is non-empty and π bounds the set $A-\pi\cdot A$ of points of A which are not in π. A plane π *separates* two sets A and B provided A and B lie one in each of the two open half-spaces of π.

THEOREM 2.1. *If $\mathbf{y}$ is a point external to a convex body K and $\mathbf{x}_0$ is the point of K closest to $\mathbf{y}$, then the plane π passing through $\mathbf{x}_0$ with normal defined by the vector $\mathbf{y}-\mathbf{x}_0$ is a plane of support of K separating y from the set $K-\pi\cdot K$.*

This is seen intuitively as follows. Suppose there is a point $\mathbf{x}_1$ of K on the same side of π as $\mathbf{y}$. Then there are points on the line segment $\mathbf{x}_0\mathbf{x}_1$ closer to $\mathbf{y}$

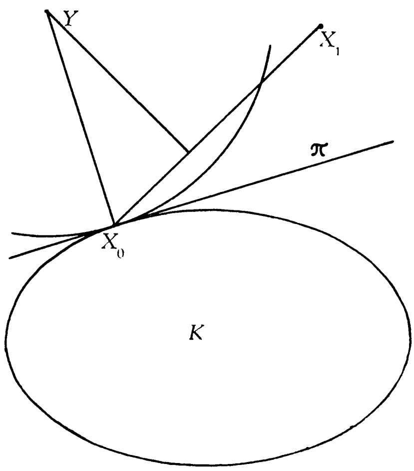

FIG. 1

than $\mathbf{x}_0$, and these points are in K by convexity. This contradicts the choice of $\mathbf{x}_0$ as the point of K closest to $\mathbf{y}$.

For notational convenience in proving the theorem we take $\mathbf{y}$ to be the origin. Then the plane π has the equation

$$x^ix_0^i - x_0^ix_0^i = 0.$$

Let $\mathbf{z}=(z^1, \cdots, z^n)$ by any point of K. Define

$$\phi(\lambda) = \left| \mathbf{x}_0 + \lambda(\mathbf{z} - \mathbf{x}_0) \right|^2.$$

Since the line segment $\mathbf{z}\mathbf{x}_0$ is contained in K and $\mathbf{x}_0$ is the point of K closest to $\mathbf{y}$, on the interval $0 \leqq \lambda \leqq 1$ the function $\phi(\lambda)$ has its least value when $\lambda = 0$. Hence

$$\phi'(0) \geqq 0.$$

Therefore if we set $\lambda = 0$ in the equation

$$\phi'(\lambda) = 2x_0^i(\mathbf{z} - \mathbf{x}_0)^i + 2\lambda \left| \mathbf{z} - \mathbf{x}_0 \right|^2,$$

it follows that

$$z^i x_0^i - x_0^i x_0^i \geqq 0.$$

And now since

$$y^i x_0^i - x_0^i x_0^i = - x_0^i x_0^i < 0,$$

the proof is complete.

COROLLARY 2.1. *Under the hypotheses of Theorem* 2.1 *there are planes separating* $\mathbf{y}$ *from K and there is a plane passing through* $\mathbf{y}$ *bounding K.*

THEOREM 2.2. *Let A be a closed set having an interior point* $\mathbf{x}$. *If through each boundary point of A there passes a plane of support of A, then A is convex.*

In case A is the entire space the conclusion is obvious. Otherwise, on the line segment joinging $\mathbf{x}$ and any point $\mathbf{y}$ not in A there is a boundary point $\mathbf{z}$ of A. By hypothesis there is a plane of support π of A passing through $\mathbf{z}$. The plane π cannot contain $\mathbf{y}$ else it would also contain $\mathbf{x}$, since $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are collinear and distinct. Hence the closed half-space of π containing $\mathbf{x}$ contains A but not $\mathbf{y}$. Therefore the intersection Π of all such closed half-spaces contains A but no point external to A. That is, $\Pi = A$. And by (2.3) Π is convex.

3. The fundamental theorem. In order to establish a modified converse of Theorem 2.2 we first consider a particular kind of convex set, the convex cone. A convex body C is called a *convex cone* with the point $\mathbf{x}_0$ as *vertex* provided C contains a point $\mathbf{x}$ different from $\mathbf{x}_0$ and for each such point $\mathbf{x}$ the set C contains the entire ray

$$(1 - \lambda)\mathbf{x}_0 + \lambda\mathbf{x} \qquad (\lambda \geqq 0).$$

THEOREM 3.1. *Every convex cone C which is not the whole space has at least one plane of support, and every plane of support of C passes through the vertex* $\mathbf{x}_0$ *of C.*

That C has a plane of support $\pi: x^i u^i - u^0 = 0$ follows from Theorem 2.2. To show that π must necessarily pass through $\mathbf{x}_0$ consider any point $\mathbf{x}_1$ in the set $\pi \cdot C$. We may as well suppose that C lies in the positive closed half-space of π. Then

$$[(1 - \lambda)\mathbf{x}_0 + \lambda\mathbf{x}_1]^i u^i - u^0 \geqq 0$$

for all $\lambda \geqq 0$. And now since $\mathbf{x}_1^i u^i - u^0 = 0$, we have

$$(1 - \lambda)(x_0^i u^i - u^0) \geqq 0 \qquad (\lambda \geqq 0).$$

Since the first factor in this expression can change sign while the second cannot, we must have $x_0^i u^i - u^0 = 0$.

If $\mathbf{x}_0$ is a boundary point of a convex body K, let T be the set of all points lying on rays from $\mathbf{x}_0$ through the points of K.

LEMMA 3.1. *The closure $\overline{T}$ of the set T is a convex cone.*

By (2.2), to show that the set $\overline{T}$ is a convex body we need only show that the set T is convex. Taking $\mathbf{x}_0$ to be the origin, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be any two points of T. We must show that any point*

$$\mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2, \quad \lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_1 + \lambda_2 = 1,$$

of the line segment $\mathbf{x}_1\mathbf{x}_2$ is also in T. By definition of T there exist positive numbers m_1 and m_2 such that the points $\mathbf{y}_1 = m_1\mathbf{x}_1$ and $\mathbf{y}_2 = m_2\mathbf{x}_2$ are in K. Writing

$$m = \left[\frac{\lambda_1}{m_1} + \frac{\lambda_2}{m_2}\right]^{-1},$$

we see that the point

$$\begin{aligned} \mathbf{y} &= m\mathbf{x} \\ &= m(\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2) \\ &= m\left(\frac{\lambda_1}{m_1}\mathbf{y}_1 + \frac{\lambda_2}{m_2}\mathbf{y}_2\right) \end{aligned}$$

is on the line segment $\mathbf{y}_1\mathbf{y}_2$ and hence is in K. Therefore the point $\mathbf{x} = m^{-1}\mathbf{y}$ is in T by definition of T.

It remains only to show that if $\mathbf{z}$ is any point of the set $\overline{T} - T$, then any point

$$\lambda\mathbf{z}, \qquad (\lambda > 0)$$

of the ray from $\mathbf{x}_0$ through $\mathbf{z}$ is in T. Choose any positive number ϵ. There is a point $\mathbf{x}$ of T such that

$$|\mathbf{z} - \mathbf{x}| < \frac{\epsilon}{\lambda}\cdot$$

Hence

$$|\lambda\mathbf{z} - \lambda\mathbf{x}| < \epsilon,$$

and now since the point $\lambda\mathbf{x}$ is in T, $\lambda\mathbf{z}$ is in $\overline{T}$.

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* This notation is seen to be equivalent to that previously used if we write

$$\lambda = \lambda_1, \; 1 - \lambda = \lambda_2.$$

THEOREM 3.2. *If* $\mathbf{x}_0$ *is a boundary point of a convex body* K *in* E_n *then there is a plane of support of* K *passing through* $\mathbf{x}_0$.

This will be an immediate consequence of Lemma 3.1 and Theorem 3.1 provided it can be shown that the set $\overline{T}$ of Lemma 3.1 is not the entire space. This is surely true if K is not n-dimensional. If K is n-dimensional, K has an interior point $\mathbf{x}$ by (2.5). Take $\mathbf{x}_0$ as the origin. Suppose $\overline{T}$ *is* the whole space. Then in particular the point $-\mathbf{x}$ is in $\overline{T}$. Hence there is a point $-\mathbf{y}$ of T so close to $-\mathbf{x}$ that $\mathbf{y}$, being equally close to $\mathbf{x}$, is interior to K. By definition of T, on the ray from $\mathbf{x}_0$ through $-\mathbf{y}$ there is a point $\mathbf{z}$ of K different from $\mathbf{x}_0$. But now by (2.5) $\mathbf{x}_0$, being on the segment $\mathbf{yz}$, is interior to K, which is a contradiction.

Professor McShane's proof of Theorem 3.2, which we mentioned in the introduction, is as follows. Let the given boundary point $\mathbf{x}_0$ of the convex body K

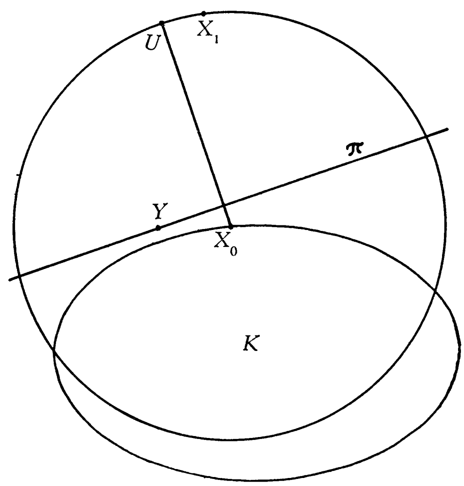

FIG. 2

be the origin, and let S be the sphere $|\mathbf{x}| = 1$ about $\mathbf{x}_0$. The distance from K to any point $\mathbf{x}$ of S, being a continuous function of $\mathbf{x}$ on S, attains its maximum at some point $\mathbf{x}_1$ exterior to K. We shall now show that $\mathbf{x}_0$ is the point of K closest to $\mathbf{x}_1$.

Choose any positive number $\epsilon < 1$ and any point $\mathbf{y}$ exterior to K for which $|\mathbf{y}| < \epsilon$. By Corollary 2.1 there is a plane $\pi: x^i u^i - u^0 = 0$ passing through $\mathbf{y}$ and bounding K. If we suppose, as we may, that the magnitude of the vector $\mathbf{u} = (u^1, \cdots, u^n)$ is 1, the distance from the plane π to the point $\mathbf{x}_0$ is $|u^0|$. And since $\mathbf{y}$ is in this plane and $|\mathbf{y}| < \epsilon$, it follows that

$$|u^0| < \epsilon.$$

Hence for the distance $1 - u^0$ from the plane to the point $\mathbf{u}$ of S the inequality

$$1 - u^0 > 1 - \epsilon$$

holds. And now since the point $\mathbf{u}$ and the set K are on opposite sides of the plane π, the distance from $\mathbf{u}$ to K is greater than $1-\epsilon$. We have thus shown that for every positive number $\epsilon<1$ there exists a point $\mathbf{u}$ of S whose distance from K is greater than $1-\epsilon$. Hence the distance of the point $\mathbf{x}_1$ from K cannot be less than 1, which is the distance of $\mathbf{x}_1$ from $\mathbf{x}_0$. It now follows from Theorem 2.1 that there is a plane of support of K passing through the point $\mathbf{x}_0$.

4. Planes tangent to convex surfaces. The proof usually given for Theorem 3.2* is essentially contained in the following lemma which we shall find useful in another connection and which we shall state without proof. We say that the sequence of planes $x^iu_j^i-u_j^0=0$, $|(u_j^1, \cdots, u_j^n)|=1$, $j=1, 2, \cdots$, converges to the plane $x^iu^i-u^0=0$ provided the sequence of points $(u_j^0, u_j^1, \cdots, u_j^n)$ in $n+1$ space converges to the point $(u^0, u^1, \cdots, u^n)$.

LEMMA 4.1. *Let $\{\pi_j\}$ denote a sequence of planes, each either bounding or supporting the closed set A, such that for some boundary point $\mathbf{x}_0$ of A the distance from the plane π_j to $\mathbf{x}_0$ tends to zero as j tends to infinity. Then there exists a subsequence of $\{\pi_j\}$ converging to a plane π, and for each such convergent subsequence the limiting plane π passes through the point $\mathbf{x}_0$ and supports the set A.*

We call a set S in E_n a surface provided for every point $\mathbf{x}$ in S there is an arbitrarily small neighborhood N of $\mathbf{x}$ such that the intersection $N\cdot S$ is homeomorphic† with the interior of an $(n-1)$ dimensional sphere whose center corresponds to the point $\mathbf{x}$. It is not difficult to show that the set of boundary points of a convex body with interior points is a surface in this sense. Let $\rho(\mathbf{x}, \pi)$ denote the distance from the point $\mathbf{x}$ to the plane π. A plane π is said to be tangent to a surface S at a point $\mathbf{x}_0$ of S provided for any sequence $\mathbf{x}_1, \mathbf{x}_2 \cdots$ of points of S which converges to $\mathbf{x}_0$ the sequence of numbers

$$\frac{\rho(\mathbf{x}_1, \pi)}{|\mathbf{x}_1-\mathbf{x}_0|}, \quad \frac{\rho(\mathbf{x}_2, \pi)}{|\mathbf{x}_2-\mathbf{x}_0|}, \cdots$$

converges to zero. Clearly when a tangent plane exists it is unique.

THEOREM 4.1. *The boundary surface S of a convex body K with interior points has a tangent plane at a point $\mathbf{x}_0$ of S if and only if the plane of support of K passing through $\mathbf{x}_0$‡ is unique. In this case the plane of support is the tangent plane.*

Assuming first that S has the tangent plane τ at $\mathbf{x}_0$, suppose there is a plane π distinct from τ, which supports K at $\mathbf{x}_0$. For convenience take $\mathbf{x}_0$ as the origin and τ as the plane $x^1=0$. Choose any point $\mathbf{y}=(y^1, \cdots, y^n)$ which is interior to K and not in τ. Say for definiteness that $y^1>0$. Choose any point $\mathbf{z}=(z^1, \cdots, z^n)$

* See the introduction and the references there given.

† Two sets are said to be homeomorphic with each other provided there is a one to one correspondence between their points which, considered as a transformation from either set to the other, is continuous.

‡ Which exists by Theorem 3.2.

of π for which $z^1>0$. Such points clearly exist, since the planes π and τ are distinct. For $j=1, 2, \cdots$, the points $2^{-j}\mathbf{y}$ are interior to K (by (2.5)) while the points $2^{-j}\mathbf{z}$ are not. Hence for each value of j there is a point in S of the form

$$\mathbf{x}_j = (1-\lambda_j)2^{-j}\mathbf{y} + \lambda_j 2^{-j}\mathbf{z}, \qquad 0 < \lambda_j \leqq 1.$$

The distance of the point $\mathbf{x}_j$ from the plane τ is x_j^1. Since for all j

$$\frac{x_j^1}{|\mathbf{x}_j|} = \frac{(1-\lambda_j)y^1 + \lambda_j z^1}{|(1-\lambda_j)\mathbf{y} + \lambda_j \mathbf{z}|} \geqq \frac{\min[y^1, z^1]}{|\mathbf{y}| + |\mathbf{z}|} > 0,$$

we have

$$\lim_{j\to\infty} \frac{x_j^1}{|\mathbf{x}_j|} > 0,$$

which contradicts the tangency of the plane τ to the surface S. This establishes half of the theorem.

Assume now that K has the unique plane of support π at $\mathbf{x}_0$. Let $\mathbf{x}_0$ be the origin, let $x^1=0$ be the equation of the plane π, and let K lie in the half space $x^1 \geqq 0$. If π is not tangent to S at $\mathbf{x}$, we can choose a positive number ϵ and a sequence of points $\{\mathbf{x}_j\}$ of S converging to $\mathbf{x}_0$ such that for all j

$$\frac{x_j^1}{|\mathbf{x}_j|} \geqq \epsilon.$$

By the Bolzano-Weierstrass theorem we may suppose the sequence $\{\mathbf{x}_j\}$ so chosen that the sequence of unit vectors

$$\left\{\frac{\mathbf{x}_j}{|\mathbf{x}_j|}\right\}$$

converges. By Theorem 3.2, through each point $\mathbf{x}_j$ there passes a plane $\pi_j: x^i u_j^i - u_j^0 = 0$ supporting K. For each j we may suppose that the magnitude of the vector $(u_j^1, \cdots, u_j^n)$ is 1 and that for all points $\mathbf{x}$ of K

$$x^i u_j^i - u_j^0 \geqq 0.$$

Further, by Lemma 4.1 we may suppose that the planes π_j converge to a plane $\pi_0: x^i u_0^i - u_0^0 = 0$ supporting K at the point $\mathbf{x}_0$. The number u_0^0 is zero since x_0 is the origin. Now if $\mathbf{u}$ is the vector $(1, 0, \cdots, 0)$,

$$\frac{x_j^i}{|\mathbf{x}_j|} u^i = \frac{x_j^1}{|\mathbf{x}_j|} \geqq \epsilon.$$

Therefore

$$(4.1)\qquad \lim_{j\to\infty} \frac{x_j^i}{|\mathbf{x}_j|} u^i \geqq \epsilon > 0.$$

Since the point $\mathbf{x}_j$ is in the plane π_j,

$$x_j^i u_j^i - u_j^0 = 0,$$

and since the origin is in K,

$$-u_j^0 \geqq 0.$$

Hence we have

$$\frac{x_j^i}{|\mathbf{x}_j|} u_j^i \leqq 0.$$

Letting j tend to infinity in this expression, we get

$$(4.2)\qquad \lim_{j\to\infty} \frac{x_j^i}{|\mathbf{x}_j|} u_0^i \leqq 0.$$

Comparing (4.1) and (4.2) we see that the unit vectors $\mathbf{u}$ and $\mathbf{u}_0$ must be different. Since K is n dimensional and all points of K satisfy both the relations

$$x^i u^i \geqq 0, \qquad x^i u_0^i \geqq 0,$$

it follows that u is also different from $-u_0$. Hence the planes $\pi: x^i u^i = 0$ and $\pi_0: x^i u_0^i = 0$ are distinct, contradicting the hypothesis that π is the only plane of support of K passing through the point $\mathbf{x}_0$. The proof of Theorem 4.1 is now complete.

We say that a plane $x^i u^i(y) - u^0(y) = 0$, $|u^1(y), \cdots, u^n(y)| = 1$, *varies continuously* in $\mathbf{y}$ on some set in n-space provided the vector

$$\mathbf{u}(\mathbf{y}) = (u^0(\mathbf{y}), u^1(\mathbf{y}), \cdots, u^n(\mathbf{y}))$$

is a continuous function of $\mathbf{y}$ on that set.

THEOREM 4.2. *If the boundary surface S of a convex body K with interior points has a tangent plane $\pi(x)$ at each point $\mathbf{x}$ of S which is in a given neighborhood N in S, then this plane varies continuously in $\mathbf{x}$ on N.*

By Theorem 4.1 at each point $\mathbf{x}$ of N the set K has the unique plane of support $\pi(\mathbf{x})$. If $\pi(\mathbf{x})$ is discontinuous at some point $\mathbf{x}_0$ of N, there is a sequence of points $\{\mathbf{x}_j\}$ of N converging to $\mathbf{x}_0$ such that the sequence of planes $\pi(\mathbf{x}_j)$ does not converge to the plane $\pi(\mathbf{x}_0)$. From these we may select a subsequence convergent to a plane π^* distinct from $\pi(\mathbf{x}_0)$. By Lemma 4.1 the plane π^* passes through the point $\mathbf{x}_0$ and supports the set K. This contradicts the uniqueness of the plane of support $\pi(\mathbf{x}_0)$.

DISCUSSIONS AND NOTES

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A THEOREM ON IMPROPER INTEGRALS

ARTHUR SARD, Queens College

Suppose that the function $f(x)$ is bounded and integrable in the sense of Riemann in each interval $[a, t]$, where a is fixed and $a \leqq t < \infty$. Consider the integral

$$I = \int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

and the series

$$S = \sum_{i=1}^{\infty} f(x_i')(x_{i+1} - x_i),$$

where

$$\sigma = \{x_1, x_1', x_2, x_2', x_3, x_3', \cdots \}$$

is a non-decreasing unbounded sequence with $x_1 = a$ such that Δ, the least upper bound of the differences $x_{i+1} - x_i (i = 1, 2, \cdots)$, is at most 1. Consider also

$$L = \lim_{\Delta\to 0} S.$$

In general any one of the limits I, S or L need not exist.

THEOREM. *The limits I, S and L exist and are finite, and $L = I$ providing that $\sum_{j=1}^{\infty} M_j < \infty$, where M_j is the least upper bound of $|f(x)|$ for x in the closed interval $[a+j-1, a+j]$, $(j=1, 2, \cdots)$.*

The proof of the theorem is given below.

COROLLARY. *If $f(x)$ is non-negative and non-increasing in $a \leqq x < \infty$, the mere convergence of I implies that S and L exist and that $L = I$.*

Proof: In this case the convergence of I implies the hypothesis of the theorem, by the integral test for the convergence of a series.*

* D. R. Dickinson has stated the following theorem, which can be considered a corollary of our theorem. *If there exists a non-increasing function $g(x)$ such that $|f(x)| \leqq g(x)$ for all $x \geqq a$ and such that $\int_a^\infty g(x)dx < \infty$, then S, L, and I exist and $L = I$.* (Quarterly Journal of Mathematics, Oxford Series, vol. 12, 1941, pp. 176–183). The present author, at the time he submitted the manuscript to the editor, was unaware of Dickinson's work.

In general the convergence of I does not even imply that S exists. To see this consider a function $f(x)$ of the following type: $f(x)$ is non-negative and continuous; the graph of $f(x)$ between $x=j-1$ and $x=j$ consists of 2^{j-1} congruent peaks each of area $1/2^{2j-1}$ and of maximum height $1/j$ $(j=1, 2, \cdots)$. Then $I=1$. Nevertheless S can diverge to ∞ with Δ arbitrarily small. Thus, let the unprimed elements of the sequence σ include all but a finite number of the positive integers and let all but a finite number of the primed elements of σ be such as to afford relative maxima to $f(x)$. Then comparison with the harmonic series shows that $S=\infty$.

Proof of the theorem: Consider the series

$$I' = \sum_{j=1}^{\infty} \int_{a+j-1}^{a+j} f(x)dx$$

and

$$S' = \sum_{j=1}^{\infty} u_j,$$

where $u_j = \sum f(x_i')(x_{i+1}-x_i)$ for all i such that $a+j-1 \leqq x_i < a+j$.

Comparison of S' with $\sum_{j=1}^{\infty} 2(M_j+M_{j+1})$ shows that S' *converges uniformly over all sequences* σ, since $|u_j| \leqq 2 \max (M_j, M_{j+1}) \leqq 2(M_j+M_{j+1})$.

Further, each term of S' approaches the corresponding term of I' as $\Delta \to 0$, by the hypothesis of Riemann integrability.

It follows that the series I' converges and that $I' = \lim_{\Delta\to 0} S'$, by a known theorem on uniformly convergent series.*

But the convergence of the series I' is equivalent to the convergence of the integral I, and $I=I'$. For, the difference between $\int_a^t f(x)dx$ and the partial sum of I' from $j=1$ to l, the largest integer such that $a+l-1<t$, is numerically not greater than M_l and $M_l \to 0$ as $t \to \infty$.

A similar argument shows that the convergence of S' is equivalent to the convergence of S and that $S'=S$.

Hence $I = \lim_{\Delta\to 0} S$ and the theorem is proved.

A NOTE ON JOINT VARIATION

R. A. ROSENBAUM, Reed College

There is a small point in connection with the topic of variation and proportionality which frequently disturbs the beginner, for many texts do not treat it adequately. Some students are still bothered by it when they take a course in thermodynamics and have to deal with relations of p, v, and T.

The trouble seems to arise from text-book statements similar to the following: "z is said to vary jointly as x and y if $z=kxy$. It is evident that, if y is held

* See, for example: W. F. Osgood, Funktionentheorie, Berlin, vol. 1, 5th ed., 1928, pp. 620–621.

constant, z varies as x, and that, if x is held constant, z varies as y." So far everything is clear, but, when the student comes to work problems, he is expected to assume the converse of the above, that is, to use the theorem: "If z varies as x, and z varies as y, then $z=kxy$." That this theorem is indeed true may be made to seem reasonable to the beginner by replacing the elliptical statement "z varies as x and z varies as y" by the complete statement "z depends on both x and y. When y is held constant, z varies as x, and, when x is held constant, z varies as y." A proof of the theorem which may be given to college students is the following:

If z varies as x when y is held constant, then $z=Rx$. For a fixed y, R is a constant, but for a different y, R may have a different value. In other words, R is a function of y; $R=f_1(y)$, say. Then

$$z = f_1(y)\cdot x. \tag{1}$$

Similarly, if z varies as y when x is held constant,

$$z = f_2(x)\cdot y. \tag{2}$$

Dividing (1) by (2), we obtain:

$$\frac{f_1(y)\cdot x}{f_2(x)\cdot y} = 1$$

or

$$\frac{f_1(y)}{y} = \frac{f_2(x)}{x}. \tag{3}$$

Since the left-hand side of (3) is a function of y and the right-hand side is a function of x, and since the equality holds for all x and y, each side must be equal to some constant, k.

i.e.

$$\frac{f_1(y)}{y} = k = \frac{f_2(x)}{x}$$

or

$$f_1(y) = ky \tag{4}$$

and

$$f_2(x) = kx. \tag{5}$$

Substituting from (4) in (1) or from (5) in (2), we have

$$z = kxy.$$

A similar type of proof may be used when more than three variables are involved, or when not all the variation is direct.

RECENT PUBLICATIONS

Edited by Virgil Snyder, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Science and Sanity. Second Edition. By Alfred Korzybski. Lancaster, Pa., The Science Press Printing Company, Distributors, 1941. $6.00.

This second edition of Korzybski's book is a reprint of the first edition, reviewed in this Monthly, Nov. 1934, with a 35-page pamphlet of comments on, and extracts from reviews of, the first edition, a new introduction of 52 pages, and a bibliography of 100 books and 32 monographs. Titles as varied as S. Zuckerman's *The Social Life of Monkeys and Apes* and Hermann Weyl's *The Mathematical Way of Thinking*, may suggest some indication of the wide scope of general semantics as presented in *Science and Sanity.*

There is nothing to add to the notice of the first edition, except one general observation: any book that was ever worth reading has been cordially damned by at least two persons. With this in mind, the author may see fit to exhibit in his third edition a select anthology of the fatuous things that have been said about general semantics, and his contribution to it, in the eight years between the two editions. Such an exhibition would be more illuminating to serious students than a hundred pages of laudatory remarks.

E. T. Bell

Mathematical Monographs. Edited by D. R. Curtiss. Northwestern University Studies in Mathematics and the Physical Sciences, No. 1. Evanston, Northwestern University, 1941. 7+172 pages. $2.25.

Maxima and Minima of Functions of Two or More Variables. By D. R. Curtiss.

The problem is the determination of proper extremes of a function $f(x_1, \cdots, x_n)$ which is real, single valued, and of class $C^{(r)}$ in a neighborhood of the origin. Taylor's expansion for the function uses the polynomials

$$G_m = \frac{1}{m!}\left(x_1\frac{\partial}{\partial x_1} + \cdots x_n \frac{\partial}{\partial x_n}\right)^m f(0, \cdots \quad 0) = \sum c_{ij} \cdots x_1^i x_2^j \cdots,$$

where $i+j+ \cdots = m$.

The first criteria are analogous to the well-known rules in case of $f(x)$. A careful analysis includes the usually slighted case in which the first non-identically vanishing G is semi-definite. The methods of Scheefer, Stolz, and Von Dantscher are reviewed with application to the semi-definite case.

The next method considers the problem of an extreme of $f(x, y)$ at the origin as a problem in conditioned extremes, namely that of an extreme of $f(x, y)$ on the curve $\partial f/\partial y = 0$. The result is expressed in terms of a set of poly-

nomials in the coefficients of the G_m. The process is a simplification of that of Stolz.

A new and most unusual process starts with the preparation theorem of Weierstrass which replaces $f(x, y)$ by $F(x, y)$, a "pseudo polynomial" in y. Then $f(x, y)$ has a proper extreme at the origin if no real branches of $F=0$ go through the origin. To test for this possibility the theorems of separation for roots of an algebraic equation are applied to $F(x, y)$ with x sufficiently small. Finally, Sturm's functions are derived directly from $f(x, y)$ by a new algorithm of greatest common divisor.

A well organized group of problems is included to show the advantages of the various methods.

The Statistics of Time Series. By H. T. Davis.

Some of the mathematical problems are presented which have arisen in the recent development of statistics in the direction of dynamics.

The important connections between the theory of integral equations and approximations by least squares is followed by an analysis of the theory of multiple regressions from the point of view of integral equations. These relations are used to establish a general significance test for the coefficients of an empirically determined regression. This followed the introduction of Schuster's periodogram $R(T)$ and spectrum of $y(t)$ in the harmonic analysis of time series, and the investigation of the distribution of the values of $R^2(T)$ in the test of the significance of maxima in $R(T)$.

The auto-correlation function, a special case of serial correlation, is related to Schuster's periodogram. The spectrum of a statistical time series can be obtained from the spectrum of a mathematical function that generates the same serial correlation function. A number of mathematical formulations of hysteresis preceded the new one in which serial correlation analysis is used. An example illustrates the process.

In the theory of random series an investigation is carried out of the effects of linear operators upon a series of random numbers; and an analogue of the random series is given in continuous variables

The problem of the dynamics of economic time series was simplified by considering only the price series. To remedy some defects of previous assumptions a new formulation of the dynamic problem requires the maximum of an integral whose integrand depends on the utility function, a function that reflects shocks of war, droughts, etc., and a function that depends on surpluses. Statistical verification of this theory is satisfactory.

Topics in Continued Fractions and Summability. By H. L. Garabedian and H. S. Wall.

Connections have been known for some time between the Stieltjes moment problem and continued fractions and between the moment problem for the interval (0, 1) and the theory of summability by matrices of finite reference.

Now the Stieltjes continued fraction has been related to the theory of summability. Some of the results are studied and presented in inspiring form.

It is assumed that the methods of summation will have finite and regular matrices of reference. The Hausdorff matrix depends on the sequence $\{c_n\}$, defined by Stieltjes integrals $c_n=\int_0^1 u^n d\phi(u)$, $n=0, 1, 2, \cdots$, where the mass function $\phi(u)$ determines the regularity of the matrix. The Hausdorff theory includes that of Cesàro, Hölder, and Euler-Knopp as special cases and extends their range of definition.

Various methods are shown by which inclusion problems in the domain of Hausdorff matrices may be studied. Sequences from row, column, or diagonal of the difference matrix Δ are essentially regular depending on the continuity at 0 and at 1 of the mass function of the base sequence. Inclusion problems are difficult when connected with the matrix Δ where the Hölder method of summation is associated with the base sequence. One special proof uses an integral equation technique to show that a moment sequence is regular and exhibits the mass function for it.

The Stieltjes moment sequence $\{c_n\}$ depends on integrals $c_n=\int_0^\infty u^n d\phi(u)$ where $\phi(u)$ is a bounded increasing function with an infinite number of points of increase. Stieltjes showed how a continued fraction determines a Stieltjes moment sequence, and conversely. When the range on the integral is (0, 1), totally monotone sequences $\{c_n\}$ are those for which the integrals hold for some monotone increasing function $\phi(u)$. The exact form of the continued fraction is found for this case and then a complete correlation exists between totally monotone sequences and continued fractions. This problem brought to light an important relation for continued fractions. A number of applications are considered.

A formulation is given of new conditions for regularity of totally monotone sequences in terms of the corresponding continued fraction. The older theorem has been provided with a new proof. A splendid group of problems includes periodic continued fractions and examples of moment sequences generated by continued fractions.

In conclusion, a class of Hausdorff methods of summation is studied with respect to its effectiveness in the analytic continuation of a power series $\sum a_n z^n$ outside its circle of convergence.

Spectra of Quadratic Forms in Infinitely Many Variables. By E. D. Hellinger.

For a class of quadratic forms in infinitely many variables, Hilbert discovered properties partly analogous to known theories but also he encountered a new phenomenon. The object here is to outline the development of this theory by means of examples.

A short resumé of some theorems of linear algebra and analytic geometry of n-dimensions includes analytical methods for deducing the theorems and applications of the theory. Then the investigation is extended to real quadratic forms in infinitely many variables of Hilbert space. For completely continuous

forms the statements about algebraic forms are virtually repeated except there is a countable infinite of characteristic values which converge to zero. This set is called the spectrum.

The Jacobi form has a continuous spectrum, $-1 \leqq \lambda \leqq 1$. Other results are analogous with such changes as sums replaced by integrals. Quadratic forms are constructed that have a segment $(-1, 1)$ as a multiple spectrum or that have both continuous and discontinuous spectra. Finally, a connection is made with continued fractions.

Bounded quadratic forms in infinitely many variables possess analogous properties. The preceding forms are bounded but are not sufficient to build all bounded forms. A suitable generalization, using Hellinger integrals, gives a new form for which all statements can be generalized. This form has a continuous spectrum, consisting of that perfect set on which the basis function increases. One example has a spectrum which is a perfect set, nowhere dense in the interval $(0, 1)$. A bounded form is found whose spectrum is continuous in the interval $(-1, 1)$ but which can not be transformed orthogonally into the Jacobi form. Conditions are discussed under which forms are combined into equivalent forms. One method is given for constructing spectrum and basis function of a given bounded quadratic form.

In case of non-bounded quadratic forms only some properties are similar to the preceding cases. A form is given whose spectrum covers the whole real axis and another where the spectrum is not uniquely determined. This indicates only a few of the differences.

MARIE M. JOHNSON

Supplement to Pandiagonal Magic Squares of Prime Order. By A. L. Candy, Author and Publisher. 1003 H Street, Lincoln, Nebraska, 1942. 3+30 pages.

In the first twenty-two pages of this pamphlet the author has shown how to extend considerably the number of pandiagonal magic squares of prime order of the type which fall under his Class II. Squares of this sort were treated at length in his book, *Pandiagonal Magic Squares of Prime Order*, reviewed in this MONTHLY, vol. 47, 1940, p. 563.

The remaining pages of the pamphlet are supplementary to the last section of the author's book, *Pandiagonal Magic Squares of Composite Order*, reviewed in this MONTHLY, vol. 48, 1941, p. 628.

G. E. RAYNOR

NEW BOOKS RECEIVED

Aircraft Mathematics. By S. A. Walling and J. C. Hill. Cambridge, University Press; New York, Macmillan Co., 1942. 189 pages. $1.25.

Analytic Geometry. By C. H. Lehmann. New York, John Wiley and Sons; London, Chapman and Hall, Ltd., 1942. 14+425 pages.

The Gist of Mathematics. By J. H. Moore and J. O. Mira. New York, Prentice-Hall, Inc., 1942. 12+726 pages. $5.35.

Differential Equations. Revised edition. By M. Morris and O. E. Brown. New York, Prentice-Hall, Inc., 1942. 11+355 pages. $3.00.

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Allegheny College, Meadville, Pa.

THE LINEAR GRAPH

M. C. SHOLANDER, Brown University

A much overlooked source of topics for mathematics clubs is the linear graph. An important branch of combinatorial topology, this theory seems to be best known as a tool for solving problems of the recreational type.* It has found, however, fundamental applications in the study of electrical systems, chemistry, logic, curve theory, and the theory of invariants.

A linear graph is a set of elements, called arcs and points, which must satisfy only the condition that each arc is associated with two points, called its end points. In most cases it may be conveniently pictured in the plane by means of diagrams such as those found in this paper. Linear graphs which have only a finite number of elements are called finite. The degree of a point is defined as the number of arcs for which that point serves as end point. An end point of first degree will be called a free end point and the arc associated with a free end point will be called a free arc. A representative theorem at this stage is, "The number of points of odd degree in a graph is even."

Euler is credited with initiating the theory with his study of the problem of the bridges of Königsberg and allied problems. This problem—to traverse the bridges in the neighborhood of the city, each once and only once, in such a way as to return to one's starting point—may be pictured graphically by taking the land areas (islands and river banks) as points and the bridges as arcs. The resulting graph (Fig. 1) may easily be checked to show that the problem has no solution. This led Euler to the formulation of a general problem—to find for what linear graphs there exists a path (called the Euler line) which traverses each arc once and only once and returns to its starting point. The solution was soon found. A graph has an Euler line if and only if it is finite, connected, and such that all its points are of even degree. This result has immediate application to many recreation problems. A simple illustration is afforded by the domino problem: to place all the dominos of a set in a closed circular array such as (01) (12) $\cdots$ (40). The trick is to represent each domino by an arc whose end points are

* Recreational problems referred to in this paper may be found in W. W. R. Ball, Mathematical Recreations and Essays, eleventh edition, 1939; namely, The Bridges of Königsberg, p. 242; The Domino Problem, p. 250; The Knight's Tour, p. 174; The Four-color Problem, p. 222; The Labyrinth Problem, p. 254.

For the consideration of these and other problems as applications of linear graph theory see Dénes König, Theorie der Endlichen und Unendlichen Graphen, Leipzig, 1936. This paper borrows material freely from the latter book.

numbered according to the values of the domino's halves. We see that the problem may be solved for a "double-n" set of dominos if and only if n is even.

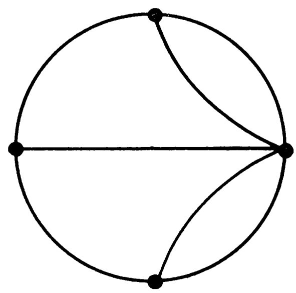

FIG. 1.

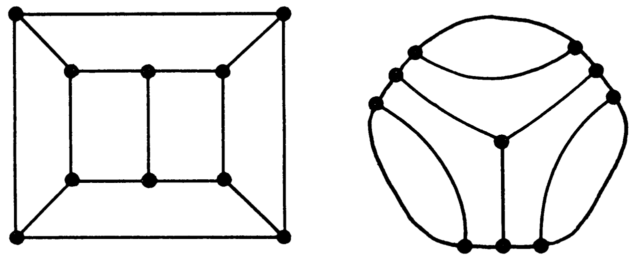

FIG. 2.

A problem seemingly of equivalent difficulty, and yet one which is as yet unsolved, is to find the necessary and sufficient conditions that a graph has a Hamiltonian line. This is a line which passes through each point of the graph once and only once and returns to its starting point. The graph of Figure 1 has such a line. The problem of the Knight's Tour may be shown to be one which requires for its solution the Hamiltonian line of a certain linear graph composed of 64 points and 168 properly placed arcs. An interesting application of this line can be made for the currently popular "four-color problem." A sufficient condition that a "map" is "colorable" is that the linear graph which it determines has a Hamiltonian line. Figure 2 will illustrate the difficulty of the problem. We have two graphs, each with 10 points and 15 arcs, only one of which admits a Hamiltonian line.

The "Labyrinth Problem" is one which has been more successfully attacked. We have space here only to remark that it may be best formulated as a problem in linear graph theory and that it, in itself, is an excellent topic for a complete mathematics club talk.

The fact that in organic chemistry we commonly picture molecules as linear graphs with atoms as points and bonds between atoms as arcs leads to other applications of linear graph theory. As background for a typical problem we make the following definitions. A "cycle" is defined as a closed circuit of arcs which passes through no point more than once—*e.g.*, $(A_1A_2)(A_2A_3)(A_3A_1)$ where A_1, A_2, and A_3 are distinct. A "tree" is a finite connected graph which contains no cycles. We shall say that the connectivity of a graph is k if it is necessary to remove k arcs from the graph in order to make a tree of it. Thus a tree has connectivity 0 and a cycle has connectivity 1. We can prove that a graph G with a_0 points and a_1 arcs has the connectivity a_1-a_0+1.

Now suppose we wish to determine all possible ways of picturing a molecule C_nH_{2n+2} in the paraffin series. Each hydrogen atom is assumed to have a single bond, so we must associate it with a free end point. The carbon atoms (having

valence four) correspond to points of degree four. We compute $a_0 = n + (2n+2) = 3n+2$, $a_1 = \frac{1}{2}(4n+2n+2) = 3n+1$, and deduce that the connectivity of the molecule is 0. Hence the molecule is a tree and there is no possibility of cycles formed by the bonds. For example, no two carbon atoms may be linked by multiple bonds. Each type of tree formed from n points yields a possible carbon configuration on which the hydrogen atoms may be "hung." No other configurations are possible.

Finally, there exist problems which because of the complexity of their conditions do not lend themselves to solution by known theorems of linear graphs, and yet whose solution may be facilitated by simply presenting the problem in graphical form. We illustrate this with the "Problem of the Missionaries and Cannibals." It is required to take three missionaries and three cannibals across a river by means of a boat which holds at most two persons. The missionaries and one of the cannibals can row. The situation is complicated by the requirement that on neither bank is it ever permitted for the cannibals to outnumber the missionaries.

To picture the problem we denote any missionary by M, the rowing cannibal by K, and either of the other two cannibals by C. A point of the graph will be assigned to any permissible grouping of the party on the first bank of the river. Specifically, we have the sixteen points:

1. $MMMCCK$
2. $MMMCC$
3. $MMMCK$
4. $MMMC$
5. $MMMK$
6. $MMCC$
7. $MMCK$
8. MMM
9. CCK
10. MC
11. MK
12. CC
13. CK
14. C
15. K
16. 0,

where 0 indicates the absence of everyone. We now place arcs on the graph between a point and any other which may result from the first by either the arrival or departure of the boat—*e.g.* between 1 and 4. This yields 25 arcs (Fig. 3) and our problem is reduced to that of going from point 1 to point 16 by means of a proper path. We have so numbered our points that—due to the boat's alternate departure and arrival—the proper path is one which zig-zags from lower to higher numbers, then from higher to lower. It is now a simple matter to inspect the graph and produce one of the few possible solutions. Such a solution is 1, 4, 3, 8, 5, 11, 7, 10, 6, 12, 9, 14, 13, 16.

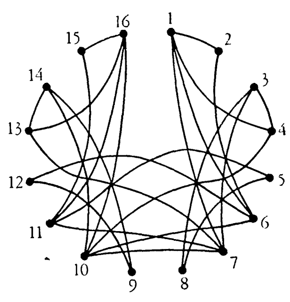

FIG. 3.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 536. *Proposed by Norman Miller, Queen's University.*

In a smooth hemispherical bowl of radius a, a needle of length $2l(l<a)$ is placed with one end projecting over the rim and is then released. Show that the needle will come to rest in a horizontal position or an inclined position according as l is less than or greater than

$$a\sqrt{\frac{2}{3}}$$

E 537. *Proposed by V. Thébault, San Sebastián, Spain.*

Let L, M, N and L', M', N' be the orthogonal projections of a point P on the sides and corresponding altitudes of a given triangle. Show that the lines LL', MM', NN' are in general concurrent, and find the locus of P when they are parallel.

E 538. *Proposed by R. V. Heath, Wall St., New York City.*

Find a perfect square whose digits form one of the permutations of eight consecutive digits. (Cf. E 532.)

E 539. *Proposed by Howard Eves, Chattanooga, Tennessee.*

Give a ruler construction for finding the centers of three given linearly independent circles, no two of which are intersecting, tangent, or concentric.

E 540. *Proposed by L. M. Kelly, U. S. Coast Guard Academy.*

Can the radius of the sixteen-point sphere ever be one-half the circumradius of the tetrahedron? (The sixteen-point sphere passes through the circumcenters of the faces.)

SOLUTIONS

The Round Table

E 500 [1941, 699]. *Proposed by S. H. Gould, University of Toronto*

In how many ways can p gentlemen and q ladies sit at a circular table if each lady has the choice, as long as gentlemen are still available, of sitting on a chair or on a gentleman's knees?

Solution by W. R. Van Voorhis, Fenn College, Cleveland, Ohio

Let m be the number of ladies who elect to sit on gentlemen's knees. Then the p gentlemen and $q-m$ disdainful ladies can take chairs in $(p+q-m-1)!$ ways. The complaisant ladies may be chosen in $\binom{q}{m}$ ways. There are p gentlemen's knees available for the first of these ladies, $p-1$ for the second, and so on. Hence they can all be seated in

$$(p+q-m-1)!\binom{q}{m}p(p-1)\cdots(p-m+1)$$

ways. Summing for the possible values of m, we find the total number of ways to be

$$\sum_{m=0}^{\min(p,q)}\binom{p}{m}\binom{q}{m}m!(p+q-m-1)!.$$

Also solved by the proposer.

Editorial Note. For the corresponding problem when the gentlemen are kept fixed, we must divide the answer by $(p-1)!$. It can then be expressed as

$$q!\sum_{m=0}^{\min(p,q)}\binom{p}{m}\binom{p+q-m-1}{p-1},$$

which is $q!$ times the coefficient of x^q in the expansion of

$$\left(\frac{1+x}{1-x}\right)^p.$$

The case where $p=q=3$ is relevant to the theory of sense in the projective line. See p. 32 of Coxeter, *Non-Euclidean Geometry*, Toronto, 1942.

The Generalized Simson Line

E 501 [1942, 61]. *Proposed by Daniel Arany, Budapest, Hungary.*

If A, B, C, I, J, X are six points on a conic, while L, M, N are points on the respective sides BC, CA, AB of the triangle ABC, and if further the three pencils $L(BXIJ)$, $M(CXIJ)$, $N(AXIJ)$ are projectively related, prove that the points L, M, N are collinear.

I. *Solution by L. M. Kelly, U. S. Coast Guard Academy.*

The notation suggests projecting I and J into the circular points at infinity. Then the conic goes into a circle, and the projected theorem is as follows:

If from any point X on the circumcircle of the triangle ABC lines be drawn making equal angles with the respective sides, the three "feet" of such lines will be collinear.

This is a well known generalization of the Simson line property, and can easily be proved by considering cyclic quadrangles.

Also solved thus by Howard Eves.

II. *Solution by R. L. Wright, University of Toronto.*

The nine points $A, B, C, I, J, X, L, M, N$ lie on two distinct cubics: one consisting of the conic $LMCXIJ$ and line ABN, another consisting of the conic $NLBXIJ$ and line CAM. These are therefore nine *associated* points (such that any cubic through eight of them goes also through the ninth). The conic $ABCIJX$ and line LM form a cubic through the first eight points, and so also through N. But $ABCIJX$ does not contain N; therefore LM does.

A Number and its Fourth Power

E 502 [1942, 61]. *Proposed by V. Thébault, San Sebastián, Spain.*

Find a number and its fourth power which together have nine digits, all different.

Solution by W. E. Buker, Pittsburgh Public Schools.

If n and n^4 together consist of nine digits, we must have $31 < n < 57$. Furthermore, n cannot end in 0, 1, 5, or 6, as then the ending of n^4 would repeat. Noting these restrictions, and leaving out 33, 44, 55, we try the following thirteen possibilities by the aid of a table of squares:

$$32, 34, 37, 38, 39, 42, 43, 47, 48, 49, 52, 53, 54.$$

We find that n and n^4 duplicate digits unless $n = 32$, in which case $n^4 = 1048576$.

We note that "casting out nines," usually of service in such problems, fails us here, as we do not know which digit is omitted.

Also solved by R. K. Allen, Paul Brock, M. L. Constable, S. T. Eriksson, Howard Eves, R. E. Greenwood, E. P. Starke, and the proposer.

The Centroid of a Tetrahedron

E 503 [1942, 61]. *Proposed by N. A. Court, University of Oklahoma.*

Through a point M lines are drawn meeting the pairs of opposite edges of a given tetrahedron in the pairs of points U, X; V, Y; W, Z. Prove that if M bisects each of the three segments UX, VY, WZ, it coincides with the centroid of the tetrahedron.

Solution by Howard Eves, Syracuse University.

Let $ABCD$ be the tetrahedron, with X on BC, Y on CA, Z on AB, U on DA, V on DB, W on DC. Since M bisects UX, VY, WZ, it follows that VW and YZ WU and ZX, UV and XY, are parallel. Therefore YZ and BC, ZX and CA, XY and AB, are parallel. But this is possible if and only if X, Y, Z are the midpoints of the edges BC, CA, AB. Similarly U, V, W are the midpoints of the edges on which they lie. Thus M is the centroid of the tetrahedron.

Also solved by Albert Furman, L. M. Kelly, W. T. Short, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4050. *Proposed by Arnold Dresden, Swarthmore College.*

If $a_1, a_2, \cdots, a_n$ are n distinct complex numbers, $n>1$, such that no two differ by a multiple of π, prove that

$$\sum_{k=1}^{n} \prod_{i=1, i\neq k}^{n} \cot (a_k - a_i) = \sin \frac{n\pi}{2} \cdot$$

4051. *Proposed by Arnold Dresden, Swarthmore College.*

If $a_1, a_2, \cdots, a_n$ are n distinct complex numbers, $n>1$, such that none of them and none of the differences is a multiple of π, show that

$$\sum_{j=1}^{n} \prod_{i=1, i\neq j}^{n} \cot (a_j - a_i) \cot a_j + (-1)^n \prod_{i=1}^{n} \cot a_i = \sin \frac{(n+1)\pi}{2} \cdot$$

4052. *Proposed by H. S. Wall, Northwestern University.*

If $\frac{1}{2}<r\leqq 1$ and $|z| \leqq (2r-1)/4r^2$, prove that

$$\left| e^z - \frac{4r^2}{4r-1} \right| < \frac{2r(2r-1)}{4r-1} \cdot$$

4053. *Proposed by E. P. Starke, Rutgers University.*

Show that all triangles inscribed in an ellipse and having their centroids at the center of the ellipse have the same area, which is the greatest possible area for an inscribed triangle.

Show that all triangles circumscribed about an ellipse and having their centroids at the center of the ellipse have the same area, which is the least possible area for a circumscribed triangle.

4054. *Proposed by V. Thébault, San Sebastian, Spain*

Find the base less than 100 for which the number 2101 is a perfect square.

4055. *Proposed by V. Thébault, San Sebastian, Spain.*

Find two numbers of ten digits of the form *aabbccddee* such that one is a perfect square and the other is a perfect square increased by 7.

SOLUTIONS

Quadrilateral and Orthopoles

3991 [1941, 214]. *Proposed by V. Thébault, San Sebastián, Spain*

Four straight lines Δ_i in a plane determine a complete quadrilateral (Q) forming four triangles $T_1\equiv(\Delta_2, \Delta_3, \Delta_4)$, $T_2\equiv(\Delta_1, \Delta_3, \Delta_4)$, *etc.*, with the orthocenters H_i. Show that the orthopoles of the straight line $\Delta\equiv(H_1, H_2, H_3, H_4)$, with respect to the four triangles, of the parallels to Δ_i through H_i are the orthogonal projections of the Miquel point on the sides of (Q).

Solution by J. W. Clawson, Ursinus College

It is well known that the circumcenters, C_i, of the triangles, T_i, are concyclic and that the Miquel (or focal) point of (Q) lies on this circle. We take this circumcentric circle as the unit circle in the complex plane and take the Miquel (focal) point, F, to be the unit point on the axis of reals. Taking the centers, C_i, to be turns, t_i, on the unit circle, it is not difficult to show that the side Δ_i has the equation $z-\bar{z}\cdot t_jt_kt_l=t_j+t_k+t_l-t_{ilj}-t_it_k-t_it_l$. In this equation z is the complex number of a point on the side and $\bar{z}$ the conjugate complex number. The equation was derived by finding the point of intersection, other than F, of the circles with centers C_j, C_k passing through F. This gives a vertex of (T), A_{il}, to be $t_j+t_k-t_jt_k$. The line Δ_i is determined by A_{il} and A_{ik} (or A_{ij}).

The foot of the perpendicular from F to Δ_i is

$$\tfrac{1}{2}(1 + t_j + t_k + t_l - t_jt_k - t_jt_l - t_kt_l + t_jt_kt_l).$$

Incidentally, these four feet lie on the pedal line, $z+\bar{z}\cdot s_4=\frac{1}{2}(1+s_1-s_2+s_3)$, where s_m is the sum of the products m at a time of t_1, t_2, t_3 and t_4.

Again, H_i is $s_1-t_i(t_j+t_k+t_l)$. Hence Δ_i', the parallel through H_i to Δ_i, is

$$z - t_jt_kt_l\cdot\bar{z} = s_1 - s_2 + (t_jt_k + t_jt_l + t_kt_l)/t_i.$$

Solving two of these equations simultaneously, we find that A_{jk}', one of the vertices of triangle T_i' determined by Δ_j', Δ_k', Δ_l', is $t_j+t_k+t_it_l-s_2$.

Again, the four points H_i lie on Δ, which is $z+s_4\cdot\bar{z}=s_1-s_2+s_3$, the orthocentric line of (Q).

To find the orthopole of Δ with respect to T_i', we find the feet of perpendiculars from A_{jk}' and A_{jl}' (or A_{kl}') on Δ, and then the equations of lines through these points perpendicular to the opposite sides of triangle T_i'. These perpendiculars intersect in the orthopole. The algebra is somewhat laborious but leads to the same point found above as the foot of the perpendicular from F to Δ_i.

Editorial Note. A solution is easily obtained by using some of the results in the solution of 3839 [1939, 604], and additional simple computations. In the above notation we have the following:

$$\Delta_i:\ m_iy - x - am_i^2 = 0; \qquad A_{ij}:\ am_im_j,\ a(m_i + m_j);$$
$$H_i:\ -a,\ a(\overset{i}{\sigma_1} + \overset{i}{\sigma_3}); \qquad F:\ a,\ 0; \qquad \Delta:\ x + a = 0;$$
$$\Delta_i':\ m_iy - x - a(1 + \sigma_2 + \sigma_4 - \overset{i}{\sigma_2}) = 0;$$
$$A_{ij}':\ -a(1 + \sigma_4 + m_im_j),\ a(m_k + m_l);$$

where σ_t is the elementary symmetric function of the four m's and σ_t^j of three m's after omitting m_j. The projection of A_{ij}' on Δ is the point $-a,\ a(m_k+m_l)$; and the equation of the straight line through this point perpendicular to Δ_k' is $y+m_kx-am_l=0$. Hence the orthopole of Δ with respect to T_l' is $0,\ am_l$, and this point is the projection of F on Δ_l.

The quadrilaterals (Q) and (Q') are symmetric with respect to the point $-a(1+\sigma_4)/2,\ a\sigma_1/2$, and this center of symmetry lies on $y=a\sigma_1/2$, the common Newton line for (Q) and (Q').

LEGENDRE TRANSFORMATION

4000 [1941, 409]. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

Given the functions $y=f(x)$, $Y=F(X)$, deduce from the transformation formulas

$$\frac{y'}{aX + bY + d} = \frac{-1}{bX + cY + e} = \frac{y - xy'}{dX + eY + f},$$

the following:

$$\frac{Y'}{ax + by + d} = \frac{-1}{bx + cy + e} = \frac{Y - XY'}{dx + ey + f},$$

where a, b, c, d, e, f are arbitrary constants. Give a geometric interpretation; and deduce from it the Legendre transformation.

Solution by Peter Chiarulli, Student, Brooklyn College.

From the given equality of three fractions we deduce from the equality of the first and third fractions that each is equal to

$$y/[X(ax + d) + Y(bx + e) + (dx + f)]; \tag{1}$$

and from the equality of this with the second given fraction we find that

$$X(ax + by + d) + Y(bx + cy + e) + (dx + ey + f) = 0. \tag{2}$$

Differentiation of (2) with respect to X gives

$$(ax + by + d) + Y'\,(bx + cy + e) + \frac{dx}{dX}\,[(aX + bY + d) + y'(bX + cY + e)] = 0.$$

Since the factor of dx/dX is zero by the given equality of the first and second fraction, we have

$$\frac{Y'}{ax + by + d} = \frac{-1}{bx + cy + e} = \frac{Y}{X(ax + by + d) + (dx + ey + f)},$$

where the third fraction results from (2). This gives the desired result by a combination of the first and third fraction in these derived equalities.

If $a=-1$, $e=1$, $b=c=d=f=0$, we have the Legendre transformations

$$y' = X, \quad xy' - y = Y; \quad \text{and} \quad Y' = x, \quad XY' - Y = y.$$

The line (2) is the polar of the point (x, y) with respect to the conic

$$\text{(A)} \qquad ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0;$$

and we now consider the polars of points on the curve $y=f(x)$ with respect to the conic (A), and the envelope $Y=F(X)$ of these polars. Regarding x as the variable parameter we find by differentiation of (2).

$$y'(bX + cY + e) + (aX + bY + d) = 0.$$

Using the above process of combination of fractions on this last result and (2) we obtain the given set of equalities. Therefore the geometric interpretation is that, if we have two curves $y=f(x)$ and $Y=F(X)$ connected by the given transformation formulas of the problem, then each curve is the envelope of the polars of points on the other curve with respect to the conic (A).

POSTPONEMENT OF THE PUTNAM COMPETITION

The William Lowell Putnam Mathematical Competition has been postponed for the present by agreement of the Putnam Trustees and the Board of Governors of the Mathematical Association. This is occasioned mainly by the preoccupation of both teachers and students with war courses in mathematics.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-seventh Annual Meeting, New York, N. Y., December 30–31, 1942.

The following is a list of the Sections of the Associations, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, State College, Pa., Oct. 23–24, 1942
ILLINOIS, Notre Dame, Ind., April 9–10, 1943
INDIANA, Notre Dame, April 9–10, 1943
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Dec. 5, 1942
METROPOLITAN NEW YORK
MICHIGAN, Notre Dame, Ind., April 9–10, 1943
MINNESOTA
MISSOURI, Kansas City, Dec. 4, 1942
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, Jan. 30, 1943
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 28, 1942
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE, fall, 1942
WISCONSIN, Milwaukee, May 7, 1943

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NEWTON AFTER THREE CENTURIES

E. T. BELL, California Institute of Technology

Christmas Day, 1942 is the three-hundredth anniversary of the birth of Isaac Newton. The two-hundredth anniversary of Newton's death was suitably commemorated in 1927 in nearly every civilized country of the world. To estimate adequately the influence of this unique mind on our present civilization would require the labors of several men, and incidentally traverse much of the histories of the astronomical and physical sciences, of industry and engineering, and of philosophic thought during the last two centuries. Until such an estimate, worthy of the man, is undertaken, a short survey of some major items of Newton's work that are still vital in science and mathematics, with a glance at what has been abandoned, and a brief indication of present problems originating in his work, may be of passing interest. Few of even the greatest men of science have left the world much that retained its full life for more than two centuries after they had died. Newton's work, though modified in detail, continues to inspire men gifted with some of his genius to carry it on, and to extend its scope beyond anything he knew or could have imagined. This is his immortality.

Newton was a man of three masterpieces: the calculus, the *Opticks*, the *Principia*. The few items of his work noted here may be conveniently classified under pure mathematics, optics, gravitation, metaphysics, all with the limitation stated above.*

Pure Mathematics

Little more than a bare list will recall certain of Newton's indispensable contributions to pure mathematics and suggest their perennial vitality.

The calculus, of course, dwarfs the rest. Without it, modern physical science and technology would have been impossible. To any scientifically literate historian this needs no argument. Though the origins of the calculus have been traced back to remote antiquity, it has not yet been denied that Newton was the first to develop and apply both the spirit and the techniques of the calculus in all their power to kinematics, dynamics, and astronomy. The example of his work started the continuing avalanche of discovery in the astronomical and physical sciences. It is now known from Newton's own testimony that he used the calculus as an implement of discovery and proof in composing the *Principia* [1]; the final form of the demonstrations was a geometrical transposition of the initial analysis.

Once of glowing interest to jealous nationals and others, the controversy between the respective partisans of Newton and Leibniz over priority in the in-

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* The following men have generously helped with the sections on optics and gravitation: I. S. Bowen, of the California Institute of Technology, who also loaned the original editions of many of the works cited; F. Zwicky, of the same Institute and the Mount Palomar Observatory; S. B. Nicholson, of the Mount Wilson Observatory. The numbered references and notes are at the end of the paper.

vention of the calculus is no longer as hot as it was. For all impartial judges it has been decided: each invented the calculus independently; Newton was first; each descended to personal attacks unworthy of him, though not entirely foreign to the temper of the time; the dispute is of no scientific interest whatever, and might well be buried in the archives of the forgettable.

The subtle difficulties at the bases of the calculus which perturbed Newton were settled in the late nineteenth century by the foremost mathematical analysts of the period, only to be more profoundly unsettled by the foremost mathematical logicians of the early twentieth century. Though the calculus still lacks a consistent foundation, the defect is perhaps of a less immediately practical character, even mathematically, than that which troubled Newton. These unresolved difficulties continue to generate a vast amount of research in mathematical logic, which in turn has reacted significantly on epistemology.

With the increasing attention paid in the twentieth century to the discrete as opposed to the continuous in the sciences, the calculus of finite differences (sum and difference calculus) has grown rapidly in scientific importance, taking its place beside the differential and integral calculus as an indispensable aid in the study of nature, from intelligence testing to statistical mechanics. Here Newton's solution of a fundamental problem in interpolation is as useful as it was when (1676) he called it [**2**] "one of the prettiest problems that I can ever hope to solve." The problem as stated by Newton is "to describe a geometrical curve which shall pass through any given points . . . . " A solution is given in the *Principia* [**3**]. Newton's interpolation formula has remained a basic result in the calculus of finite differences, especially in its scientific and technological applications.

A more recondite calculus, of even greater scientific utility and deeper mathematical interest, is at least implicit in Newton's problem [**4**] of the solid of revolution offering the least resistance in moving through a medium in the direction of its axis. This would be attacked today by the calculus of variations. Newton merely stated his solution without indicating how he obtained it. Discussed many times, the problem has continued to suggest successive refinements in analysis since Legendre (1786) used it as an example on which to test his criterion for discriminating between maxima and minima in the calculus of variations.

Newton is said to have been attracted to this problem by his early interest in exterior ballistics; he mentioned its possible application to naval architecture. His disposal of it in the *Principia* is typical of his attitude toward pure mathematics: get the result by any method that will work, and let the method take care of itself. A pure mathematician might have thought it worth recording that the problem is not amenable to the differential calculus; it is of another genus than the problems of maxima and minima solvable by comparatively elementary means. Newton was content (1687) to state his solution. Later (c. 1694), to oblige a correspondent, he wrote out a draft of a proof by a mixture of geometry, variations, and fluxions. Recast in modernized form, Newton's proof

amounts to finding a first integral of the appropriate Eulerian differential equation. That he was able by his methods to complete the solution and obtain the minimizing arc explicitly, looks like a happy mathematical accident. But he solved his problem.

Newton's attitude toward mathematics was that of the scientific engineer or theoretical physicist today: mathematics for him was an efficient tool without intrinsic interest. But lest anyone be tempted to enlist Newton as an ally in an attack on the value of mathematics for other than immediately practical ends, it may be recalled that not even the unsurpassable splendors of the *Principia* elicited his own highest esteem. That was reserved for his works in theology and sacred chronology, both long since hopelessly antiquated.

Yet such was the universality of Newton's mathematical talent that he could do pure mathematics with the best of his contemporaries, and do it as well or better than most of them. The binomial theorem alone would have made a lesser reputation. In geometry, Newton was the first to make a comprehensive study [**5**] of any class of plane curves beyond the long-familiar conics and a handful of transcendental curves; and it was he who first demonstrated the full power of analytical methods in geometry. Again, though it is only a minor detail, 'Newton's polygons' are more useful today than when he invented them, appearing in regions far beyond the mathematics of his day, for example, in one recent theory of algebraic numbers.

The penetration of his mathematical intuition is perhaps as well illustrated by 'Newton's rule' in the theory of algebraic equations as by any other of his purely mathematical discoveries. In the words of the great algebraist [**6**] who first (1864) gave a proof of the rule: "In the *Arithmetica Universalis*, in the chapter *De Resolutione Equationum*, Newton has laid down a rule, admirable for its simplicity and generality, for the discovery of imaginary roots in algebraical equations, and for assigning an inferior limit to their number. He has given no clue towards the ascertainment of the grounds upon which this rule is based, and has stated it in such terms as to leave it quite an open question whether or not he had obtained a demonstration of it." It would be difficult to find a juster evaluation of Newton's attitude toward pure mathematics and the intuitive quality of his genius as a pure mathematician.

Optics

In a paragraph [**9**] that might profitably be framed and hung in the study of the mystical astrophysicist [**10**] who today occupies the chair of astronomy and experimental philosophy in Newton's alma mater, Newton stated (1717) his scientific philosophy with matchless clarity and forthright common sense. As a radically different conception of the scientific method must be noticed later [**51**], the passage may be transcribed here. It is a perfect synopsis of the Newtonian method.

"As in Mathematics, so in Natural Philosophy, the Investigation of difficult Things by the Method of Analysis, ought ever to precede the Method of Com-

position. This Analysis consists in making Experiments and Observations, and in drawing general Conclusions from them by Induction, and admitting of no Objections against the Conclusions, but such as are taken from Experiments, or other certain Truths. For Hypotheses are not to be regarded in experimental Philosophy. And although the arguing from Experiments and Observations by Induction be no Demonstration of general Conclusions; yet it is the best way of arguing which the Nature of Things admits of, and may be looked upon as so much the stronger, by how much the Induction is more general. And if no exception occur from Phaenomena, the Conclusion may be pronounced generally. But if at any time afterwards any Exception shall occur from Experiments, it may then begin to be pronounced with such Exceptions as occur. By this way of Analysis we may proceed from Compounds to Ingredients, and from Motions to the Forces producing them; and in general, from Effects to their Causes, and from particular Causes to more general ones, till the Argument end in the most general. This is the Method of Analysis: And the Synthesis consists in assuming the Causes discovered, and established as Principles, and by them explaining the Phaenomena proceeding from them, and proving the Explanations."

Newton's positive, experimental contributions to optics were the analysis of white light (sunlight) into lights of different colors, separated in the visible spectrum according to their different refrangibilities, and the reverse synthesis of these colors into white light; the quantitative study of colors of thin films; the invention of the reflecting telescope. In theoretical optics he explained the colors of the rainbow, and imagined that all colors can be produced by properly combining primary colors, of which he named seven. This hypothesis was backed by experiments. His investigation of the chromatic aberration of lenses having decided him against refracting telescopes [11] in favor of reflectors, he constructed with his own hands (1668, 1671) the first reflecting telescopes in history. The second was exhibited before the Royal Society in January, 1672. Though a puny mite, Newton's diminutive reflector is the ancestor of the 102-inch and 200-inch giants of the twentieth century.

Before Newton, it was supposed that color resides in matter; he showed it to be a quality of light, and he instituted its quantitative analysis. There is no need to review here the fluctuating popularities of the wave theory of light invented by Huygens and the corpuscular theory proposed by Newton. However, the relative standings of the two theories in 1942 is perhaps the most appropriate item as a commemorative tribute to Newton's physical insight on the three-hundredth anniversary of his birth. Though there are no grounds for believing that either theory will satisfy all physicists three centuries hence, it is a matter of fact that Newton's theory is in closer accord with the quantum mechanics of 1942 than is its recent rival. Whatever is to be the fate of either theory, there is no question that Newton's experimental observations will remain substantially unchallenged—unless a greater and more muddled Goethe

shall take it upon himself to teach the intelligentsia the rudiments of optics. The artless simplicity of Newton's analysis and synthesis of white light marks his work as one of the timeless classics of experimental science; and the sharp precision of his measurements of the colors of thin films contributed decisively to the undulatory theory of the nineteenth century, one of the most suggestive hypotheses that ever accompanied and guided an experimental science.

Though he disdained speculation, and strove in his finished work to reduce it to a minimum, Newton could, on occasion, let his scientific imagination roam as unrestrainedly as did any theoretical physicist of the nineteenth century, or as any mathematical astrophysicist of the twentieth. Probably the boldest of Newton's conjectures, apart from some of his fanciful chronology, are among the more enlivening queries with which he rounded out the experimental science of his *Opticks*. One of these [**12**] is particularly relevant for twentieth-century theories regarding the nature of light. For its prophetic suggestiveness it may be transcribed in full [**13**].

"Qu. 17. If a Stone be thrown into stagnating Water, the Waves excited thereby continue some time to arise in the place where the Stone fell into the Water, and are propagated from thence in concentrick Circles upon the Surface of the Water to great distances. And the Vibrations or Tremors excited in the Air by percussion, continue a little time to move from the place of percussion in concentrick Spheres to great distances. And in like manner, when a Ray of Light falls upon the Surface of any pellucid Body, and is there refracted or reflected: may not Waves of Vibrations or tremors be thereby excited in the refracting or reflecting Medium at the point of Incidence, and continue to arise there, and to be propagated from thence as long as they continue to do so, when they are excited in the bottom of the Eye by the Pressure or Motion of the Finger, or by the Light which comes from the Coal of Fire in the Experiments above mention'd? And are not these Vibrations propagated from the point of Incidence to great distances? And do they not overtake the Rays of Light, and by overtaking them successively, do they not put them into the Fits of easy Reflexion and easy Transmission described above [**13**]? For if the Rays endeavour to recede from the densest part of the Vibration, they may be alternately accelerated and retarded by the Vibrations overtaking them."

To appreciate the peculiar significance of this query, it is necessary to remember a detail of Huygens' wave theory of light. Huygens had merely the conception of a wave front, and had no idea of a train of waves. According to Huygens' theory, the velocity of light in water should be less than it is in air; Newton's corpuscular theory predicted the opposite. Experiments by J. L. B. Foucault [**14**] (1853) and others confirmed Huygens. Though this seemed to dispose of the corpuscular theory, the passage quoted shows that Newton's general idea of light was closer to the twentieth-century conception than was Huygens'. Light for Huygens was merely an impulse; he had no idea of a wave length. Newton, on the other hand, in his explanation of 'Newton's rings,' [**14a**] imagined that the impinging corpuscle, having passed the first surface of a thin film or

plate, started a splash or a vibration there, which sent out a train of waves. The train then caught up with the particle; and at the second surface the train was reflected or transmitted according as there was a crest or a trough.

The resemblance to the explanation by quantum mechanics is striking enough: a particle (corpuscle) is accompanied by a wave, and the behavior of the particle is isomorphic to that of the wave. Photons doubtless would have been acceptable to Newton but not to Huygens.

Though Newton rejected the wave theory, it was partly his accurate measurements of the colors of thin plates or films reflecting the different colors that were responsible for Young's undulatory theory in the early nineteenth century. Newton gave all the data [**15**] for determining wave lengths; Young only interpreted it [**16**]. His was the first (1802) table of wave lengths [**17**], and he was the first to consider the wave train as the principal thing in light. It would be interesting to know whether 'Qu. 17' started Young thinking about wave trains. In any case, the best data available for his deductions of 1802 were those in Newton's *Opticks* of 1704.

As there is sometimes a temptation to read the present into the past, it may be recalled that the now famous 'Qu. 17' was in no way responsible for the quantum-mechanical wave theory. Newton's query acquired its air of rationality only after this theory of the twentieth century, suggested by quite different considerations, had been fully elaborated, when the coincidental agreement with Newton's undeveloped conjecture was noted.

If one science more than another has extended positive knowledge of the stellar and extra-galactic universes beyond the utmost imaginable in Newton's time, it is spectroscopy. Only a century ago, the parochial mind found comfort in the illusory assurance that though science might overtake the farthest stars in their courses, and unravel the tangle of their wanderings, it could never guess the chemical composition of even the nearest and brightest star of all; and this ignoramibus was flaunted as final and irrefutable evidence of the superiority of revelation over science as a guide to understanding "the wonder of the heavens." Like Newton on the shore of "the great ocean of truth," [**18**] scientists are acutely aware that they have found but little in comparison with what may yet be found; but at the limits of the spectroscopically visible universe they have diverted themselves "in now and then finding a smoother pebble or a prettier shell than ordinary" [**18**]. Among the pebbles and shells they have picked up are millions of galaxies akin to our own, and clusters of galaxies, undreamed of when Newton reduced the motions of the planets in the solar system to order with his law of universal gravitation. All this almost inconceivably vast access of verifiable knowledge had its inception in Newton's analysis of light. Unaccountably—for him—he failed to take the last, short steps which would have brought the chemistry of the stars, the sun among them, within his reach.

To see how close Newton came to the crucial discoveries, it will be necessary to recall briefly the principal stages in the development of spectroscopy. An early

recorded hint of the chemical possibilities was Jabir's [**18a**] observation (eighth century A.D.) that copper heated in a flame colors the flame blue-green. Agricola [**19**] next suggested (1556) that the different colors produced in a similar manner be used for chemical analysis. These facts were readily fitted into the theory of emission spectra when finally it arrived in the nineteenth century.

Newton's theory of light and colors [**20**] was laid before the Royal Society in 1672. It had been supposed that the object did something to light in reflecting it; Newton's experiments with prisms showed for the first time that white light is a composite of lights of different refrangibilities, and that the only thing the prism does is to spread out these lights. In reflection from a colored object, only some of these lights are reflected, the others are absorbed. As far as Newton's experiments went, they were closely similar to those of Wollaston [**21**] (1802), who also, however, saw the absorption lines in the solar spectrum which Newton with moderate attention might have seen, but which, surprisingly, he did not see. All the setting for observing the absorption lines was in order when Newton passed sunlight through a slit and lens and focussed the spectrum [**22**].

Again, in the matter of emission spectra, Newton amplified the observations of Agricola and others on the colors of flames [**23**], but did not look at a flame through a prism. Had he done so, he would have seen an emission spectrum. The first to observe an emission spectrum was Newton's fellow Briton, the gifted young clergyman Thomas Melvill [**24**], whose paper (read, February, 1752) was published posthumously in 1756, Melvill having died (December, 1753) at the age of twenty-seven.

The first statement that emission lines could be used for chemical analysis was in 1830 by J. F. W. Herschel [**25**]. The historic first reversal (Feb. 7, 1849) of a line in the laboratory was by J. B. L. Foucault. Meanwhile the 'Fraunhofer lines' that Wollaston had noted were rediscovered and located (1814–15) with high precision in the solar spectrum by Fraunhofer [**26**]. The motivation of this decisive work was the practical improvement of refracting telescopes; the discovery itself, of greater significance for astrophysics than the objective sought, was an unsought and under-appreciated by-product. By 1850 at the latest, all the basic experimental facts of spectroscopy needed for 'the chemistry of the stars' were available. It may not be too much to claim that none of this would have come about had not Newton, or some equally acute experimentalist and equally gifted theoretician, observed that light has certain properties which are not put into it by reflection or refraction or passage through colored matter.

Finally it may be noted that a remark by Huygens (1690) on double refraction in Iceland crystal provided Newton with a fact which he found inexplicable on the hypothesis that "Light be nothing else than Pression or Motion propagated through Aether" in analogy with the propagation of sound waves in air [**27**]. For he inferred from the experimental facts that a ray in double refraction must have "two opposite Sides," while an ordinary ray exhibits no such polarization. He could not reconcile the facts with the wave theory.

Light as a transverse vibration of an elastic solid (the ether) was an invention of the nineteenth century. This took account of double refraction till, to the distress of some who persisted in regarding their scientific theories as creeds instead of working hypotheses, the elastic solid collapsed under the weight of its own hypothetical attributes. James Clerk Maxwell's electromagnetic theory of light [**28**] next (1861, 1864) reigned almost unchallenged till the twentieth century, when it was amplified by the electron theory and the quantum theory to take account of the interaction between light and matter. With these successive modifications of theories which had done the work for which they were created, and had become obsolete in the process, an ancient philosophy of science [**51**] returned from the remote past to dispute the scientific method advocated by Newton. But whatever philosophies and theories of the physical universe are to prevail in the next three centuries, it seems probable that Newton will remain an unchallenged witness to the historical truth that theories pass but experiments abide.

Gravitation

"You some times speak of gravity as essential and inherent to matter. Pray, do not ascribe that notion to me; for the cause of gravity is what I do not pretend to know, and therefore would take more time to consider of it" [**29**].

So Newton wrote (1692) in self-defense to a friend who had endeavored to interpret Newton's conception of gravity. Earlier (1679), in a letter [**29**] to Boyle, Newton had permitted himself to speculate on "the cause" of gravity, seeking an explanation of the mutual attraction of bodies in the mechanics of an "aether." Many subsequent attempts by others to explain gravitation mechanically—Le Sage's corpuscular bombardments (1782) seemed promising at a first unmathematical glance—proved equally abortive until, in the twentieth century, Newton's gravity having been explained away in the geometry of space-time, nothing about it was left to be explained. Nevertheless it remained indispensable in both theory and practice. The 223 years from Newton's "I do not pretend to know" of 1692 to Einstein's equivalence principle of 1915 are crowded with scientific discoveries and mathematical theories, no inconsiderable number of which originated, either directly or not too remotely, in Newton's law (or hypothesis) of universal gravitation.

The postulated universality of Newtonian gravitation is perhaps the item of most vital interest for current science. To provide a background for a few major problems of Newtonian gravitation today, it is necessary to summarize very briefly what Newton himself derived from his grand hypothesis. As he formulated it in the *Principia* [**30**], Book III, Proposition VII, Theorem VII: "That there is a power of gravity tending to all bodies, proportional to the several quantities of matter which they contain. . . . Cor. 2. The force of gravity towards the several equal particles of any body, is reciprocally as the square of the distance of places from the particles [**31**]."

This general law, or hypothesis, or principle was not imagined as a mathe-

matical diversion. It was proposed as a logical equivalent, adapted to analysis by mathematical reasoning, of the gist of Kepler's three laws, empirically discovered, of planetary orbits. The cardinal point here is that Newton based *The System of the World* in Book III of his *Principia* on established scientific fact. He did not, as some had done before him and others have done since, spin his own world out of cerebration, and then proceed to write reams of high metaphysics or regiments of impressive formulas about a preposterous universe that existed only in his own head. He listened to what nature had to say, and then made it tell more than it may have meant to disclose. It is no miracle then, when the power of Newton's mathematical genius is taken into account, that the *Principia* is the unsurpassed masterpiece of both scientific coordination and the art of scientific prediction that it is.

From Kepler's first and second laws, and his own three laws of motion Newton deduced [**32**] that each planet is attracted by a central force directed toward the sun, the intensity of the force varying inversely as the square of the distance between the two bodies; from the third law he deduced that all the planets are similarly attracted, the intensities depending on the sun's mass.

Almost as corollaries of these general conclusions, Newton showed how the sun's mass can be calculated in terms of the earth's mass: the length of any planet's year and its distance from the sun are the sufficient data. The mass of any planet having a satellite can be similarly computed. The same force of gravity that accounts for the fall of an apple was then shown to be sufficient for holding the moon to its orbit. Explicit definitions of 'mass,' 'gravity,' 'force,' 'attraction' are unnecessary if the objective of these deductions is correlation of known facts and prediction from them; the mathematical equivalents of the verbal statements suffice. Thus metaphysical disputes are short-circuited to their proper function.

The roll of the memorable conclusions Newton deduced from his law is only begun. The Newtonian theory of gravitation accounted for the tides. From the sun's mass, deduced from the theory, the height of the solar tide was calculated; as a sort of converse, from the observed heights of the spring and neap tides, the lunar tide was calculated, whence an estimate of the moon's mass was obtained. From Newton's dynamics of a gravitating rotating body, it followed that the earth is not a sphere as had been supposed since ancient times, but an oblate spheroid, and the measure of flattening at its poles was calculated. Conversely, from the observed oblateness of any planet the length of its day was shown to be calculable. Another easily verifiable prediction deduced from the polar flattening of the earth and centrifugal force was the variation of the weight of a body with the latitude. The attraction of the sun and moon on the earth's equatorial bulge, with similar attractions by the other planets, was proved to perturb the earth's axis of rotation by calculable amounts; and thus it was possible to follow "the wandering of the pole" and precession of the equinoxes. In these and other problems, Newton inaugurated the theory and calculation of planetary perturbations.

The comets which for ages had evoked the superstitious fear of savage and civilized man alike, were proved to be law-abiding members of the solar system, differing from the friendly planets chiefly in the smallness of their masses and the relatively high eccentricity of their orbits. (Their tails were explained only much later; their probable composition also was not determined until long after Newton was dead.) Further, it was possible to predict with high accuracy the dates at which some would return. Here was a deliverance from superstition that any lettered person could appreciate; and of all the deductions from Newton's hypothesis of universal gravitation, the accurately predicted return of a comet [**33**] has been the most popular. Scientifically, however, it is not comparable with what followed from Newton's hard thinking on the problem of the moon's motion. Among its numerous claims to perpetual remembrance, this problem has the unique distinction of being the only one of which Newton confessed that it gave him a headache.

The motion of the moon presents a special case of the problem of three bodies. The moon's orbit is perturbed by the attractions of the earth and the sun, and also, to a lesser degree, by the attractions of the other planets. The perturbations cause irregularities in the moon's orbit, some of which had been observed by the Babylonians (?) and the Greeks. None had been accounted for when Newton [**34**] deduced them as consequences of universal gravitation, and in addition uncovered two more. In historical order from Hipparchus (2nd century B.C.) to Newton's contemporary Flamsteed, these were the equation of the center, the evection, the variation, and the annual equation; retrogression of the nodes, variation of the inclination; progression of the apses (only half the observed amount was given by Newton's calculation); and the inequalities of apogee and of nodes, Newton's discoveries. It is at this point that Newtonian gravitation makes one of its most direct contacts with the science of the twentieth century. The problem of three bodies [**35**] remains an astonishingly prolific source of new mathematical methods and refined astronomical-physical observations.

A concise history of this problem would fill a large book. Here, only a detail or two can be noted. To state the problem: Three particles, free to move in space, attract one another according to the Newtonian law of gravitation: from arbitrarily assigned initial motions of the particles, to determine their subsequent motions. From Newton in the seventeenth century, through Clairaut, D'Alembert, Euler, Lagrange, Laplace, and Poisson in the eighteenth and early nineteenth centuries, through Weierstrass, G. W. Hill, E. W. Brown, Bruns, Poincaré, Levi-Civita, and Sundman from thence to the present, this problem has exercised the highest talents of some of the most powerful mathematicians of the last two centuries. Only a few of the great names associated with the three-body problem have been recalled, none from the living.

Three items will suffice to indicate the inexhaustible fertility of this venerable problem, and generally of the theory of planetary perturbations initiated by Newton. First is the prediction from the theory of the existence of a planet

which, until the perturbations of the orbit of Uranus were analyzed mathematically almost simultaneously (1845–46) and independently by Adams and Leverrier, was no more than a vague conjecture. The subsequent discovery (1846) of the hypothetical planet (Neptune), almost in the calculated place, is one of the half dozen or so masterpieces of scientific prediction. Uncritically viewed, it was acclaimed as a conclusive demonstration of the universality of Newton's law of gravitation, until critical mathematicians hinted that fortunate accident had also contributed to the discovery. Benjamin Peirce even proceeded fearlessly to the limit, declaring that if the calculations had been more accurately performed, Neptune would not have been observed. There was no question of minimizing the magnitude of the discovery; criticism was motivated solely by a desire to uncover the scientific facts regardless of the great personalities involved. Nature plays no favorites: was the discovery of Neptune a final confirmation of the Newtonian law, or was it not? Criticizing the mathematical critics, astronomical critics set aside the strictures of Peirce and others by an argument of a type which a mathematical logician might dismiss as specious and ad hoc, in spite of its frequent reproduction in standard textbooks. However, the prediction is now generally accepted as genuine.

A more exacting and less debatable test lay nearer to hand in the moon's motion. Laplace had imagined that gravitation alone would explain the slow approach of the moon to the earth; refining the calculations, Adams showed that gravitation accounted for only half the observable amount. Though at first sight a serious setback for Newton's conception of "the System of the World," this discrepancy between theory and observation helped to precipitate the far-reaching theory of tidal friction in the solar system and in stellar evolution, to-day of livelier scientific interest for the dynamics of the nebulae than most of the particle dynamics of classical celestial mechanics. Thus, in the end, impartial scrutiny of nature by the observational-mathematical method followed to the exclusion of all others by Newton himself, sustained the spirit of his law of gravitation, and broadened its scope more widely than strict adherence to its exact letter could ever have done. The question of the universality of Newton's law was still open. The more recent discovery [**36**] (1930) of Pluto, the ninth planet of the solar system, raised similar doubts. How much of the credit should be assigned to calculation from theory and how much to chance? The question is still debated. If, as has been conjectured from possible perturbations of Halley's comet, a trans-Plutonian planet exists, expert opinion is that it will be found by photographic methods rather than by calculation.

The second item of current interest that can be traced quite directly to attempts to solve the three-body problem is the development of modern topology. One of the most active departments of twentieth-century pure mathematics, topology has a main source in the new methods introduced into celestial mechanics in the 1890's by Poincaré [**37**].

The third item disposed of a possible doubt that had haunted some expert analysts from Lagrange to Weierstrass. To the mathematical tyro it may seem

self-evident that the highly idealized mathematical representation of a concrete physical situation must lead to a system of equations that are solvable. There is not space to present the matter here; but it may be noted that competent differences of opinion have arisen from examination of the relation of mathematical prediction to observationally verifiable fact. In the three-body problem, it had been suspected that a solution by convergent power series might be impossible. It was therefore a mathematical event of more than ordinary interest when, in 1912, the work (1906, 1909) of Sundman [**38**] became known outside his native Finland. His capital result is as follows (translation): "If the constants of areas in the motion of three bodies with reference to their common center of gravity are not all zero, a variable τ may be found such that the coördinates of the bodies, their mutual distances, and the time are developable in convergent power series in τ which represent the motion for all real values of the time, and do so whatever collisions may occur between the bodies." A note by Sundman recalls that Weierstrass in a letter of 1889 to Mittag-Leffler had shown that in triple collision the constants of areas vanish simultaneously. Sundman's solution is not (yet?) adapted to numerical computation. But that was not its motivation, nor is computability its present interest.

On the side of greater significance for practical astronomy, that of mutually checking observations and computations, the special three-body problem of the lunar theory also has made remarkable progress in the twentieth century. Before stating what may seem a rather disconcerting conclusion for the Newtonian theory of universal gravitation, it may be emphasized that authorities on the subject—men who live with telescopes and who supervise the calculations based on their own observations—anticipate no fatal discrepancy between theory and observation. The labor of adding another decimal place, or of taking account of further planetary or tidal perturbations, is very great; and until more effective means of calculation are devised, it would be premature to assert that the Newtonian theory has encountered an irremovable obstacle of fact. On the other hand, nothing is to be gained and a universe may be lost by ignoring or minimizing indisputable discrepancies between irreproachable calculation and precise observation. The advance of the perihelion of Mercury (to be noted in another connection) offers a famous historical instance of the advantages of total scientific honesty.

A serious disagreement between fact and prediction appears in the irregular differences between the moon's observed place and its place as calculated on the lunar theory. In the opinion of experts [**39**], no significant term can possibly have been neglected. On the numerical side, the accuracy of the calculations has been checked mechanically. The conclusion is that the observed discrepancies are much too large to be attributed to observational errors. There is also an unexplained secular acceleration. Among possible hypotheses within the Newtonian theory to account for these irregularities of the moon's motion are tidal friction, undescribed inhomogeneities in the density of the earth, irregularities in its shape, and variations in its rate of rotation. It is now supposed that if tidal

friction is significant, oceanic tides are of comparative unimportance, the major effects being caused by tides in shallow inland seas and tidal basins. An interesting by-product of these irregularities in the moon's motion is the revision they necessitate of dates of ancient eclipses. As some of these dates have been used as points of reference for human history, the lunar theory may yet further confound the confused annals of our race. In any event, this special case of the three-body problem suggests more unsolved problems in 1942 than Newton could possibly have imagined in 1687, when he published the epoch-making results of his most intense thinking in the *Principia.*

The three-body problem is a special case of the problem of n bodies attracting one another according to the Newtonian law of gravitation. On the mathematical side, "there remain still a great number of unsolved problems . . . , for example, the problems of stability and transitivity [associated with the names of Poincaré and G. D. Birkhoff in one method of attack], and it seems that the solution of the main problems will require new methods of analysis" [**40**]. This pronouncement by a mathematical expert need not necessarily discourage research on the observational side. Paradoxically, mathematical solutions of the three-body problem may still be waiting objective verification ten thousand years after the problem of hundreds of millions of bodies has been solved mathematically and checked observationally. New methods (to be noted presently) are already being applied to the larger problem with what, in 1942, looks like promising success.

It is merely one of the minor misfortunes of the human race that it was assigned to a planet from which no decent embodiment of the general three-body problem is visible, even spectroscopically [**41**]. Anything that can be seen as an individual three-body configuration is too far away, or too complicated, or too meanly specialized for an observational check of any mathematical theory in any span of years our race is likely to survive. Neither Jupiter, Saturn, nor the Sun has zero mass; yet the three have furnished mathematicians with an instance of the special case of the three-body problem in which one mass is 'zero.' Again, each of the six Trojan planets (asteroids with the same period as Jupiter) with Jupiter and the Sun is ideal, with some slight forcing, for the Lagrangian special case in which the bodies are initially at the vertices of an equilateral triangle. Yet even this problem is at present too difficult when rigid idealization is only slightly relaxed to approximate roughly the reality of actual planets subject to librations [**39**]. The planets in the sky are more than mathematical points endowed with mass on paper. As for Jupiter himself, his ephemeris is significantly in error; and his nine [**42**] satellites pose a problem that not only is beyond mathematical solution at present, but also is of great difficulty observationally on account of the faintness of the satellites contrasted against their planet's brightness. The only known way of keeping track of this too-numerous family is by mechanical integration; an analytical solution would demand an explicit form of the perturbations. As a final scandal in our erratic solar system, the latest transit of Mercury produced an unanticipated error that amused as-

tronomers for weeks. Here, however, the discrepancy between prediction and observation may be reduced farther than it already has, when all the corrections of the ephemeris are included and the diameters of the bodies concerned are satisfactorily defined. In precise measurement, what exactly is the diameter of a bright luminous disc? From all this and more of the same general character, it seems unlikely that the universality of Newton's law of gravitation will receive conclusive observational confirmation in the immediate future from the three-body problem for any configuration in our solar system. Fortunately for possible progress, advances in astronomical instruments and in mathematical technique have made possible the verification of Newton's law in regions tens, or hundreds, or thousands, or even millions, of light years beyond "the World," whose "System" he explored and reduced to order in the most inclusive synthesis of natural phenomena in the history of science.

To return for a moment to pure mathematics, some of Newton's greatest triumphs in the *Principia* would have been impossible without the theorem, which he discovered and proved, that the attraction of a homogeneous gravitating sphere on an external mass-particle can be calculated as if the mass of the sphere were all concentrated at its center. It has been argued [**43**] that the lack of this master theorem delayed the publication of the law of gravitation for twenty years. Today it is a students' exercise in the calculus. *Sic itur ad astra.*

Last, it should be noted that the identity of inertial and gravitational mass, long a puzzle in Newtonian mechanics, was satisfactorily accounted for only when that mechanics was supplemented by relativity.

To resume the main question, how 'universal' is Newton's law of universal gravitation? As sometimes stated in textbooks, the law asserts that "Every particle of matter in the universe attracts every other particle with a force which, for any two particles, is directly proportional to their masses and inversely proportional to the square of the distance between them, mass, distance, and force being measured in the appropriate units." Taken literally, this statement is as wild an extrapolation as any pseudo-scientific generalization that ever confused theology with science. Neither in space nor in time, nor yet in space-time, is anything whatever known of "every particle of matter in the universe"; nor is it likely that human beings will ever be able to make any scientifically verifiable statement about "the universe." Such statements of course are frequent enough; but the more cautious, chastened by experience, leave these tremendous nothings to mystics and belated stragglers from the nineteenth century. In that heyday of theology masquerading as physics, the conservation of energy was as everlasting and as pervasive as the will of the deity, and the laws of thermodynamics were eternal truths throughout all space.

Actually, so far as gravitation was concerned until quite recently, it was satisfactorily verified only for a negligibly small time span in that speck within our own system of stars which we call the solar system, and beyond that only out to binary stars well within our own stellar neighborhood. Analysis of the

orbits of numerous binaries, both visual and spectroscopic, since Savary [**43a**] (1828) suggested that certain double stars observed by W. Herschel (1780) offered a test of the Newtonian theory, have confirmed the Newtonian law of gravitation well within the probable observational errors. But even here, so elementary a question as the following cannot be decided observationally and no other way is known: How accurately can the motions of a swarm of stars, considered as a dynamical system, be described by linear differential equations —a capital assumption of Newtonian celestial mechanics? There has not been time enough to obtain the necessary data. Once more necessity drives observation out of our solar system, out far beyond our own galaxy, to seek the facts on which the celestial mechanics of the future may be based, in the extra-galactic nebulae. As Newton reared his *System of the World* on scientific facts obtained by laboriously painstaking observation, so some spiritual successor of his may find in the rapidly accumulating observations of the nebulae the building stones of a vaster system. But he will not, if he is a worthy disciple of Newton, misname it a system of the universe.

An early hint that methods beyond those of classical celestial mechanics were about to emerge was Poincaré's suggestion, near the turn of the century, that star-streaming in our galaxy be investigated by the methods of the kinetic theory of gases. In this daring program, the individual stars were the 'atoms' of the 'gas' which was the Milky Way. As the proposal was put forward about a quarter of a century before Hubble inaugurated the modern era in the study of the extra-galactic nebulae [**44**], it has not had much influence, if any, on subsequent progress. The discovery of clusters of nebulae, and the revolutionary advance in certain departments of observational astronomy consequent on the recent use of Schmidt telescopes for rapid surveys of the whole sky, have transformed the problem completely. As mathematicians occasionally overlook a humble but important source of some of their sublimest imaginings, it may be recalled that progress in technology and in the mere machining of precision instruments often precedes the decisive turning points in science from which new mathematical theories start. Without modern spectroscopes, rapid photographic films, and Schmidt telescopes, current advances in the application of Newtonian gravitation to the extra-galactic nebulae would be a dream for the distant future.

Since it is impossible to observe millions of stars or thousands of nebulae individually, statistical mechanics is indicated as the appropriate mathematics for the newer problems of astronomy [**45**]. Theoretical progress is possible partly because Emden elaborated (1907) the mathematical theory of gravitational gas spheres [**46**]. The gravitation in this theory is Newtonian. From statistical mechanics it is deduced [**45**] that a non-rotating cluster of stars or of nebulae must be spherically symmetrical to within calculable fluctuations in the distribution of the objects composing the cluster. There are available for observational tests of the predictions (at least) three large clusters of nebulae, located respectively in the constellations Hydra, Perseus, and Coma Berenices. In millions of light

years the respective distances of these are twenty-four, thirty-six, and forty-five. Compared to the least of the three, the distance of the farthest binary star ever observed is practically zero. Thus if theory and observation check, Newton's law will have been extended enormously. The extension is, as it were, a second-order mathematical effect; it is achieved indirectly, through the assumption of Newton's law in the derivation of the statistical formulas. The check [**45**] is sufficiently good to be most encouraging to anyone who is discourteous enough to abandon his companions on the beaten path and let obsolescence overtake the hindmost.

The Coma cluster consists of some 2,000 nebulae, probably more, of which about 650, each with a luminosity exceeding a hundred million times that of the Sun, have been identified. These are distributed throughout a sphere whose diameter is of the order of five million light years. If the cluster is actually stationary, statistical mechanics predicts that the distribution of nebulae throughout this sphere is not random; and application of the Emden theory [**45, 46**] gives the rate of decrease in the density of nebulae as the radial distance from the center of the sphere increases. The agreement found by Zwicky [**45**] between his calculated and observed radial distribution of nebulae in the Coma cluster is most remarkable, and likewise for the Perseus and Hydra clusters. These agreements between theory and observation can be interpreted as an extension of Newtonian gravitation to the realm of the nebulae.

It is important to notice that the Newtonian law of the inverse square is a sufficient, but not necessary, hypothesis to account for the observations. It is possible that from further counts of nebulae some power other than the second, say the 2.01th, may be demanded; but at present this seems improbable. The situation is analogous to that confronting Kepler when he stated his three laws which were later to be interpreted by Newton on the basis of the law of the inverse square. It has been said, and it appears to be no more than the truth, that with more exact observational data at his disposal, no man of Kepler's undeviating intellectual integrity would ever have found his laws. Mathematically, Kepler's laws are both necessary and sufficient for the deduction of Newton's inverse square law of gravitation. It is indeed fortunate for science, especially for the *Principia*, that the data which Kepler ground through the mill of his interminable arithmetic were no better than they were; and it is to be hoped that no catastrophic refinement in observational technique will obliterate a most promising mechanics of the nebulae before it is fully born. The outstanding mathematical problem here is to give a rigorous discussion of Emden's equation and to investigate the domains of existence of any new solutions ($n \neq 2$) that may be found.

Another recent confirmation [**47**] of the Newtonian law from the extra-galactic nebulae is the observational and theoretical work of Mayall, Aller, and Wyse on the spirals Messier 31 (the Andromeda nebula) and Messier 33. Slipher in 1914 proved that the former is rotating—the first such proof for any nebula. The spectroscopically observed distribution of velocities for each of these spirals at

different distances from its centre is almost weirdly irregular. Yet, analyzed mathematically, the distribution fits the Newtonian law when a reasonable distribution of the spiral's mass is postulated.

Surely there could be no more fitting gift for Newton on the three-hundredth anniversary of his birth than all these extra-galactic confirmations of his law of gravitation, unless it be one or other of the many new problems which they suggest. Among these are the distribution of the nebulae in space, the distribution of their velocities, and the shift toward the red of the spectral lines in the light from distant nebulae. Attempts to explain the latter are still in progress. The theory of the expanding universe, once attractive to theoretical astrophysicists as an explanation of the red shift, is not yet refuted or sustained by the experimental evidence. It appears that the decision must wait for more observational data, to be obtained by other instruments than those immediately available. Star counts of nebulae and counts of nebulae in clusters will doubtless figure in the decision, and these are not yet problems for pencil and paper [**48**]. It is the belief of some whose work entitles them to an opinion on the matter, that in any serious attack on these problems and others of the new astronomy, the question of the scope of Newtonian gravitation will recur again and again.

Metaphysics

At the six-hundredth commemoration of Newton's birth, if anything worth remembering is remembered then, some curious historian may inquire what change in Newton's standing as of 1942 against 1687 most deeply affected the so-called common man. A backward glance at the enlarging conception of the physical universe in 1942 will suffice to reject it from further consideration. The common man of 1942 has all he can do to make a living, or even to keep alive, in the world he can see, touch, hear, and smell with his naked senses. The historian almost certainly will be aware of the common man's profoundly disturbed state of mind in the 1930's; and doubtless some of the scathing rebukes squandered on him for his cynicism, his poverty of ideals, his callous disregard of this, that, or the other sanctity of the past, will have survived in the world museum of mental pathology. A note in the catalogue may even state that these musty relics have been preserved because they are the first hint, all but imperceptible, that the mass of mankind in the early twentieth century was beginning to develop a mind of its own, and might some day be able to think for itself.

Trying to recall what first roused this mind to a subconscious distrust of all self-constituted authority, the historian may dimly remember something about the perihelion of Mercury, the deflection of a ray of starlight grazing the sun, and the shift toward the red of the spectral lines of sunlight. It will all come back to him then, if he is an orthodox historian of his age. The total collapse in 1915–16 of Newton's absolute space, absolute time, absolute motion, that had been believed in for two centuries by the majority of thinking mankind, common and uncommon, carried down with it more than one absolute that had stood unshaken for a thousand years in the popular mind—absolute right, absolute wrong,

absolute justice, absolute truth, and all the other hoary old absolutes that summed up to absolute authority over what human beings may think and what they shall not think.

The abolition of Newtonian metaphysics is the change of deepest significance for the common man that has occurred since the *Principia* in 1687 fixed the absolutes of space and time in the popular mind no less firmly than in the scientific. Though the man who knows little or nothing of science, and who has heard of relativity only as a month's sensation in the press, may be unaware of what changed his mind on so many things in less than a generation, it was the collapse of Newtonian absolutism that started the radical transformation of his attitude toward all absolutes, all authority, and all tradition. The change is still stubbornly resisted, especially by the humanistically educated; but not even unteachable reactionaries and vestigial mediaevalists believe in their hearts that a return to the Newtonian metaphysics of science is possible. Whatever the future of relativity is to be, it is unthinkable, to a generation reared in its conception of space-time, that theoretical physics can ever again follow the strict Newtonian way.

The first prediction of general relativity, which accounted for the large discrepancy between observation and calculation by the Newtonian theory in the advance of Mercury's perihelion, was almost decisive. Observational verifications of the other predictions of relativity settled the matter, except for those who argued that, as the quantitative differences between Newtonian and relativistic physics are small, therefore one is no better a description of natural phenomena than the other. It is not a question of quantity but of quality, a fact that no amount of fatuous casuistry can quibble away. As the author of relativity has repeatedly declared, a single new observed fact may destroy the theory of relativity any day. So far the destructive fact has not appeared. But if it should, it would not necessarily be followed by an immediate return to Newtonian metaphysics. That, apparently, has gone forever.

Newton attempted in the first scholium of the *Principia* to clarify the then acceptable notions of absolute time, absolute space, place, and absolute motion. The scholia of the *Principia*, it may be recalled, were designed as a gloss on the definitions and propositions to assist the reader's understanding. A student having some acquaintance with the modern postulational method in mathematics might find this first scholium obscure and hopelessly confused, and wonder why readers of the *Principia* for so long treated the Newtonian absolutes with respect. Newton appears to have been at his worst as a metaphysician, and that worst, supreme intellect as Newton was, seems to modern students to be no better than that of his contemporaries. Servile respect for authority, the bane of science, may have stifled objective criticism, perpetuating ideas that had better have been forgotten before they were printed. However, an independent mind of first-rate intelligence here or there in the century following Newton did venture to reject the patent circularities and obvious inconsistencies in the famous scholium of the absolutes. Thomas Young [49], for example, in

1807, having stated as a definition that "motion, . . . , is the change of rectilinear distance between two points," continued after some argument as follows: " . . . therefore if a single atom existed alone in the universe, it could neither be said to be in motion nor at rest. This may seem in some measure paradoxical, but it is a necessary consequence of our definition." Newton himself was dissatisfied with the evident incompatibility between the spatial relativity of his dynamics (with reference to unaccelerated motion) and his metaphysics of absolutes. Young showed that absolute rest and absolute motion are illusory. But more than good logic against bad was required to dispose once for all of an impossible metaphysics; and it was only when experiment verified the predictions of general relativity that the Newtonian absolutes were abolished.

If relativity destroyed one metaphysics, it and the quantum theory between them generated another. This is of particular interest for Newton's three-hundredth anniversary, as it is the complete antithesis of his teachings [**50**] regarding the office of experiment in the physical sciences. The most vigorous proponent of this anti-Newtonian philosophy of science is Sir Arthur Eddington, professor of astronomy and experimental science in the same University of Cambridge where Newton lectured on optics and philosophized about the part of experiment in the search for positive knowledge. An adequate presentation of Eddington's scientific philosophy is out of the question here, and any short notice of it may do it injustice. However, as he himself has presented it with his usual clarity and charm in places readily accessible [**51**], any injustice is not as serious as it might be.

Though not a full return to Pythagorean numerology and Platonic realism, the anti-experimental philosophy is hauntingly reminiscent of both. Two much-quoted professions of faith will suffice to indicate the chasm that separates the new from the old. "An intelligence unacquainted with our universe, but acquainted with the system of thought by which the human mind interprets to itself the content of its sensory experience, should be able to attain all the knowledge of physics that we have attained by experiment. He would not deduce the particular events and objects of our experience, but he would deduce the generalizations we have based on them. For example, he would infer the existence and properties of sodium, but not the dimensions of the earth." Again, "I believe . . . [that] all the laws of nature that are usually classed as fundamental can be foreseen wholly from epistemological considerations." Sublime, if true. Also one of the perfect ironies of history.

In support of his belief, Eddington claims to deduce the value 137 for the fine-structure constant of spectroscopy. While not pretending to follow the proof, experimental physicists admit that '137' (it was 136 at first) [**52**] was the direct occasion for certain refined experiments in electronics. The men who performed these experiments state that 137 is factually correct to well within the probable errors of measurement. Until Eddington claimed to have proved that this constant must be an integer, and stated 137 as the integer, no experimental physicist had suspected as much. "Now," as one competent authority said, "you

have got to accept 137 whether you like it or not." To this extent, at least—two other consequences of the theory are substantiated by previous laboratory tests—the theory has discharged its scientific function: it has predicted an experimentally verified fact. Should it continue to be as successful, the new natural philosophy will be of quite another order than Newton's, and all that followed from his will be a trivial detail in the most astounding scientific generalization since Pythagoras succeeded in astonishing himself with the pseudo-discovery that everything is number. Possibly a decision between the Newtonian and the Eddingtonian philosophies of science will have been reached by the year 2242. Until then, bystanders can only watch and hope.

If nothing has been said about Newton's life and character, it is because both were exhaustively discussed in connection with the two-hundredth anniversary of his death and for a decade thereafter. From all that discussion Newton emerged as a human being. The insipid saint of the nineteenth century was a pious myth; Newton could be as angry, as petty, and as generous as any normal man. It is no longer reputable for a retrospective generation to endow its heroes with all the virtues with which it might like to be credited, but which it lacks. In regaining the common humanity of which apologetic sentimentalists had robbed him, Newton lost nothing of the one thing that distinguished him from his fellow men. The supremacy of his intellect has not been challenged.

By 1942, western civilization had experienced three centuries, more or less, of the modern science which developed from the experimental-mathematical method of Galileo and Newton. Among other things this science has taught open minds is a decent humility in the presence of nature. The old assurances and arrogances are gone; the universe is not a book to be read in a cloister, nor is the solar system the simple parish it was in the middle ages. If little is known now, less was known then. Yet the majority, if the choice were theirs, would probably return to the centuries before Galileo and Newton were born. Not all are envious reactionaries; many sincerely and ignorantly believe that science has showered the world with a wealth of material comforts while robbing it of what they call spiritual values.

If they ever think at all about their place and function in society, scientists may be inclined to overestimate their importance as shapers of public opinion and educators of the mass of mankind. The mass of mankind knows next to nothing of them or their work, and if it knew more, might think even less. What little it does know fosters a sullen distrust. Men who hold their hypotheses lightly or who, like Newton [**53**], glory that they frame none, are not popular. They never were. And while science goes its indifferent way, the world it would serve yearns for the futilities of a nostalgic humanism that knew better days three hundred years ago, and surrenders its intelligence to the unreason of credulous mysticisms.

If the world is to abandon science and return to the past, somewhere on its way in the next three centuries Newton's estimate of his scientific work will be

confirmed; and his *Observations upon the Prophecies of Daniel and the Apocalypse of John*, with *The Chronology of Ancient Kingdoms amended*, on which he lavished his intellectual powers in the latter half of his life, will outlive the *Opticks* and the *Principia*.

References, Notes

1. *Philosophiae Naturalis Principia Mathematica*, Auctore Is. Newton; London, 1687. (a) *The Mathematical Principles of Natural Philosophy*. By Sir Isaac Newton, Translated into English by Andrew Motte; etc., vols. 1, 2, London, MDCCXXIX. (b) Florian Cajori's revision of (a), Berkeley, California, 1934. Unless otherwise stated, references are to (a).

2. In a letter to Oldenburg.

3. Book 3, Lemma 5.

4. *Principia*, Book 2, Proposition 34, (=35 in first edition).

5. *Opticks* (see reference 7), *Enumeratio Linearum Tertii Ordinis*, pp. 139–162+6. Tab. omitted in second edition (see reference 8).

6. J. J. Sylvester, *Philosophical Transactions of the Royal Society of London*, 154, 1864, p. 579.

7. *Opticks: or, a Treatise of the Reflexions, Refractions, Inflexions and Colours of Light. Also Two Treatises of the Species and Magnitude of Curvilinear Figures.* [Author's name not given.] London MDCCIV.

8. *Opticks: or, a Treatise on the Reflections, Refractions, Inflections and Colours of Light.* The Second Edition, with Additions. By Sir Isaac Newton, Knt. London, 1717.

9. *Opticks*, (reference 8), pp. 380–381. Not in first edition.

10. Sir Arthur Stanley Eddington.

11. For diagram, reference 7, Book I, Part I, Plate V.—It has been claimed that Newton was misled by what, for him, was a singularly maladroit experiment, and that the great weight of his mistaken authority in this instance retarded progress in refractors for three-quarters of a century.

12. Reference 8, Book 3, pp. 322–323. The queries in reference 7 end with Qu. 16.

13. If omission of the context to which Newton refers obscures his meaning here, see the account of his theory in almost any textbook of college physics. Recent texts contain sufficient on the quantum theory to bring out the analogy noted.

14. *Thèse de Physique: Sur les vitesses de la lumière dans l'air et dans l'eau. Thèse*, etc., Paris, 1853. Par M. Léon Foucault. Soutennue le 25 avril 1855. pp. 3–35+1 plate.

14a. So-called because Newton's work made them famous in optics. Due to R. Hooke, *Micrographia or some Physiological Descriptions of Minute Bodies*, etc., London, MDCLXV. *Observation IX.*, *Of the Colours observable in the Muscovy Glass and other thin Bodies*, pp. 47–67+1 plate.

15. *The thicknesses of colored Plates and Particles*, reference 7, p. 35; reference 8, p. 206.

16. The Bakerian Lecture, *On the Theory of Light and Colors*. By Thomas Young, M.D., F.R.S., Professor of Natural Philosophy in the Royal Institution. Philosophical Transactions of the Royal Society of London for the year MDCCCII. Part 1, pp. 12–48+1 plate.

17. Reference 16, p. 39.

18. "I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."—Newton.

18a. *Jabir ibn Haiyan al-Tarasusi, works of Geber*; Englished by Richard Russell, 1678; a new edition, edited with introduction, by E. J. Holmyard, London, 1929. [It is not settled whether Jabir was the same man as Geber.]

19. Georgii Agricolae, *De Re Metallica*, etc., 1556. English translation, *Georgius Agricola, De Re Metallica*, etc., . . . ; by Herbert Clark Hoover and Lou Henry Hoover, etc., San Francisco, 1912, p. 235.

20. *A Letter of Mr. Isaac Newton, Mathematick Professor in the University of Cambridge; con-*

taining his New Theory about Light and Colors: etc. Philosophical Transactions of the Royal Society of London, Number 80, February 19, 1671–72.

21. William Hyde Wollaston, M.D., F.R.S.; *A Method of explaining refractive and dispersive Powers by prismatic Reflection.* Transactions of the Royal Society of London, 1802. Read, June 24, 1802, pp. 365–380. One plate, p. 378, with picture of the lines.

22. Reference 7, Book I, Plate V.

23. Reference 7, Qu. 10, p. 134.

24. Thomas Melvill, M. A., *Observations on Light and Colours.* Essays and Observations, Physical and Literary. No. IV. Read before a Society in Edinburgh and published by them; Vol. II, Edinburgh, 1756. [Recent references in the literature are almost invariably wrong. The date is important.]

25. J. F. W. Herschel, Art , *Light* in Encyclopaedia Metropolitana, Second Division, vol. 2., London, 1830, p. 438. [H. Kayser's criticism (*Handbuch der Spectroscopie*, Leipzig, 1900, Bd. 1, Kap. 1, §11, pp. 12–14. especially p. 14) of Herschel is in error; he evidently did not have available the early copy of the Encyclopaedia.]

26. Joseph Fraunhofer, *Bestimmung des Brechungs- und- Farbenzerstreuungs Vermögens verschiedenen Glasarten in Bezug auf die Vervollkommnung achronalischer Fernöhe.* Denkschriften der königlichen Akademie der Wissenschaften zu München für die Jahre 1814 und 1815, pp. 193–326+3 plates. See especially plate 2, and p. 202.

27. See also reference 8, Qu. 26, Qu. 28.

28. *The Electromagnetic Theory of Light*, Cambridge, 1864, in the Scientific Papers of J.C.M., Cambridge, 1890, vol. 1, pp. 577–588. See also ibid., 1861–2, pp. 451–488.

29. Quoted from reference 1(b), pp. 633–4.

30. Reference 1(a), p. 225.

31. Reference 1(a), p. 226.

32. Reference 1(a), Book III.

33. The error for the predicted date of return (at perihelion) of Halley's Comet in 1910, after an absence of 75 years, was 3.03 days.

34. Reference 32.

35. See reference 1(a), Book III, Propositions XXII, XXV–XXXV for Newton's treatment of the special case of the lunar theory.

36. Percival Lowell claimed to have predicted the planet.

37. H. Poincaré, *Les Méthodes nouvelles de la Mécanique Céleste*, 3 vols., Paris, 1892, 1893, 1899.

38. Karl F. Sundman, *Recherches sur le problème des trois corps*; Acta Societatis Scientiarum Fennicae, vols. 34–5; reproduced, revised, in Acta Mathematica, vol. 36, 1912–1913, pp. 105–179; result stated, p. 107.

39. Of whom E. W. Brown was the leader on the computational side till his death in 1938. His works include *Tables of the Moon's Motion*, 1920; *An Introductory Treatise on the Lunar Theory*, 1896; *The Evidence for Changes in the Rate of Rotation of the Earth*, etc., 1926; *Planetary Theory*, 1933; *Tables for the Development of the Disturbing Function*, 1933. These contain much material for the present status of Newtonian gravitation in the solar system.

40. C. L. Siegel, *On the modern development of celestial mechanics*, this MONTHLY, vol. 48, 1941, p. 435.

41. There is an interesting historical resonance effect in the application of perturbative analysis, originating with Newton, to the modern outgrowths of classical spectroscopy, which also developed from Newton's work, in the quantum theory, specifically in the four-body problem of the hydrogen molecule.

42. The ninth was discovered only in 1914, by S. B. Nicholson. Like the eighth, it has retrograde motion, which might prove embarrassing for any resurrection of the Kant-Laplace nebular hypothesis.

43. Florian Cajori, *Newton's twenty years' delay in announcing the law of gravitation*, in *Sir Isaac Newton, 1727–1927*, The History of Science Society, pub. Baltimore, 1928, pp. 127–188.

43a. *Connaissance des Temps*, Paris, 1830, pp. 56, 163. The binary concerned is the fourth magnitude star ξ Ursae Majoris.

44. An early and decisive item of Hubble's work was the first resolution of any nebula (Messier 31, the great spiral in Andromeda) into stars. Thus Herschel's conjecture that the spiral nebulae are 'island universes,' comparable to our own galaxy, became scientific fact. This work bears the same relation to the mechanics of the nebulae that Newton's optics bears to theories of light. See Edwin Hubble, *The Realm of the Nebulae*, New Haven, MDCCCCXXXVI.

45. F. Zwicky, *Hydrodynamics and the structure of stellar systems*, Applied Mechanics, Th. von Kármán Anniversary Vol., 1941, pp. 137–153; *Reynold's number for extragalactic nebulae*, Astrophysical Journal, vol. 93, 1941, pp. 411–416; *On a cluster of nebulae in Hydra*, Proceedings of the National Academy of Sciences, vol. 27, 1941, pp. 264–269; *On the physical characteristics of the Hydra cluster of nebulae*, ibid., vol. 28, 1942, pp. 151–155; *On the clustering of nebulae*, I, Astrophysical Journal, vol. 95, 1942, pp. 555–564; *Newton's law of gravitation*, Astronomical Society of the Pacific, Leaflet No. 163, September, 1942; further work in press.

46. R. Emden, *Gaskugeln*, Leipzig, 1907.

47. See Henry Norris Russell, The Scientific American, June, 1942, pp. 274–5, Wyse was killed recently in the line of military duty.

48. But see reference 51.

49. *A course of Lectures on Natural Philosophy and the Mechanical Arts.* By Thomas Young, M.D., F.R.S., etc., vol. I, London, 1807; Lecture II, On Motion, p. 19.

50. Reference 9.

51. A. S. Eddington, *Relativity Theory of Protons and Electrons*, Cambridge 1936, especially p. 327. [This theory did not predict the mesotron, which was in process of experimental discovery while the book was in press.] *The Philosophy of Physical Science*, Cambridge, 1939, especially p. 56. [The whole context of the first reference should be read.]

52. The fact that every group has an identity element was overlooked, as it well might be by physicists interested only in getting something new. This was the root of the discrepancy.

53. "But hitherto I have not been able to discover the cause of gravity from phaenomena, and I frame no hypotheses."—Newton, reference 1(a), p. 392. The sense in which Newton intended his "hypotheses non fingo" is plain from his writings.

THE TWENTY-FIFTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The twenty-fifth summer meeting of the Mathematical Association of America was held at Vassar College, Poughkeepsie, New York, on Monday to Wednesday, September 7–9, 1942, in conjunction with the summer meeting and colloquium of the American Mathematical Society and the meeting of the Institute of Mathematical Statistics. Two hundred eighty-three were in attendance at the meetings, including the following one hundred thirty-nine members of the Association:

C. R. Adams, Brown University
R. P. Agnew, Cornell University
R. C. Archibald, Brown University
L. A. Aroian, Hunter College
W. L. Ayres, Purdue University

Frances E. Baker, Vassar College
Aaron Bakst, Columbia University
Grace M. Bareis, Ohio State University
Walter Bartky, University of Chicago
E. G. Begle, Yale University
Brother Bernard Alfred, Manhattan College
Felix Bernstein, New York University
Henry Blumberg, Ohio State University
Julia W. Bower, Connecticut College
R. W. Brink, University of Minnesota
A. B. Brown, Queens College
J. A. Bullard, University of Vermont
F. J. H. Burkett, Union College
Hobart Bushey, Hunter College
Jewell Hughes Bushey, Hunter College

S. S. Cairns, Queens College
W. D. Cairns, Oberlin College
B. H. Camp, Wesleyan University
W. B. Carver, Cornell University
F. L. Celauro, Loyola College
W. F. Cheney, Jr., University of Connecticut
L. W. Cohen, University of Kentucky
Nancy Cole, Sweet Briar College
E. P. Coleman, U. S. Military Academy
J. A. Cooley, University of Tennessee
T. F. Cope, Queens College
A. H. Copeland, University of Michigan
Richard Courant, New York University
C. C. Craig, University of Michigan
J. H. Curtiss, Cornell University

D. R. Davis, State Teachers College, Montclair, N. J.
F. F. Decker, Syracuse University
R. P. Dilworth, Yale University

C. D. Firestone, Cornell University
M. M. Flood, Princeton University
L. R. Ford, Illinois Institute of Technology
R. M. Foster, Bell Telephone Laboratories

B. P. Gill, College of the City of New York
R. E. Gilman, Brown University
A. M. Ginsburg, Bronx Vocational High School
Michael Goldberg, Bureau of Ordnance, Navy Department
Cornelius Gouwens, Iowa State College
H. S. Grant, Rutgers University
R. E. Greenwood, Jr., University of Texas
T. N. E. Greville, Bureau of the Census
C. C. Grove, College of the City of New York
V. G. Grove, Michigan State College

D. W. Hall, Brown University
Olive C. Hazlett, University of Illinois
G. A. Hedlund, University of Virginia
E. R. Hedrick, University of California at Los Angeles
L. S. Hill, Hunter College
Einar Hille, Yale University
T. R. Hollcroft, Wells College
Grace M. Hopper, Vassar College
W. A. Hurwitz, Cornell University

Dunham Jackson, University of Minnesota
Fritz John, University of Kentucky
R. A. Johnson, Brooklyn College
B. W. Jones, Cornell University
Harris Jones, U. S. Military Academy
Margaret E. Jones, Ohio State University

Aida Kalish, Columbia University
Wilfred Kaplan, University of Michigan
Edward Kasner, Columbia University
L. M. Kelly, U. S. Coast Guard Academy
J. R. Kline, University of Pennsylvania
Maurice Kraitchik, New School for Social Research

A. E. Landry, Catholic University of America
R. E. Langer, University of Wisconsin
Gillie A. Larew, Randolph-Macon Woman's College
Solomon Lefschetz, Princeton University
C. I. Lubin, University of Cincinnati

C. C. MacDuffee, Hunter College
Saunders Mac Lane, Harvard University
H. F. Mac Neish, Brooklyn College
N. H. McCoy, Smith College
Morris Marden, University of Wisconsin at Milwaukee
D. May Hickey Maria, Brooklyn College
Margaret P. Martin, Columbia University
A. E. Meder, Jr., New Jersey College for Women
F. H. Miller, Cooper Union
E. B. Mode, Boston University
E. C. Molina, Bell Telephone Laboratories
Deane Montgomery, Smith College
C. N. Moore, University of Cincinnati
Eugenie M. Morenus, Sweet Briar College
Richard Morris, Rutgers University
D. S. Morse, Union College
Marston Morse, Institute for Advanced Study
F. C. Mosteller, Princeton University

Abba V. Newton, Hartwick College

E. G. Olds, Carnegie Institute of Technology
Oystein Ore, Yale University

Gordon Pall, McGill University
C. R. Phelps, U. S. Naval Academy
A. E. Pitcher, Lehigh University
G. B. Price, University of Kansas

S. E. Rasor, Ohio State University
L. L. Rauch, Princeton University
Mina S. Rees, Hunter College
C. F. Rehberg, New York University
W. T. Reid, University of Chicago
R. G. D. Richardson, Brown University
J. F. Ritt, Columbia University
H. A. Robinson, U. S. Military Academy
Selby Robinson, College of the City of New York
R. E. Root, U. S. Naval Academy

Arthur Sard, Queens College
Henry Scheffé, Princeton University
Abraham Schwartz, Pennsylvania State College
A. J. Smith, Philadelphia, Pa.
Virgil Snyder, Cornell University
Vivian E. Spencer, U. S. Department of Commerce
Abraham Spitzbart, University of Minnesota
Otto Szász, University of Cincinnati
Gabor Szegö, Stanford University

J. D. Tamarkin, Brown University
C. J. Thorne, University of Michigan
L. V. Toralballa, Michigan State College
A. W. Tucker, Princeton University
J. W. Tukey, Princeton University

R. M. Walter, New Jersey College for Women
Louis Weisner, Hunter College
Mary Evelyn Wells, Vassar College
E. T. Welmers, Michigan State College
G. T. Whyburn, University of Virginia
R. L. Wilder, University of Michigan
S. S. Wilks, Princeton University
Clement Winston, Office of Price Administration
Jack Wolfe, Brooklyn College
Euphemia R. Worthington, University of California at Los Angeles

R. C. Yates, U. S. Military Academy
C. H. Yeaton, Oberlin College

The mathematicians found it very convenient to have dormitory rooms, social rooms, and the dining room in Main Building of Vassar College, with only a short distance to the lecture rooms in Rockefeller Hall. The social parlors were continually in use throughout the week. On the opening afternoon a tea was given by the Department of Mathematics in the Aula; a notably large number were present at the very outset of the meetings. At eight-thirty on Wednesday evening a delightful concert was given in Skinner Recital Hall by members of the faculty of the Department of Music.

One hundred sixty-two shared in the joint dinner on Wednesday evening. The toastmaster, Professor H. S. White, introduced President McCracken who welcomed the mathematicians as he had done nineteen years before on the occasion of the previous Vassar meeting, and he gave an interesting characterization of the student body of Vassar College. As the chief speaker of the evening, Professor Langer described the distinct change of mind that has occurred in the past year in regard to mathematics. Instead of strong objections from high school educators to college requirements in preparatory mathematics, we see a strong support of mathematical training and a recognition of its great advantages, a greater responsiveness and greater appreciation on the part of the students. There is now presented to all of us a great challenge, whether we are adapting mathematics to war efforts or are continuing to teach the mathematical material which is needful in the present emergency. And we face an even greater challenge in carrying on our courses in advanced mathematics for the sake of the more distant future. On motion of Professor Reid a resolution was adopted by a rising vote thanking the administration of Vassar College and the mathematics staff for their hearty welcome, their many provisions of the facilities and conveniences which go so far toward making a success of our meetings.

The American Mathematical Society held sessions beginning Tuesday forenoon and continuing through Thursday forenoon. Professor R. L. Wilder gave four lectures on "Topology of manifolds" as the twenty-fourth Colloquium. On Wednesday afternoon Professor W. L. Ayres gave an invited lecture on "Transformations with periodic properties."

The Institute of Mathematical Statistics held sessions for the reading of papers on Tuesday afternoon and Wednesday forenoon. On Wednesday afternoon a joint session with the Society was held on the general topic "The applicability of mathematical statistics to war efforts."

The Mathematical Association held sessions on Monday morning and afternoon. The thanks of the Association are due the program committee, which consisted of Professors Jewell Hughes Bushey, Philip Franklin, and R. E. Langer, chairman. The program follows, together with abstracts of some of the papers numbered in accordance with their place on the program.

First Session of the Association

1. "The nature of the applications of mathematics in meteorology" by Professor Bernard Haurwitz, Massachusetts Institute of Technology.
2. "A method in cryptography" by Professor L. S. Hill, Hunter College.
3. "The Postgraduate School of Mathematics and Mechanics at Annapolis" by Professor R. E. Root, U. S. Naval Academy.

1. Professor Haurwitz's paper will appear in an early issue of the Monthly.

3. Certain significant events in the history of the Naval Postgraduate School at Annapolis were briefly discussed by Professor Root in explaining the

development of mathematics and mechanics in the curricula of the school. From the beginning in 1909 and the layout of a single one-year curriculum for mechanical and electrical engineers by the first three professors in 1913–14, main features were: (a) investigation and recommendations by a committee of prominent engineering educators, appointed by the President of the Society for Promotion of Engineering Education, in 1916, at the request of the late Rear Admiral John Halligan, then Head of the School; (b) reorganization of the school after the war, in 1919, under Captain E. J. King, with one year curricula in mechanical, electrical, radio and aeronautical engineering and in ordnance, naval construction and civil engineering; (c) a transition, completed about 1933, to curricula covering two years at Annapolis, the first year given largely to non-technical "general line" subjects, but including five hours per week of mathematics and mechanics for pre-technical groups.

Emphasis was placed on the cooperation of civilian institutions and on the arrangements with certain schools to which special groups may be sent for specialized courses in the second or third year.

Just before the present national emergency there were five major curricula each running through two years at Annapolis and one year at another institution. Each had a general objective, and each course had its specific objective, the extent, content, and treatment of the work in each subject being a matter of special planning in relation to these objectives. With the increased diversity of subject matter in the several curricula, naval engineering, radio engineering, ordnance engineering, aeronautical engineering, and aerological engineering, the professors in mathematics and mechanics developed special fields of interest, each major sequence of courses in a curriculum being the particular responsibility of some one professor.

Under present war conditions the time schedules are revised, non-technical courses are dropped, and the time of each curriculum reduced as much as possible without sacrificing the general objective. New curricula are also established for the training of new reserve officers, graduates in engineering, for certain naval specialties.

Second Session of the Association

1. "The mathematical consultant, past, present and future" by Professor Walter Bartky, University of Chicago.

2. "Applied mathematics" by Professor H. B. Phillips, Massachusetts Institute of Technology.

Meeting of the Board of Governors

Nine members of the Board were present at the meeting Monday evening, including four regional governors.

The following twenty-eight persons were elected to membership on applications duly certified:

H. W. BECKER. Electrician, Mare Island Navy Yard, Vallejo, Calif.
F. C. BIESELE, Ph.D.(Texas) Instr., Univ. of Utah, Salt Lake City, Utah
B. K. BROWN, A.M.(Colorado) Instr., Colorado School of Mines, Golden, Colo.
RAPHAEL CERINO, A.B.(Brooklyn) Apprentice Shipfitter, Navy Yard, New York, N. Y.
E. P. COLEMAN, M.S.(Iowa) Capt., Instr., U. S. Military Acad., West Point, N. Y.
H. D. COLSON, A.B.(Minnesota) Teaching asst., Univ. of Minnesota, Minneapolis, Minn.
W. H. COULTER, Licentiate of instruction (Nashville Nor. Coll.) Retired, Railway mail clerk-in-charge, Decatur, Ill.
F. T. FRANK, A.B.(Stanford) Engineer, Board of Fire Underwriters of the Pacific, San Francisco, Calif.
W. J. FRY, M.S.(Pennsylvania State) Sound Division, Naval Research Lab., Washington, D. C.
MARJORIE J. GROVES, A.M.(Chicago) Librarian, Eckhart Library, Univ. of Chicago, Chicago, Ill.
H. T. GUARD, M.S.(Colorado) Instr., Colorado State Coll. of A. and M.A., Fort Collins, Colo.
M. H. HEINS, Ph.D.(Harvard) Asst. Prof., Illinois Inst. of Tech., Chicago, Ill.
ADA F. JOHNSON, Ph.D.(Minnesota) Prof., Math. and Physics, Rockford Coll., Rockford, Ill.
ERNEST JOHNSTON, M.S.(Illinois) Instr., Austin Junior Coll., Austin, Minn.
AIDA KALISH, A.B. (Brooklyn) Grad. student, Columbia Univ., New York, N. Y.
L. N. KALLFELZ, B.S. in Educ.(Syracuse) Teacher, The Peddie School, Hightstown, N. J.
L. C. KNIGHT, Jr., A.M.(Kent) Instr., Northeastern Oklahoma Junior Coll., Miami, Okla.
H. E. NELSON, Ph.M.(Wisconsin) Instr., Gustavus Adolphus Coll., St. Peter, Minn.
H. C. PARRISH, M.S.(N. Texas St. T. C.) Asst., Ohio State Univ., Columbus, Ohio
A. M. PEISER, A.M.(Cornell) Instr., Cornell Univ., Ithaca, N. Y.
L. L. RAUCH, A.B.(Southern California) Grad. student, Princeton Univ., Princeton, N. J.
C. F. REHBERG, A.M.(Columbia), M.E.E.(New York Univ.) Instr., Elec. Eng., New York Univ., New York, N. Y.
Sister M. ROSALIN SCHAEFFER, A.M.(Catholic Univ.) Instr., Ursuline Coll., Louisville, Ky.
A. J. SMITH, Ph.D.(Pennsylvania) Philadelphia, Pa.
R. S. SPENCER, M.S.(Michigan) Physicist, Dow Chemical Co., Midland, Mich.
C. J. THORNE, Ph.D.(Iowa State) Instr., Univ. of Michigan, Ann Arbor, Mich.
H. W. WILLIAMS, A.M.(Missouri) Asst. Prof., Colorado State Coll. of A. and M.A., Fort Collins, Colo.
Sister GERTRUDE MARIE ZIEROFF, M.S.(St. Louis Univ.) Instr., Marian Coll., Indianapolis, Ind.

President Marston Morse of the Society addressed the Board with reference to requesting deferment for those students of mathematics who (1) give promise of becoming research scientists, (2) plan to become college teachers and are competent persons, or (3) plan to be trained for high school teaching. The members of the Board agreed in the judgment that we might well ask deferment for the first two groups but not for the third even in states where there is a shortage of high school teachers.

On recommendation of the Executive Committee it was voted, among other measures, (1) to postpone the Putnam Competition for the present, as already announced; (2) to extend the terms of office of regional governors so as to terminate on July 1 instead of at the time of the annual meeting, the regional governors thereby having a better opportunity to acquaint themselves with the duties of the office before the annual meeting; (3) to appropriate $400 from the

1940 and 1941 income from the Chace Fund for aid in the publication by the American Oriental Society of a volume on Babylonian mathematical tablets by Professor Otto Neugebauer and Doctor A. J. Sachs; (4) to participate formally in symposia Wednesday, December 30, as planned by the Secretaries of Sections A and L, in observance of the 300th anniversary of the death of Galileo and the birth of Newton; (5) to make an appropriation of $400 in 1943 toward the expense of printing and distributing the National Mathematics Magazine for 1942–43, either as a provision in the 1943 budget or from one of the special funds.

On recommendation of the Finance Committee it was voted (1) to continue to employ the Cleveland Trust Company as our financial adviser for 1943; (2) to utilize the appropriation of $200 in the 1942 budget, together with any necessary addition, for the expenses of the regional governors to the 1942 annual meeting, to the extent of one-third of the first-class railroad fare to and from the meeting.

The Board adopted formal resolutions empowering the Cleveland Trust Company to sell certain registered bonds for the sake of a desirable reinvestment and empowering the Cleveland Trust Company to collect interest coupons.

Because of the war conditions it was voted to recall our acceptance of the invitation to meet at Boulder in the summer of 1943, and to express our hope that we may meet there when normal times return.

Miss Marjorie J. Groves was appointed an associate editor for 1942.

W. D. Cairns, *Secretary-Treasurer*

NINETEENTH ANNUAL MEETING OF THE INDIANA SECTION

The nineteenth annual meeting of the Indiana Section of the Mathematical Association of America was held Friday and Saturday, April 24 and 25, 1942, at Wabash College, Crawfordsville, Indiana.

Sixty registered at the meetings, including the following thirty-three members of the Association: W. C. Arnold, Emil Artin, W. L. Ayres, Juna L. Beal, L. G. Black, I. W. Burr, G. E. Carscallen, K. W. Crain, W. E. Edington, P. D. Edwards, B. C. Getchell, E. L. Godfrey, G. H. Graves, H. E. H. Greenleaf, C. T. Hazard, Cora B. Hennel, F. H. Hodge, H. K. Hughes, M. W. Keller, W. C. Krathwohl, Cornelius Lanczos, Karl Menger, G. T. Miller, Paul Muehlman, P. M. Pepper, J. C. Polley, C. K. Robbins, L. S. Shively, D. R. Shreve, M. S. Webster, F. J. Weyl, K. P. Williams, H. E. Wolfe.

At the business meeting on Saturday the following officers were elected for next year: Chairman, J. C. Polley, Wabash College; Vice-Chairman, P. M. Pepper, Notre Dame University; Secretary, M. W. Keller, Purdue University. The twentieth annual meeting will be held April 9 and 10, 1943, at Notre Dame University.

At the annual dinner on Friday evening the chairman, Professor P. D. Edwards of Ball State Teachers College, acted as toastmaster and introduced Dr. F. Sparks, President of Wabash College, who welcomed the visitors.

Following the dinner the first session of the Section was held with Professor H. T. Davis of Northwestern University as guest speaker. His subject was "Dinner with Archimedes." Professor Davis invited the audience to have dinner with him at the request of King Ptolemy Philadelphus, the second ruler of Alexandria. The dinner was in honor of Archimedes, a distinguished visitor from Syracuse. At the dinner the audience met the various guests King Ptolemy had invited. The amazingly modern work of the Alexandrian Museum of this golden period was revealed in the conversations between these people, and the court of Ptolemy Philadelphus was shown to equal both in luxury and learning that of any in more modern history.

Saturday morning Professor Davis gave a second lecture. His subject was "A mathematical theory of income and its consequences." He considered the problem of representing mathematically the frequency function which describes the distribution of the national income among income recipients. The distribution is characterized by an abnormally large standard deviation, and by the fact that the modal income is very close to the wolf-point, that is to say, the income of subsistence level. For large incomes the distribution must give asymptotically the Pareto law, which asserts that in normal economies the distribution of income is represented by the formula, $y=ax^{-v}$, where y is the number of people having the income x or greater, and v is approximately 1.5. The function which most satisfactorily describes the distribution is given by

$$\phi(x) = \frac{a}{z^n} \cdot \frac{1}{e^{b/z} - 1}$$

where $z=x-c$, $n-v=2$, c is the wolf point, and $\phi(x)$ is the number of people with incomes between x and $x+dx$.

National industrial production, P, represented by the Douglass-Cobb formula, $P=AL^pC^q$, $p+q=1$, where L measures labor and C measures capital, is found to be a function of the concentration of wealth, represented by the ratio, $p=1/(2v-1)$. The relationship between industrial production and the distribution of income is thus exhibited, and certain consequences derived.

At the two sessions on Saturday the following seven papers were presented:

1. "The work of the Indiana Section" by Professor P. D. Edwards, Ball State Teachers College, retiring chairman of the Indiana Section.

2. "On the curvature of surfaces" by Professor Karl Menger, Notre Dame University.

3. "On the value distribution of meromorphic functions" by Dr. F. J. Weyl, Indiana University.

4. "Results of a diagnostic testing and remedial teaching program" by Dr. D. R. Shreve and Dr. M. W. Keller, Purdue University.

5. "Linear and almost linear sets" by Professor P. M. Pepper, Notre Dame University.

6. "Remarks on a problem of Kakeya" by Dr. J. W. T. Youngs, Purdue University, introduced by Professor Ayres.

7. "On curves in 3-space" by Dr. Peter Scherk, Indiana University, introduced by Professor Williams.

Abstracts of papers follow.

1. Professor Edwards gave a summary of the mathematical work presented at the meetings of the Indiana Section of the Association since its organization in 1924. A statistical summary of papers presented, attendance, etc., was included. The influence of the Indiana Section as an organization was noted and attention was called to needs of the immediate future that must be met by the members.

2. Professor Menger showed how the curvature of a curve C at a point P may be defined by a direct limit process, *viz.*, as the reciprocal of the limit of the radii of circumcircles formed for triples of points of C converging towards P. This definition is exclusively based on the distance between the points of C and, hence, is applicable to any curve contained in a general metric space in the sense of Frechét. In an analogous way, the Gauss curvature of a surface S at a point P may be defined by considering quadruples of points of S converging towards P. However, instead of the circumsphere of a quadruple of points, one has to study the radius of a sphere containing four points whose six distances are respectively equal to those between the points of the quadruple. By a sphere of positive, infinite or negative radius we mean an ordinary sphere in which the distance of two points is the length of the minor arc of the great circle joining the two points, or the euclidean plane, or the hyperbolic plane, respectively.

3. Dr. Weyl's report dealt with R. Nevanlinna's defect relation for meromorphic functions. As the sharpest and most natural generalization of E. Picard's classical theorem that a meromorphic function cannot leave out more than two values, this relation is sufficiently attractive to warrant the search for a proof of greatest possible simplicity. The one presented by Dr. Weyl was based upon ideas of L. V. Ahlfors and might well be considered simple enough to make Nevanlinna's result a possible candidate for inclusion in any graduate course on Complex Variables.

4. Dr. Shreve gave a report on a three-year experiment in the teaching of trigonometry and elementary algebra to freshmen students of engineering in Purdue University. The control group was taught in the usual manner. The experimental group was given seven lessons in review of elementary algebra preceding the study of trigonometry. In teaching the experimental group the concepts and skills to be given particular emphasis were determined from an error analysis of the difficulties which the students in the control group had encountered. He reported that by this procedure it had been possible to reduce the number of failures by more than thirty-five per cent and increase the number

of A's by fifty per cent although the control group was initially slightly superior in ability as measured by preliminary tests.

5. Professor Pepper stated that Menger had shown each semimetric space of five or more points all of whose triples are linear is linear and the existence of four point non-linear sets whose triples are all linear. Professor Pepper made a survey for each k of all $(5+k)$-point semimetric spaces with exactly $k+1$ non-linear triples and showed that each semimetric space of $5+k$ points with at most k non-linear triples is actually linear. He also showed that in any non-linear semimetric space of more than four points at most two points can escape lying in at least one non-linear triple. He gave the following corollary to this: Each non-linear semimetric space which has non-denumerably many points must have non-denumerably many non-linear triples. Congruent imbeddings into function space were given for some of those non-linear spaces which are metric.

6. Dr. Youngs presented an outline of the history of a celebrated problem of Kakeya with a sketch of the elegant solution by Perron.

7. Dr. Scherk showed how some concepts of algebraic geometry can be translated to real non-analytical curves which are assumed to have tangents and osculating planes everywhere. Thus, the local behavior of such a curve can be described by means of a set of three numbers that is the precise generalization of the characteristic (mod 2) of algebraic curves. Since the points of a non-analytic curve are bound together by no tie, one has introduced beside the order, rank, class in the large of such a curve (equal number respectively, of points of the curve in a plane, tangents through a straight line, osculating planes through a point) corresponding local concepts. Assuming reasonable smoothness, he proved that the local order, class, and rank can be expressed through the characteristic in a simple way, and that, especially, the first two are equal. This last result followed at once from the deeper theorem that the first and last of the characteristic numbers are dual to one another, while the second is self-dual.

M. W. KELLER, *Secretary*

THE TWENTY-THIRD ANNUAL MEETING OF THE ILLINOIS SECTION

The twenty-third annual meeting of the Illinois Section of the Mathematical Association of America was held at James Millikin University, Decatur, Illinois, on Friday and Saturday, May 8 and 9, 1942. Professor R. N. Johanson, chairman of the Section, presided at all sessions.

The attendance at the sessions was approximately forty-five, including the following twenty-eight members of the Association: Beulah Armstrong, Edith I. Atkin, S. F. Bibb, O. K. Bower, Laura E. Christman, W. H. Coulter, D. R. Curtiss, J. E. Davis, W. W. Denton, Elinor B. Flagg, A. E. Gault, B. H.

Gere, G. D. Gore, M. R. Hestenes, Mildred Hunt, R. N. Johanson, E. C. Kiefer, J. M. Kinney, W. C. Krathwohl, H. J. Miles, C. N. Mills, G. E. Moore, E. J. Moulton, Margaret Olmsted, F. C. W. Olson, E. W. Ploenges, Ruth B. Rasmusen, E. H. Taylor.

At the annual business meeting the following officers of the Section were elected: Chairman, E. W. Ploenges, James Millikin University; Vice-Chairman, C. N. Mills, Illinois State Normal University; Secretary, E. C. Kiefer, James Millikin University. The members of the Section voted to join with the Indiana and the Michigan Sections in a joint meeting in 1943 to be held at Notre Dame University, the details of this meeting to be announced early next spring. The next regular meeting of the Illinois Section will be held in 1944 at Illinois State Normal University, Normal, Illinois.

The following twelve papers were presented:

1. "Mathematics for the consumer" by Laura E. Christman, Senn High School, Chicago.

2. "The construction and use of a mathematics placement test" by Dr. B. H. Gere, Herzl Junior College, Chicago.

3. "Determinant theory without the use of inversions" by Dr. I. E. Perlin, Illinois Institute of Technology.

4. "Trigonometry for the Navy V-7 program" by Professor G. E. Moore, University of Illinois.

5. "Teaching college geometry from the teacher-training point of view" by Professor C. N. Mills, Illinois State Normal University.

6. "Determinants and Taylor's Theorem" by Dr. Bernard Friedman, Woodrow Wilson Junior College, introduced by the Secretary.

7. "Report of meetings of Board of Governors" by Professor W. C. Krathwohl, Illinois Institute of Technology.

8. "Mathematics and war" by Professor E. J. Moulton, Northwestern University.

9. "A page of vector calculus for sophomores" by Professor G. D. Gore, Central Y.M.C.A. College, Chicago.

10. "Critical points of functions" by Professor M. R. Hestenes, University of Chicago.

11. "Mathematics in the canning industry" by F. C. W. Olson, American Can Company, Maywood, Illinois.

12. "An analogue of Pascal's arithmetical triangle" by Professor S. F. Bibb, Illinois Institute of Technology.

Abstracts of some of the papers follow:

1. High school mathematics may benefit three groups of future citizens: members of professions, members of skilled trades, and consumers. The classical development of high school mathematics aims to help the future member of a profession, essential mathematics and shop mathematics do the same thing for the skilled worker (as far as we can start to grade such needs at this early date) but the consumer is seldom considered in our mathematics program.

Senn High School, Miss Christman stated, is offering a course called "Mathematics for the Consumer" based on the text of the same name by Anna Louise Cowan, published by Stackpole Sons. Decimals and percentage are the mathematical background of the course. Some pupils continue into solid geometry from this course.

2. Dr. Gere outlined a procedure for constructing a placement test. The labor involved in the construction is small but the results are satisfactory for many placement problems. A number of results obtained from the use of a test actually constructed according to this procedure were presented.

4. A three semester hour course designed for men near graduation, with no college mathematics, except perhaps algebra, was described by Professor Moore to show how the University of Illinois meets the needs of the Navy V-7 program. Special emphasis is placed upon computation; half of the course deals with the trigonometry of the earth and the celestrial sphere. With the cooperation of Professor R. H. Baker of the Department of Astronomy students are given lectures, outside class time, on celestial coördinates, time (siderial, solar, mean sun, civil), the sextant and its use, star charts, etc.

5. Professor Mills stated that the golden thread which binds many of the different topics in college geometry is "Harmonic Ratio," saying that by means of analytical relations between the various topics usually considered, the student is given a broader point of view of college geometry and projective geometry. Paper folding exercises and properly designed construction plates afford an interesting approach to the general theorems.

6. Dr. Friedman presented a new method for obtaining old and well-known results. If a determinant is considered as a function of any set of its elements and then Taylor's Theorem is applied to this function, the determinant can be expanded into a sum of terms depending upon the particular elements chosen. In this way, the expansion by minors, the characteristic equation of a determinant, Laplace's expansion, Cauchy's expansion and Cayley's expansion (see Muir and Metzler's Theory of Determinants) can be quickly and conveniently found.

9. Professor Gore adapts to several kinds of motion in the plane the derivative of a vector with respect to a scalar. The object is to give to students of the sophomore levels of calculus, mechanics, and engineering kinematics a common language in which to study motion; and to give them greater facility with such entities as displacement, velocity, and acceleration than has been attained by methods that employ only the usual Cartesian and polar coördinates. The simplification of these subjects, which is achieved by a modicum of vector calculus, seems to warrant the introduction of the concept of the vector derivative at an earlier stage than is now customary in mathematical education.

10. Professor Hestenes considered the historical development of the theory of critical points. He made a survey of various definitions of critical points and their indices, and discussed their relation to restricted maxima and minima, considering these topics from an analytic and a topological point of view. He

pointed out how these results can be extended to obtain similar results in the calculus of variations.

11. Mr. Olson discussed some of the mathematical problems involved in determining the proper processing time and temperature to sterilize canned foods. Bacterial death rates as a function of the temperature are combined with the heating equation to form a criterion of sterility whose solution, although of formidable complexity, has been successfully accomplished by tabulation of auxiliary functions, and more recently by nomograms. Fundamental studies of the properties of the heating equation and its solutions have materially increased the usefulness and scope of mathematics as applied to canning problems.

12. Professor Bibb showed how $y_n=f(x)$, obtained by eliminating the parameter t from the pair of equations $x=t+t^{-1}$, $y_n=t^n+t^{-n}$, $[t\neq 0,\ n=1, 2, 3, \cdots,]$, might be written as $\Sigma_{s=0}^{r}(-1)^s({}_nD_s)x^{n-2s}$, where the ${}_nD_s$ are determinants with elements of the form ${}_nC_p$. He then pointed out the coefficients, ${}_nD_s$, of the polynomials for $n=1, 2, 3, \cdots$, could be arranged as a triangle analogous to that of Pascal.

C. N. MILLS, *Secretary*

THE SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The Spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Randolph-Macon College at Ashland, Virginia, on Saturday, May 2, 1942, with a morning session, luncheon, and afternoon session. Professor E. J. McShane, chairman of the Section, presided at the sessions.

The attendance was twenty-eight including the following sixteen members of the Association: M. W. Aylor, C. C. Bramble, R. E. Gaines, Isabel Harris, G. A. Hedlund, Evelyn M. Kennedy, A. E. Landry, E. J. McShane, P. W. A. Raine, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, R. E. Root, T. McN. Simpson, Jr., C. H. Wheeler III, G. T. Whyburn.

At the invitation of the Section, Dr. G. A. Hedlund of the University of Virginia gave an address on "Symbolic dynamics and topological transformations." A motion was passed expressing the appreciation of the Section to the authorities of Randolph-Macon College for their generous hospitality. The following officers for the ensuing year were elected: Chairman, J. H. Taylor, George Washington University; Secretary, W. K. Morrill, Johns Hopkins University; Members of the Executive Committee, G. A. Hedlund, University of Virginia, O. J. Ramler, Catholic University. The following invitations were accepted for future meetings: Loyola College, Baltimore, Fall meeting, 1942; Johns Hopkins University, Baltimore, Spring meeting, 1943; Trinity College, Washington, Fall meeting, 1943.

After an address of welcome by Dr. J. E. Moreland, President of Randolph-Macon College, the following papers were read:

1. "Quadratic and cubic equations with complex coefficients whose roots have unit modulus" by Professor O. J. Ramler, Catholic University of America.

2. "An application of the calculus of variations to a problem in mechanics" by W. A. Blankinship, University of Virginia, introduced by Professor Whyburn.

3. "Linear velocity fields in a baratropic atmosphere" by Professor R. E. Root, United States Naval Academy.

After these papers there was open discussion on the teaching of college mathematics.

4. "Symbolic dynamics and topological transformations" by Professor G. A. Hedlund, University of Virginia.

Abstracts of the papers follow:

1. Professor Ramler showed that the necessary and sufficient conditions for the roots of the quadratic $z^2+pz+q=0$ to have unit modulus are $p/\bar{p}=q$ and $|p| \leqq z$. When $p/\bar{p}=q$ and $|p|>2$ the roots are inverse points with respect to the unit circle in the Argand diagram. He also showed that the necessary conditions for the roots of the cubic $z^3+pz^2+qz+r=0$ to lie on the unit circle are $|p|=|q|$ and $pq/\bar{p}\bar{q}=r^2$. It was also pointed out that when these conditions are satisfied the roots of the Hessian of the cubic are either on the unit circle or inverse points with respect to it.

2. Mr. Blankinship discussed the problem: "To determine the shape that a rod of uniform cross-section and elasticity will asssume if forced to pass freely through the three points, $(a, 0)$, $(-a, 0)$, and $(0, b)$," He set it up as a Lagrange problem and obtained an explicit solution in terms of elliptic integrals.

3. Dr. Root stated the general equations of motion relative to a system of axes fixed to the moving earth and discussed some of their general implications. Horizontal motion in which the velocity components are linear functions of displacement coordinates were considered in relation to the requirements of the equations of motion.

4. Dr. Hedlund stated that the methods and examples of symbolic dynamics can be applied to the construction of topological transformations on compact metric spaces. He showed that it was relatively simple to obtain an example which displays most of the properties of the geodesic flow on a closed surface of negative curvature in that there exist transitive orbits, the periodic orbits form an everywhere dense set, and there is a continuum of orbits asymptotic to any given orbit. With the aid of known examples of non-periodic recurrent symbolic trajectories, some of the possibilities in the behavior of non-regular minimal sets can be explored.

C. H. WHEELER III, *Secretary*

NOTE ON AUTOPOLAR SURFACES*

MALCOLM FOSTER, Wesleyan University

1. Introduction. The purpose of this paper is to discuss those surfaces which are autopolar with respect to the paraboloid $2\zeta=\xi^2+\eta^2$. The method for the determination of these surfaces is essentially the same as that used by the author in a recent paper in which a study was made of curves autopolar with respect to the parabola $2\eta=\xi^2$.† These autopolar surfaces are considered as special solutions of those partial differential equations which are invariant under the dual transformation for which the above paraboloid is the quadric of reference. It will be obvious that this method may be readily modified for the study of surfaces autopolar with respect to any given quadric.

2. The dual transformation for $2\zeta=\xi^2+\eta^2$. When this paraboloid is taken as the quadric of reference, the equations for the dual transformation are‡

$$\begin{gathered} x = P, \quad y = Q, \quad z = PX + QY - Z, \quad p = X, \quad q = Y, \\ r = T/RT - S^2, \quad s = - S/RT - S^2, \quad t = R/RT - S^2, \quad etc. \end{gathered} \tag{1}$$

Under this transformation a differential equation

$$f(x, y, z, p, q, r, s, t, \cdots) = 0, \tag{2}$$

becomes

$$f(P, Q, PX + QY - Z, X, Y, \cdots) = 0; \tag{3}$$

if a solution of (3) be denoted by

$$F(X, Y, Z) = 0, \tag{4}$$

the corresponding solution of (2) is found by eliminating X, Y, Z, from (4) and the following three equations,§

$$\begin{gathered} x\frac{\partial F}{\partial Z} + \frac{\partial F}{\partial X} = 0, \qquad y\frac{\partial F}{\partial Z} + \frac{\partial F}{\partial Y} = 0, \\ - z\frac{\partial F}{\partial Z} = Z\frac{\partial F}{\partial Z} + X\frac{\partial F}{\partial X} + Y\frac{\partial F}{\partial Y}\cdot \end{gathered} \tag{5}$$

If this solution of (2) be denoted by

$$\phi(x, y, z) = 0, \tag{6}$$

the surfaces (4) and (6) are polar reciprocals.

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* Presented to the American Mathematical Society, February 22, 1941.

† Malcolm Foster, Note on autopolar curves, Bulletin of the American Mathematical Society, vol. 47, No. 4, April, 1941, pp. 247–253.

‡ A. R. Forsyth, A Treatise on Differential Equations, third edition, pp. 371–375. See also H. T. H. Piaggio, An Elementary Treatise on Differential Equations, 1926, p. 189.

§ Forsyth, op. cit.

When (2) is invariant under (1), the general solution, say,

$$z = \phi(u, v),^* \tag{7}$$

must include pairs of polar reciprocal surfaces; that is, for any choice of ϕ, say $z=\phi_1 (u, v)$, the polar reciprocal surface will be given by some other choice of ϕ, say $z=\phi_2 (u, v)$. The problem here is to determine those functions $\phi (u, v)$ which will define autopolar surfaces, that is, surfaces which are their own polar reciprocals.

3. Condition that $z=f(x, y)$ be autopolar. The equation of the plane which is the polar of any point (x, y, z) on this surface is

$$\xi x + \eta y - \zeta - f(x, y) = 0; \tag{8}$$

this is a two-parameter family, and the equation of the polar reciprocal surface, which is the envelope of the planes (8), will be found by eliminating x and y from (8) and the following equations,

$$\xi - p = 0, \qquad \eta - q = 0. \tag{9}$$

The necessary and sufficient condition that $z=f(x, y)$ be autopolar is that this elimination shall give us the equation $\zeta=f(\xi, \eta)$. Hence from (8) and (9) we get

$$f(x, y) + f(p, q) = px + qy. \tag{10}$$

We have, therefore, the following theorem:

THEOREM 1. *A necessary and sufficient condition that $z=f(x, y)$ be autopolar with respect to the paraboloid $2\zeta=\xi^2+\eta^2$ is that* (10) *be satisfied.*

It is of interest to note that (10) is of the Clairaut type.

Without repeating the argument as given above, we merely quote the following theorem:

THEOREM 2. *A necessary and sufficient condition that a surface $f(x, y, z)=0$ be autopolar with respect to the paraboloid $2\zeta=\xi^2+\eta^2$ is that*

$$f(p, q, px + qy - z) = 0. \tag{11}$$

4. Conjugate pairs. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be a pair of points on any autopolar surface Σ such that the polar of P_1 is tangent to Σ at P_2; then the polar of P_2 is tangent to Σ at P_1. We shall call such a pair of points, a conjugate pair, and say that each is the conjugate of the other. Since the equation of the polar of P_1, $\xi x_1+\eta y_1-\zeta-z_1=0$, must be satisfied by the coordinates of P_2, we have

$$x_1x_2 + y_1y_2 = z_1 + z_2. \tag{12}$$

* Here u and v are certain functions of x and y.

It is obvious from (1) that for a conjugate pair,

$$
(13) \qquad \begin{aligned} &x_1 = p_2, \qquad x_2 = p_1, \qquad y_1 = q_2, \qquad y_2 = q_1, \\ &z_1 = p_2x_2 + q_2y_2 - z_2, \qquad z_2 = p_1x_1 + q_1y_1 - z_1, \qquad \textit{etc.}, \end{aligned}
$$

where, for example, p_2 denotes the value of p at P_2.

5. Self-conjugate points $\overline{P}$. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be any conjugate pair on an autopolar surface Σ, $z=f(x, y)$; we shall also assume that P_1, P_2 lie within a simply connected region of Σ wherein $f(x, y)$ and its first and second partial derivatives are defined. Let C be an arbitrarily selected path on Σ which joins P_1, P_2, and let $P(x, y, z)$ denote a general point on C between P_1 and P_2. The conjugate of $P(x, y, z)$ we shall denote by $P'(x^{(1)}, y^{(1)}, z^{(1)})$. Since P_1, P_2 constitute a conjugate pair, it is obvious that as $P(x, y, z)$ moves along C from P_1 to P_2, its conjugate P' must describe a corresponding path C' from P_2 to P_1.* Hence, as x varies from x_1 to x_2, $x^{(1)}$ varies from x_2 to x_1. There is, therefore, a self corresponding value $x=x^{(1)}$ between x_1 and x_2. Consequently, there exists a point $P(x, y, z)$, the conjugate of $P'(x, y^{(1)}, z^{(1)})$, for which (10) may be written

$$f(x, y) + f(x, y^{(1)}) = x^2 + yy^{(1)},$$

wherein $\partial f(x, y)/\partial x=x$ and $\partial f(x, y)/\partial y=y^{(1)}$. If this equation be differentiated partially with respect to x, we obtain $\partial f(x, y^{(1)})/\partial x=x$, since $\partial f(x, y)/\partial x=x$ at P. Hence, corresponding to an arbitrary surface curve C through P_1, P_2 the points $P(x, y, z)$, $P'(x, y^{(1)}, z^{(1)})$ lie on the curve defined by Σ and $\partial f(x, y)/\partial x=x$. Let us call this curve C''. Hence, as C is varied so that $P(x, y, z)$ tends toward $P'(x, y^{(1)}, z^{(1)})$, P varies along C'', and the point $P'(x, y^{(1)}, z^{(1)})$ approaches $P(x, y, z)$ along the same curve. There is, therefore, a self-conjugate point $\overline{P}$ on the surface Σ somewhere along C''. Consequently we have the following theorem:†

THEOREM 3. *On any autopolar surface there exists at least one self-conjugate point.*

If $\overline{P}(x, y, z)$ be self-conjugate we see from (12) that $2z=x^2+y^2$, and hence $\overline{P}$ must also be on the quadric of reference. Consequently the polar of $\overline{P}$ must be tangent to Σ and to the paraboloid at the same point $\overline{P}$. If there are an infinite number of self-conjugate points on Σ, the quadric of reference and Σ are tangent all along the curve made up of these points.‡ We have, therefore, the following theorem:

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* It is assumed that the paths C and C' lie within the simply connected region on Σ.

† Hereafter we shall make no reference to a particular quadric of reference unless the property of Σ is not invariant under a collineation.

‡ A specific example of such a surface will be given later.

THEOREM 4. *Every autopolar surface is tangent to the quadric of reference at at least one point.*

6. Properties of a conjugate pair.

1. The necessary and sufficient condition that the tangent planes at a conjugate pair be perpendicular is that $p_1p_2+q_1q_2+1=0$; and from (12) and (13) this reduces to

$$z_1 + z_2 = -1. \tag{14}$$

2. For any conjugate pair we have from (1),

$$\begin{aligned} &r_1 = t_2/r_2t_2 - s_2^2, \qquad s_1 = -s_2/r_2t_2 - s_2^2, \qquad t_1 = r_2/r_2t_2 - s_2^2, \\ &r_2 = t_1/r_1t_1 - s_1^2, \quad etc. \end{aligned} \tag{15}$$

Hence $s_1/s_2 = -1/r_2t_2 - s_2^2 = -(r_1t_1 - s_1^2)$, or

$$(r_1t_1 - s_1^2)(r_2t_2 - s_2^2) = 1; \tag{16}$$

that is, at a conjugate pair the values of the function $rt - s^2$ are reciprocals.

3. The equations of the normals to Σ at a conjugate pair are

$$\begin{aligned} \xi + x_2\zeta - x_1 - x_2z_1 = 0, \qquad \xi + x_1\zeta - x_2 - x_1z_2 = 0, \\ \eta + y_2\zeta - y_1 - y_2z_1 = 0, \qquad \eta + y_1\zeta - y_2 - y_1z_2 = 0. \end{aligned} \tag{17}$$

The necessary and sufficient condition that these normals be coplanar is that the determinant of the system in (17) shall vanish. After some reduction this condition becomes

$$(x_1y_2 - x_2y_1)(z_1 - z_2) = 0. \tag{18}$$

4. The equation of the tangent plane to the quadric of reference at the point where $x = x_1$, $y = y_1$ is evidently

$$x_1x + y_1y - z - 1/2(x_1^2 + y_1^2) = 0; \tag{19}$$

and the coordinates of the mid-point of the line which joins a conjugate pair are, on using (12),

$$\left(\frac{x_1 + x_2}{2}, \quad \frac{y_1 + y_2}{2}, \quad \frac{x_1x_2 + y_1y_2}{2}\right).$$

Since these coordinates satisfy (19), we see that the tangent plane to the quadric of reference at the point where $x = x_1$, $y = y_1$, (or $x = x_2$, $y = y_2$), passes through the mid-point of the segment which joins a conjugate pair.

5. Let us consider the Gaussian curvatures at a conjugate pair, P_1, (x_1, y_1, z_1) and $P_2(x_2, y_2, z_2)$. The principal radii of curvature at P_1 and P_2 are the roots of

$$(r_1t_1 - s_1^2)R^2 - \sqrt{1 + x_2^2 + y_2^2}[(1 + x_2^2)t_1 + (1 + y_2^2)r_1 - 2x_2y_2s_1]R + (1 + x_2^2 + y_2^2)^2 = 0,$$

and

$$(r_2t_2 - s_2^2)R^2 - \sqrt{1 + x_1^2 + y_1^2}[(1 + x_1^2)t_2 + (1 + y_1^2)r_2 - 2x_1y_1s_2]R + (1 + x_1^2 + y_1^2)^2 = 0,$$

respectively. Hence the Gaussian curvatures at P_1 and P_2 are

$$K_1 = (r_1t_1 - s_1^2)/(1 + x_2^2 + y_2^2)^2, \qquad K_2 = (r_2t_2 - s^2)/(1 + x_1^2 + y_1^2)^2; \tag{20}$$

and on using (16), the product of the Gaussian curvatures at a conjugate pair is

$$K_1K_2 = 1/(1 + x_1^2 + y_1^2)^2(1 + x_2^2 + y_2^2)^2. \tag{21}$$

It is readily seen that this is identical with the product of the Gaussian curvatures of the quadric of reference at those points whose x- and y-coordinates are (x_1, y_1) and (x_2, y_2).

6. For any conjugate pair we also have*

$$\begin{aligned} r_1r_2 + s_1s_2 &= 1, & r_2s_1 + s_2t_1 &= 0, \\ r_1s_2 + s_1t_2 &= 0, & s_1s_2 + t_1t_2 &= 1. \end{aligned} \tag{22}$$

7. Properties of self-conjugate points.

1. From (22) it is evident that at all self-conjugate points $\overline{P}$,

$$r^2 + s^2 = 1, \qquad s(r + t) = 0, \qquad s^2 + t^2 = 1; \tag{23}$$

hence at $\overline{P}$, either $s=0$ or $r=-t$. If $s=0$, it follows that $r=\pm 1$, $t=\pm 1$. Also, from (16), $rt-s^2=\pm 1$ at all self-conjugate points.

2. We have at once the following theorem from §6, 5:

THEOREM 5. *For any autopolar surface the Gaussian curvature at a self-conjugate point is the same, except for sign, as the Gaussian curvature at this point of the quadric of reference.*†

3. In §8, in which we shall give several examples of autopolar surfaces, it will be shown that for many of these surfaces the principal radii of curvature at a self-conjugate point are the same, except for sign, as the principal radii of curvature of the quadric of reference; and at these points the lines of curvature of the two surfaces are sometimes, but not always in tangency.

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* Goursat-Hedrick, Mathematical Analysis, vol. 1, p. 78.

† Cf. R. Mehmke, Einige Sätze über die räumliche Collineation und Affinität, welche sich auf die Krümmung von Kurven und Flächen beziehen, (Schlömilchs Zeitschrift, 36, 1891, pp. 56–60); also, R. Mehmke, Über zwei die Krümmung von Curven und das Gauss'sche Krümmungsmass von Flächen betreffende characteristische Eigenschaften der linearen Punkttransformationen (Ibid., 36, 1891, pp. 206–213)

8. Examples of autopolar surfaces.

Example 1. Let us consider the partial differential equation $pq+xy+py+qx=1$, invariant under (1), and a solution of which is

$$z = c_1x - \frac{x^2}{2} + \frac{y}{c_1} - \frac{y^2}{2} + c_2. \tag{24}$$

We ask: Are there any members of (24) which are autopolar?

Let us apply the test (10). We find $p=c_1-x$, $q=1/c_1-y$, and these values in (10) satisfy this relation identically, provided $c_2=-1/4c_1^2-c_1^2/4$. Hence within the two-parameter family (24) there exists the following one-parameter family of autopolar surfaces:

$$z = c_1x - \frac{x^2}{2} + \frac{y}{c_1} - \frac{y^2}{2} - \frac{1}{4c_1^2} - \frac{c_1^2}{4}. \tag{25}$$

As indicated in Theorem 4, the quadric of reference is the envelope of this family.

It is readily verified that at $\overline{P}(c_1/2,\ 1/2c_1,\ c_1^4+1/8c_1^2)$, the only self-conjugate point on any member of (25), the principal radii of curvature are the same, except for sign, as the principal radii at this point for the quadric of reference.

From the equation of the lines of curvature,

$$\frac{(1+p^2)dx+pqdy}{rdx+sdy} = \frac{pqdx+(1+q^2)dy}{sdx+tdy}, \tag{26}$$

it is also readily shown that the lines of curvature on the two surfaces are in tangency at $\overline{P}$. This is also evident from the fact that the quadric of reference and any member of (25) are paraboloids of revolution.

Example 2. $px-qy=0$. The general solution of this equation is

$$z = F(xy); \tag{27}$$

and our problem is to determine what functions $F(xy)$ define autopolar surfaces. This is wholly a matter of testing one function after another in (10).

If we test $z=xy+c$ in (10) we find that this condition of autopolarity is satisfied when $c=0$. Hence $z=xy$ is autopolar. In a like manner we find that $z=\log(xy)+1$ is autopolar.

Let us consider the hyperbolic paraboloid $z=xy$. It is tangent to the quadric of reference along the curve $z=xy$, $x=y$, and every point on this curve is a self-conjugate point. Moreover, the principal radii of the quadric and of $z=xy$ are for *any* self-conjugate point, the same except for sign.

We also find that for *any* self-conjugate point (26) reduces to $dy^2-dx^2=0$ for each surface; hence the lines of curvature are in tangency at *every* point of contact of the two surfaces.

The surface $z=\log(xy)+1$ has but two self-conjugate points, (1, 1, 1) and (−1, −1, 1). The lines of curvature of the two surfaces are in tangency at each

of the above points, and the principal radii of the two surfaces are numerically equal.

No doubt there are many other choices of $F(xy)$ which in (27) define autopolar surfaces.

Example 3. $px=1$. The general solution of this invariant equation is

$$z = \log x + F(y). \tag{28}$$

If we take $F(y)=c_1y^2+c_2$, and test (28) in (10), we see that the condition of autopolarity is satisfied if $c_1=\pm 1/2$, $c_2=1/2$. Let us therefore consider the autopolar surface

$$z = \log x + y^2/2 + 1/2. \tag{29}$$

This surface is tangent to the quadric of reference all along the curve in which each surface is cut by the plane $x=1$; and every point on this curve is a self-conjugate point. In this case, however, the principal radii of the quadric and the autopolar surface are *not* numerically equal at all self-conjugate points. It is readily shown that the principal radii are numerically equal *only* for the self-conjugate point $(1, 0, 1/2)$. It is evident, however, that, as stated in Theorem 5, the Gaussian curvatures of the two surfaces are equal, except for sign, at *all* self-conjugate points.

In like manner we find upon making use of (26) that the lines of curvature on the two surfaces are *not* in tangency at every self-conjugate point; they are in tangency, however, at $(1, 0, 1/2)$.

Another choice of $F(y)$ in (28) which defines an autopolar surface is $F(y)=\log y^2-\log 2+3/2$.

9. Other examples. We shall conclude by listing briefly a few more autopolar surfaces.

1. $r-t=0$. From the solution $z=\phi_1(x+y)+\phi_2(x-y)$ of this equation we may derive several autopolar surfaces, but shall note but one of these, $z=\log(x+y)+\log(x-y)+1-\log 2$.

2. $3r-8s-3t=0$. The general solution $z=\phi_1(y-x/3)+\phi_2(y+3x)$ of this equation also yields many autopolar surfaces, one of which is $z=\log(y-x/3)+\log(y+3x)+1-\log .3$.

3. $py-qx=0$. This is the differential equation of surfaces of revolution, and from its general solution, $z=f(x^2+y^2)$, we may derive many autopolar surfaces. In fact, any plane curve which is autopolar with respect to a meridian of the quadric of reference, if revolved about the z-axis will generate an autopolar surface. We give but the one example, $z=\log(x^2+y^2)+1-\log 2$.

It is of course, a simple matter to write down a differential equation which is invariant under (1). On one side of an equation we write *any* function of x, y, z, p, q, $\cdots$, and set it equal to the expression into which it is transformed by (1) when the notation in capitals is ignored. If a solution of the equation can be found, we may then test certain particular solutions for autopolarity by using (10) or (11).

A THEOREM ON THE TANGRAM

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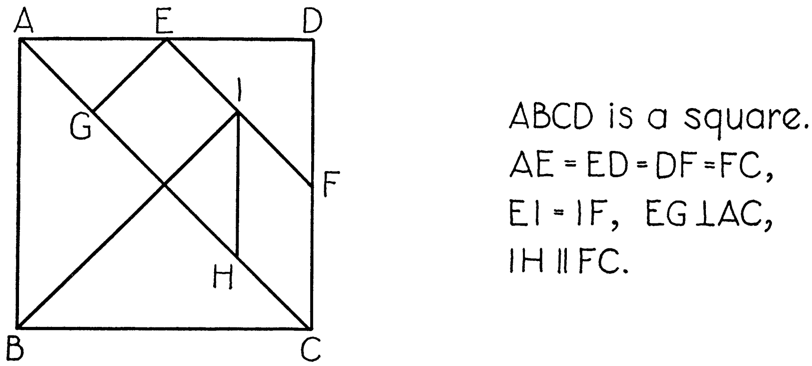
1. Introduction. The tangram is a Chinese puzzle consisting of a square card or board cut by straight incisions into different-sized pieces (five triangles, a square, and a lozenge) as shown in the following figure. These seven pieces may be combined to form many different figures. It is natural to inquire



how many convex polygons may be formed by the tangram. We propose in the present note to give a solution of this problem by proving the following theorem:

THEOREM. *By means of the tangram exactly thirteen convex polygons can be formed.*

2. Lemmas. It is easily seen that the tangram can be divided into sixteen equal isosceles right triangles. For the sake of convenience, we call the legs and the hypotenuses of these right triangles respectively *the rational* and *the irrational sides.* Then we may prove our theorem by finding out first the convex polygons formed by these sixteen triangles and then discarding those cases impossible for the tangram. For this purpose we introduce the following four lemmas.

LEMMA 1. *If sixteen equal isosceles right triangles are combined into a convex polygon, then a rational side of one triangle does not lie along an irrational side of another.*

Proof: First of all, let us suppose that of the sixteen given triangles two, denoted by ABC and $A'B'C'$, are arranged so that the irrational side $A'C'$ of the triangle $A'B'C'$ lies along the rational side AB of the triangle ABC. Since the given triangles are combined into a convex polygon, we may, without loss of generality, further suppose that the vertex A' coincides with the vertex A. In this case, at least another pair of the given triangles, denoted by DEF and $D'E'F'$, is such that one rational side $D'E'$ of the triangle $D'E'F'$ lies along the

irrational side DF of the triangle DEF, and $D \equiv B$, $D' \equiv C'$, $E' \equiv F$. If we fill the angle CDE or $B'D'F'$ with one or more of the given triangles, the case where one rational side of one triangle lies along the irrational side of another triangle will occur again. Repeatedly applying the above discussion, it may be easily seen that the polygon formed by the given triangles can not be convex. This contradicts the hypothesis, and establishes the lemma.

From Lemma 1, follows immediately the following lemma:

LEMMA 2. *If sixteen equal isosceles right triangles are combined into a convex polygon, then the sides of the polygon are formed by sides of the same kind (rational or irrational) of the triangles. Moreover, if a side of the polygon which is formed by the rational or the irrational sides of the triangles is said to be a rational or an irrational side (respectively) of the polygon, then in general the rational and the irrational sides of the polygon alternate. In particular, if an angle of the polygon is a right angle, the two adjacent sides are both rational or both irrational.*

LEMMA 3. *If sixteen equal isosceles right triangles are combined into a convex polygon, then the number of the sides of the polygon does not exceed eight.*

Proof: Since the sum of all the angles of a convex polygon of n sides is equal to $(n-2)\pi$, and the maximal value of the angles formed by the given triangles is $3\pi/4$, we have $(n-2)\pi \leqq 3\pi n/4$. It follows that $n \leqq 8$.

Since the angles of the convex polygon formed by the given triangles are $3\pi/4$, $\pi/2$, or $\pi/4$, by means of Lemma 2 and Lemma 3 we easily obtain

LEMMA 4. *If sixteen equal isosceles right triangles are combined into a convex polygon, then this polygon can be inscribed in a rectangle with all the rational or the irrational sides of the polygon as the sides of the rectangle.*

3. Proof of the theorem. For the purpose of proving our theorem, we have to find out the convex polygons formed by sixteen equal isosceles right triangles. First of all, we may assume, that this convex polygon is an octagon, denoted by $ABCDEFGH$. From Lemma 2 and Lemma 4 we may, further, assume that this polygon is inscribed in a rectangle $PQRS$ and that all the rational sides BC, DE, FG, HA of the polygon lie along the sides PQ, QR, RS, SP of the rectangle respectively. If the number of irrational sides of the given triangles on AB, CD, EF, GH are respectively a, b, c, d, and PQ, QR have equal lengths, respectively, with the lines composed of x and y rational sides of the given triangles, then a, b, c, d, x, y satisfy the equation

$$a^2 + b^2 + c^2 + d^2 = 2xy - 16, \tag{1}$$

with the conditions

$$\begin{cases} a + b \leqq x, & c + d \leqq x, \\ a + d \leqq y, & b + c \leqq y, \end{cases} \tag{2}$$

Hence our problem is reduced merely to finding the integral solutions of the equation (1) and the inequalities (2). For this purpose, we denote $x=\alpha$, $y=\beta$ by (α, β), and divide our discussion into the following cases, which may be easily proved to be sufficient.

a. *The case* $y>x$, $y>5$.

(i) $x>1$. Noticing that

$$9/x + x < 2 + x \leqq y + 1, \quad \text{whenever} \quad x \geqq 5,$$

we easily obtain

$$x(y + 1) > x^2 + 9, \quad \text{for} \quad x > 1 \quad \text{and} \quad y > 5. \tag{3}$$

By means of (1), (3) and the inequality $c^2+d^2 \leqq (c+d)^2 \leqq x^2$, follows immediately

$$a^2 + b^2 > (x - 1)^2 + 1. \tag{4}$$

It follows that a and b are not both zero. On the other hand a and b are not both different from zero, for if $a \geqq 1$, $b \geqq 1$, then from the first inequality of (2),

$$a^2 + b^2 \leqq (a + b - 1)^2 + 1 \leqq (x - 1)^2 + 1, \quad \text{for} \quad x > 1, \tag{5}$$

which contradicts (4). Whence either a or b (and not both) equals zero. Let, for instance, $b=0$; then $a \leqq x$. If, further, $a<x$, then $a \leqq x-1$, which contradicts (4). Therefore $a=x$.

Similarly, we know that $c=0$, $d=x$, or $c=x$, $d=0$. Thus we can easily show that (2, 6), (4, 6), (8, 9) are solutions.

(ii) $x=1$. In this case $a+b \leqq 1$, $c+d \leqq 1$. Therefore $a=b=c=d=0$, or $a=c=1$, $b=d=0$, or $a=c=0$, $b=d=1$, and hence we obtain the solutions (1, 8), (1, 9).

b. *The case* $x=y$. In this case we shall prove that $x \leqq 5$. First of all, it is easily seen that if $a=b=c=d=0$, there is no solution. Secondly, if $a, b, c, d<x$, then it is seen that

$$a^2 + b^2 \leqq (x - 1)^2 + 1, \qquad c^2 + d^2 \leqq (x - 1)^2 + 1. \tag{6}$$

From (1) and (6), we obtain

$$2x^2 - 16 \leqq 2(x - 1)^2 + 2,$$

which shows $x \leqq 5$.

Thirdly, we consider the case where one of a, b, c, d is equal to x. If, for instance, $a=x$, then from (2), $b=0$ and $d=0$. Whence (1) becomes $x^2=16+c^2$, which gives $x=5$ or 4.

Finally, it is not difficult to show that when $a=b=0$ or $c=d=0$, then $x<4$; and when $a=b=c=0$, then $x=d=4$.

c. *The case* $5 \geqq y > x$. In this case we may test for every set of integral values of x, y, a, b, c, d directly from the equation (1) and the inequalities (2) and easily obtain the required solutions.

In conclusion, we summarize the complete solution of our problem related to sixteen equal isosceles right triangles as follows:

x	y	a	b	c	d
*1	8	0	0	0	0
*1	9	1	0	1	0
*1	9	1	0	0	1
*8	9	8	0	8	0
*4	6	4	0	4	0
2	6	2	0	2	0
2	6	2	0	0	2
*5	5	4	1	4	1
*5	5	5	0	3	0
3	5	3	0	1	2
3	5	3	0	2	1
2	5	1	1	1	1
2	5	2	0	0	0
4	4	2	2	2	2
4	4	4	0	0	0
3	4	2	0	2	0
3	4	2	0	0	2
2	4	0	0	0	0
3	3	1	0	1	0
3	3	1	0	0	1

It is easy to show that the solutions indicated by asterisks in the above table are useless if the given sixteen equal isosceles right triangles are supposed to form a tangram. Thus we have completely proved our theorem.

4. Remark. It should be noted that the thirteen convex polygons (four hexagons, two pentagons, six quadrangles, and a triangle) obtained in the previous section are familiar to us. We can arrange the tangram into any of them by one or more methods, but space does not permit their inclusion here.

Moreover, in these thirteen convex polygons the perimeters of the first, the second, and the eighth, according to the order in the above table, are maximum and that of the last two are minimum.

BOUNDED LAPLACE TRANSFORMS*

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1. Introduction. Since the development of the classical theory of power series during the nineteenth century much attention has been devoted to the theory of such series but with the emphasis in quite a different direction. In recent decades investigators have been interested not so much in existence theorems and convergence properties as in the structure of power series under certain general and simple hypotheses. General treatments of the principal results are available† and Szegö has given a summary of these investigations and references to the literature where they may be found in detail.‡

A typical example is the following theorem of Fejer: let

$$f(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n + \cdots, \qquad |z| < 1 \tag{1.1}$$

and

$$\sigma_n(z) = \frac{s_0(z) + s_1(z) + \cdots + s_{n-1}(z)}{n} \tag{1.2}$$

be the arithmetic mean of the partial sums $s_k(z) = a_0 + a_1z + \cdots + a_kz^k$, then a necessary and sufficient condition that $|f(z)| \leqq M$ for $|z| < 1$ is that $|\sigma_n(z)| \leqq M$ for all n and $|z| < 1$.

Now it is of some interest and importance to investigate whether results such as the above are true for other representations of analytic functions besides the power series (1.1). One such other representation is the Laplace transformation

$$f(s) = \int_0^\infty e^{-st}\alpha(t)dt, \tag{1.3}$$

where $s = \sigma + i\tau$ and $\alpha(t)$ is a function of the real variable t. That the integral in (1.3) may be expected to have properties analogous to those of the series in (1.1) may be seen from purely formal considerations as follows. To generalize the series the sequence of integers may be first replaced by any discrete sequence of real numbers $\lambda_0, \lambda_1, \lambda_2, \cdots$ where $\lambda_n \to \infty$ as $n \to \infty$. To avoid multiple valued functions when λ_k is not an integer one can replace z by s by means of the conformal transformation

$$z = e^{-s}. \tag{1.4}$$

* Presented to the Southeastern Section of the Mathematical Association of America at Chapel Hill, N. C., March 28, 1941.

† E. Landau, Darstellung und Begründung Einiger Neuerer Ergebnisse u.s.w. 2nd edit., 1929. P. Dienes, The Taylor Series. Oxford Univ. Press, 1931.

‡ G. Szegö. Some recent investigations concerning the sections of power series, Bulletin of Amer. Math. Soc., vol. 42, 1936

Finally, replacing λ's by the continuous variable t, the coefficient $\{a_k\}$ by the function $\alpha(t)$ and the summation by integration, one obtains the integral in (1.3). However, since the transformation (1.4) is not reciprocal one-to-one it is not to be expected that all properties of the series and of the integral will be the same.

The region of convergence of the integral in (1.3) is a half plane $\sigma > \sigma_0$ and $f(s)$ is an analytic function of s there. Analyticity in a half plane is *not*, however, a sufficient contition for $f(s)$ to have the representation (1.3). The fundamental properties of the transformation (1.3) have been studied by Widder, Doetsch, and others.*

2. Bounded transforms. We now obtain the complete analogue for the Laplace integral of the Fejer boundedness theorem above. To this end we first define $F_R(s)$ thus

$$F_R(s) = \frac{1}{R}\int_0^R dR_1 \int_0^{R_1} e^{-st}\alpha(t)dt,$$

which is equal† to

$$F_R(s) = \int_0^R \left(1 - \frac{t}{R}\right) e^{-st}\alpha(t)dt. \tag{2.1}$$

With this notation we have the

THEOREM. *Let $f(s)$ converge absolutely for $\sigma > 0$, then a necessary and sufficient condition that $|f(s)| \leqq M$ for $\sigma > 0$ is that $|F_R(s)| \leqq M$ for $\sigma > 0$ and all $R \geqq 0$.*

Proof. The proof is based upon the identity

$$\int_0^R e^{-st}\alpha(t)dt = \frac{2}{\pi}\int_0^\infty dx \int_0^R \cos xt\, dt \int_0^\infty e^{-st_1}\alpha(t_1) \cos xt_1 dt_1. \tag{2.2}$$

We first prove this identity. This identity is then used to prove the condition necessary. The proof of sufficiency is immediate, requiring only the comment that the method of summation defined by (2.1) as $R \to \infty$ is regular.

To prove the identity (2.2) observe first that the right member is

$$\frac{2}{\pi}\int_0^\infty \frac{\sin Rx}{x} dx \int_0^\infty e^{-st_1}\alpha(t_1) \cos xt_1 dt_1.$$

But

$$\begin{aligned}\int_0^N \frac{\sin Rx}{x} dx \int_0^\infty e^{-st_1}\alpha(t_1) \cos xt_1 dt_1 \\ = \int_0^\infty e^{-st_1}\alpha(t_1)dt_1 \int_0^N \frac{\sin Rx \cos t_1 x}{x} dx,\end{aligned} \tag{2.3}$$

* D. V. Widder, A generalization of Dirichlet's series and Laplace's integrals, etc., Transactions of Amer. Math. Soc., vol. 31, 1929.

† See, for example, Goursat-Hedrick, Mathematical Analysis, vol. 1, p. 295, ex. 11.

the interchange of the order of integration being justified by the uniform convergence of the inner integral with respect to x on the interval $0 \leqq x \leqq N$. But the inner integral (on the right) is bounded for all $t_1 \geqq 0$ and $N \geqq 0$ as one sees from

$$\int_0^N \frac{\sin Rx \cos t_1 x}{x} dx = \frac{1}{2}\int_0^N \frac{\sin (R+t_1)x}{x} dx + \frac{1}{2}\int_0^N \frac{\sin (R-t_1)x}{x} dx$$

which gives

$$\int_0^N \frac{\sin Rx \cos t_1 x}{x} dx = \frac{1}{2}\int_0^{N(R+t_1)} \frac{\sin y}{y} dy \pm \frac{1}{2}\int_0^{N|R-t_1|} \frac{\sin y}{y} dy$$

by a change of variable in each integral, the sign of $R-t_1$ being taken for the second integral in the right member. But

$$\int_0^K \frac{\sin y}{y} dy$$

is a bounded function of K, hence by the theorem of bounded convergence* one may allow $N \to \infty$ in (2.3). Thus one obtains

$$\frac{2}{\pi}\int_0^\infty \frac{\sin Rx}{x} dx \int_0^\infty e^{-st_1}\alpha(t_1) \cos xt_1 dt_1 = \frac{2}{\pi}\int_0^\infty e^{-st_1}\alpha(t_1)dt_1 \int_0^\infty \frac{\sin Rx \cos tx}{x} dx.$$

Dividing the range of integration in the right member above gives

$$\begin{aligned}(2.4)\qquad &\frac{2}{\pi}\int_0^R e^{-st_1}\alpha(t_1)dt_1 \int_0^\infty \frac{\sin Rx \cos t_1 x}{x} dx \\ &\qquad + \frac{2}{\pi}\int_R^\infty e^{-st_1}\alpha(t_1)dt_1 \int_0^\infty \frac{\sin Rx \cos t_1 x}{x} dx.\end{aligned}$$

But the inner integral here is a well known discontinuous integral, namely,

$$\int_0^\infty \frac{\sin Rx \cos t_1 x}{x} dx = \begin{cases} \dfrac{\pi}{2}, & t_1 < R, \\ 0, & t_1 > R. \end{cases}$$

Hence the second term of (2.4) vanishes, the first term reduces to the left member of (2.2), and the identity is established.

Returning now to (2.2) to prove necessity we have on integrating both members of the equation

* See, for example, Titchmarsh, The Theory of Functions, p. 337.

$$\frac{1}{R_1}\int_0^{R_1} dR \int_0^R e^{-st}\alpha(t)dt$$

$$= \frac{2}{\pi}\,\frac{1}{R_1}\int_0^{R_1} dR \int_0^\infty dx \int_0^R \cos xt\,dt \int_0^\infty e^{-st_1}\alpha(t_1)\cos xt_1 dt_1,$$

or

$$F_{R_1}(s) = \frac{2}{\pi}\int_0^\infty dx \int_0^{R_1}\left(1 - \frac{t}{R_1}\right)\cos xt\,dt \int_0^\infty e^{-st_1}\alpha(t_1)\cos xt_1 dt_1.$$

But by elementary methods

$$\int_0^{R_1}\left(1 - \frac{t}{R_1}\right)\cos xt\,dt = \frac{1 - \cos R_1 x}{R_1 x^2}\,.$$

Thus

$$F_{R_1}(s) = \frac{2}{\pi}\int_0^\infty \frac{1 - \cos R_1 x}{R_1 x^2}\,dx \int_0^\infty e^{-st_1}\alpha(t_1)\cos xt_1 dt_1. \tag{2.5}$$

But the outer integral in the right member of (2.5) is *non-negative*, hence

$$\left| F_{R_1}(s) \right| \leqq \frac{2}{\pi}\int_0^\infty \frac{1 - \cos R_1 x}{R_1 x^2}\,dx \left| \int_0^\infty e^{-st_1}\alpha(t_1)\cos xt_1 dt_1 \right|. \tag{2.6}$$

But we have

$$\begin{aligned}\left| \int_0^\infty e^{-st_1}\alpha(t_1)\cos xt_1 dt_1 \right| &= \left| \int_0^\infty e^{-st_1}\alpha(t_1)\,\frac{e^{ixt_1} + e^{-ixt_1}}{2}\,dt_1 \right| \\ &\leqq \frac{1}{2}\left| \int_0^\infty e^{-\sigma t_1}e^{-i(\tau - x)t_1}\alpha(t_1)dt_1 \right| \\ &\quad + \frac{1}{2}\left| \int_0^\infty e^{-\sigma t_1}e^{-i(\tau + x)t_1}\alpha(t_1)dt_1 \right| \\ &\leqq \tfrac{1}{2}M + \tfrac{1}{2}M \\ &= M.\end{aligned}$$

Hence (2.6) gives when combined with the above

$$\left| F_{R_1}(s) \right| \leqq M\cdot\frac{2}{\pi}\cdot\frac{1}{R_1}\int_0^\infty \frac{1 - \cos R_1 x}{x^2}\,dx.$$

However, by familiar methods of contour integration* we have

$$\int_0^\infty \frac{1 - \cos R_1 x}{x^2}\,dx = \frac{R_1 \pi}{2},$$

* Whittaker and Watson, Modern Analysis, 4th ed., p. 116, ex. 2.

and hence

$$|F_{R_1}(s)| \leqq M,$$

and the necessity of the condition has also been demonstrated.

A theorem similar to the one above was obtained by Stacho for the Stieltjes integral* by different methods. Stacho does not use the identity (2.2) but a method based on contour integration analogous to Landau's proof for series. The identity (2.2) is an example of Parseval's relation for Fourier-cosine transforms.

EVALUATION OF SURFACE INTEGRALS BY ELECTRICAL IMAGES

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Given a rigid system, T, consisting of electrical charges $q_1, q_2, \cdots, q_n$ located at points $\mathbf{R}_1, \mathbf{R}_2, \cdots, \mathbf{R}_n$ which are such that the distances $|\mathbf{R}_i - \mathbf{R}_j| = R_{ij}$ are constant for all i and j. When this system is placed in the neighborhood of the surface, S, of a grounded conductor, the surface acquires a charge of density $\sigma(\mathbf{r})$ which is everywhere such that the electric field within the conductor vanishes and its surface remains at ground potential. Here $\mathbf{r}$ is a vector to any point in the surface.

The potential energy, U, of the entire system thus formed consists of three parts. The first is the self energy of the system T

$$U_{TT} = \frac{1}{2} \sum_{i,j=1}^{n}{}' q_i q_j / R_{ij} = U_0 \tag{1}$$

where the prime indicates that all terms having $i = j$ are omitted from the summation. Since all the R_{ij} remain constant, this contribution does not vary with the separation between T and the conductor. Next, we have the self energy of the induced surface charge

$$U_{\sigma\sigma} = \frac{1}{2} \int\int \frac{\sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} dS_1 dS_2 \tag{2}$$

where each integration extends over the entire surface of the conductor. Finally there is the mutual energy of the system T and the surface charge

$$U_{T\sigma} = \sum_{i=1}^{n} \int \frac{q_i \sigma(\mathbf{r})}{|\mathbf{r} - \mathbf{R}_i|} dS. \tag{3}$$

* T. Stacho, Über die Riemann-Stieltjesschen Integrale u.s.w. Ber. aus Ungarn, vol. 33, 1923–25.

Combining all these terms one has

$$U = U_{T\sigma} + U_{\sigma\sigma} + U_0. \tag{4}$$

The present note concerns a method for obtaining indirectly an evaluation of surface integrals of the form (2) and (3). This method can be applied whenever it is possible to replace the surface distribution, σ, by a suitably constructed electrical image, T', of T without altering the electric field exterior to the surface. When this is the case, the interaction $U_{T\sigma}$ between T and the surface charge is the same as the interaction U_{TT}' between T and its image, and this can be written down directly. Moreover, the surface charge density is given in terms of the surface electric field (which is always normal to the surface) by

$$\sigma(\mathbf{r}) = E(\mathbf{r})/4\pi \tag{5}$$

In this case $E(\mathbf{r})$ can be computed from the combined fields of T and T' at the surface. Thus (3) may be written

$$\sum_{i=1}^{n} \int \frac{q_i E_{TT'}(\mathbf{r})}{|\mathbf{r} - \mathbf{R}_i|} dS = 4\pi U_{TT'}. \tag{6}$$

In order similarly to express Eq. 2, we consider the force of attraction $\mathbf{F}_{T\sigma}$ between the system and the conductor. In the special case we are considering this force is the same as the force $\mathbf{F}_{TT'}$ between the system and its image. The total potential energy U may be written in terms of this force as

$$U = -\int_C \mathbf{F}_{TT'} \cdot d\boldsymbol{\lambda} + U_1 \tag{7}$$

where U_1 is constant and C is an arbitrary path extending to an infinite separation between T and S. As this separation increases without limit, $U_{T\sigma}$, $U_{\sigma\sigma}$, and the integral in (7) approach zero. Hence, comparing Eqs. 4 and 7, we see that

$$\begin{aligned} U_1 &= U_0, \\ U_{T\sigma} + U_{\sigma\sigma} &= -\int_C \mathbf{F}_{TT'} \cdot d\boldsymbol{\lambda}. \end{aligned} \tag{8}$$

Substituting (5) and (8) in Eq. 2 and replacing $U_{T\sigma}$ by $U_{TT'}$, we obtain

$$\iint \frac{E_{TT'}(\mathbf{r}_1) E_{TT'}(\mathbf{r}_2)}{r_{12}} dS_1 dS_2 = -32\pi^2 \left[U_{TT'} + \int_C \mathbf{F}_{TT'} \cdot d\boldsymbol{\lambda} \right] \tag{9}$$

In the following sections, the procedure outlined here is applied to the well known cases of electrical images in plane and spherical conductors. As a result, integrals of the form of (6) and (9) taken over plane and spherical surfaces are evaluated without the necessity of performing any quadratures.

1. Integrals over a plane. Let the surface of the conductor form a plane of infinite extent and choose the origin of coordinates in this plane with the Z-axis normal to it. Each of the charges q_i with coordinates (X_i, Y_i, Z_i) composing the system T has an image $-q_i$ with coordinates $(X_i, Y_i, -Z_i)$ and these compose the image T'. The potential energy of the interaction between T and T' is then

$$U_{TT'} = -\sum_{i=1}^{n} \frac{q^2}{2Z_i} - \sum_{i,j=1}^{n}{}' \frac{q_i q_j}{[R_{ij}^2 + 4Z_i Z_j]^{1/2}} \tag{10}$$

The force $\mathbf{F}_{TT}'$ is normal to the surface and is easily found to be

$$\mathbf{F}_{TT'} = -\left\{ \sum_{i=1}^{n} \frac{q_i^2}{4Z_i^2} + \sum_{i,j=1}^{n}{}' \frac{q_i q_j (Z_i + Z_j)}{[R_{ij}^2 + 4Z_i Z_j]^{3/2}} \right\} \mathbf{k} \tag{11}$$

where $\mathbf{k}$ is a unit vector along the Z-axis. Substituting (11) in (7) and comparing the result with (10), we find

$$\int_C \mathbf{F}_{TT'} \cdot d\boldsymbol{\lambda} = -\tfrac{1}{2} U_{TT'}. \tag{12}$$

The electric field at the surface of the conductor produced by T and T' is

$$\mathbf{E}_{TT'}(\mathbf{r}) = -\sum_{j=1}^{n} \frac{2q_j Z_j}{|\mathbf{r} - \mathbf{R}_j|^3} \mathbf{k} \tag{13}$$

where $\mathbf{r}$ is now the vector (x, y).

The expressions given by Eqs. 10, 12, and 13 may now be substituted in the general equations 6 and 9. Since all potential energy terms in this problem are quadratic in $q_i q_j$, the results may be represented by the quadratic form

$$\sum_{i=1}^{n} A_{ii} q_i^2 + \sum_{i,j=1}^{n}{}' A_{ij} q_i q_j = 0. \tag{14}$$

This equation must be satisfied for all values of q_i and q_j since the charges composing the system T are arbitrary both in number and in their individual magnitudes. Thus, we are led to the set of equations

$$A_{ij} + A_{ji} = 0, \qquad (i, j = 1, 2, \cdots, n). \tag{15}$$

Inspection shows that the integrals in the A_{ii} are merely special cases of those in the A_{ij} with $i \neq j$, while the latter are of the same form for all i and j. We, therefore, confine the discussion to the two equations originating from (6) and (9) and represented by $A_{12} + A_{21} = 0$. These are

$$\int \frac{Z_1 dS}{|\mathbf{r} - \mathbf{R}_2| \cdot |\mathbf{r} - \mathbf{R}_1|^3} + \int \frac{Z_2 dS}{|\mathbf{r} - \mathbf{R}_1| \cdot |\mathbf{r} - \mathbf{R}_2|^3} = \frac{4\pi}{[R_{12}^2 + 4Z_1 Z_2]^{1/2}}$$

from Eq. 6, and

$$2\int\int \frac{Z_1Z_2dS_1dS_2}{|\mathbf{r}_1-\mathbf{R}_1|^3|\mathbf{r}_2-\mathbf{R}_2|^3r_{12}} = \frac{8\pi^2}{[R_{12}^2+4Z_1Z_2]^{1/2}}$$

from Eq. 9. Here $\int dS$ stands for $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dxdy$ and $r_{12}=[(x_1-x_2)^2+(y_1-y_2)^2]^{1/2}$.

If we now put $X_1=\alpha_x$, $Y_1=\alpha_y$, $Z_1=A$ and $X_2=\beta_x$, $Y_2=\beta_y$, $Z_2=B$, the above integrals may be written in the form

$$\begin{aligned}(16)\qquad &\int\frac{AdS}{[A^2+(\mathbf{r}-\boldsymbol{\alpha})^2]^{3/2}[B^2+(\mathbf{r}-\boldsymbol{\beta})^2]^{1/2}}\\ &+\int\frac{BdS}{[A^2+(\mathbf{r}-\boldsymbol{\alpha})^2]^{1/2}[B^2+(\mathbf{r}-\boldsymbol{\beta})^2]^{3/2}} = \frac{4\pi}{[(\boldsymbol{\alpha}-\boldsymbol{\beta})^2+(A+B)^2]^{1/2}}\end{aligned}$$

and

$$(17)\qquad \int\int\frac{dS_1dS_2}{[A^2+(\mathbf{r}_1-\boldsymbol{\alpha})^2]^{3/2}[B^2+(\mathbf{r}_2-\boldsymbol{\beta})^2]^{3/2}r_{12}} = \frac{4\pi^2}{AB[(\boldsymbol{\alpha}-\boldsymbol{\beta})^2+(A+B)^2]^{1/2}}.$$

The first of these is probably too specialized to be of any particular use in problems involving integrals of this type. However, one special case has been found which may be of some interest. This is obtained by taking $A=B$ and $\boldsymbol{\beta}=-\boldsymbol{\alpha}$, so that (16) reduces to

$$(18)\qquad \int\frac{(A^2+\alpha^2+r^2)dS}{[A^2+(\mathbf{r}-\boldsymbol{\alpha})^2]^{3/2}[A^2+(\mathbf{r}+\boldsymbol{\alpha})^2]^{3/2}} = \frac{\pi}{A[A^2+\alpha^2]^{1/2}}$$

This may be further simplified by means of the substitutions $\zeta=A^2+\alpha^2$, $\eta=\alpha^2$, and $z=r^2$ with the result

$$(19)\qquad \int_0^{\infty}\int_0^{2\pi}\frac{(z+\zeta)dzd\theta}{[(z+\zeta)^2-4\eta z\cos^2\theta]^{3/2}} = \frac{2\pi}{\sqrt{\zeta(\zeta-\eta)}}$$

A number of additional integrals can be obtained by performing various operations on Eq. 17 with respect to the parameters A, B, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$. Some of the most interesting of these seem to be the following:

$$(20)\qquad \begin{aligned}&\int\int\frac{dS_1dS_2}{[A^2+r_1^2]^{3/2}[B^2+r_2^2]^{3/2}r_{12}} = \frac{4\pi^2}{AB(A+B)},\\ &\int\int\frac{(\boldsymbol{\alpha}\cdot\mathbf{r}_1)(\boldsymbol{\beta}\cdot\mathbf{r}_2)dS_1dS_2}{[A^2+r_1^2]^{3/2}[B^2+r_2^2]^{3/2}r_{12}} = \frac{2\pi^2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{A+B},\\ &\int\int\frac{dS_1dS_2}{[A^2+r_1^2]^{5/2}[B^2+r_2^2]^{5/2}r_{12}} = \frac{8\pi^2[(A+B)^2+AB]}{9A^3B^3(A+B)^3},\\ &\int\int\frac{(\boldsymbol{\alpha}\cdot\mathbf{r}_1)(\boldsymbol{\beta}\cdot\mathbf{r}_2)dS_1dS_2}{[A^2+r_1^2]^{5/2}[B^2+r_2^2]^{5/2}r_{12}} = \frac{4\pi^2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{9AB(A+B)^3}.\end{aligned}$$

The first of these is obtained by putting $\boldsymbol{\alpha}=\boldsymbol{\beta}$ in Eq. 17 and then removing $\boldsymbol{\alpha}$ from the denominator by means of the translation $\mathbf{r}=\mathbf{r}'+\boldsymbol{\alpha}$. The third is obtained from the first by means of the operation $\partial^2/\partial A\partial B$. The second and fourth are obtained in a somewhat more involved manner by applying the operator $(\boldsymbol{\alpha}\cdot\mathrm{grad}_\alpha)$ $(\boldsymbol{\beta}\cdot\mathrm{grad}_\beta)$ to Eq. 17 and then substituting $\boldsymbol{\alpha}=\boldsymbol{\gamma}$ and $\boldsymbol{\beta}=\boldsymbol{\gamma}$ in the result. The vector $\boldsymbol{\gamma}$ is then removed from the denominator of the integrand by means of the translation $\mathbf{r}=\mathbf{r}'+\boldsymbol{\gamma}$. The resulting expression can then be generalized by substituting $\boldsymbol{\alpha}'+\boldsymbol{\beta}'$ for $\boldsymbol{\gamma}$ where $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$ are arbitrary vectors in the x, y plane. Of the four resulting integrals it can be shown that the one involving $(\boldsymbol{\alpha}'\cdot\mathbf{r}_1')$ $(\boldsymbol{\beta}'\cdot\mathbf{r}_2')$ differs from the one involving $(\boldsymbol{\beta}'\cdot\mathbf{r}_1')$ $(\boldsymbol{\alpha}'\cdot\mathbf{r}_2')$ only by an integral which includes

$$\int_0^{2\pi}\frac{\sin\theta_{12}d\theta_{12}}{[r_1^2+r_2^2-2r_1r_2\cos\theta_{12}]^{1/2}}\,.$$

Since this is zero, the two terms are equivalent. Combining them and dropping the primes, the fourth of Eqs. 20 is obtained. The second of these equations is obtained from it by multiplying both sides by $ABdAdB$ and integrating with respect to A and B.

2. Integrals over a sphere. On applying considerations similar to those for images in a plane to the case of the electrical image of an arbitrary system T in a sphere, Eqs. 14 and 15 are again obtained so that no loss in generality is entailed by choosing for the system T only two charges q_1 and q_2. Choose the origin of coordinates at the center of the conducting sphere and let the charges q_1 and q_2 be located at the termini of two vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ drawn from this origin. The image system T' consists of two image charges q_1' and q_2' placed at points $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ respectively where

$$(21)\qquad \begin{aligned} q_1' &= -(r/R_1)q_1, & q_2' &= -(r/R_2)q_2,\\ \boldsymbol{\rho}_1 &= (r/R_1)^2\mathbf{R}_1, & \boldsymbol{\rho}_2 &= (r/R_2)^2\mathbf{R}_2. \end{aligned}$$

Here r denotes the radius of the sphere and $\mathbf{r}$ is a vector to some point on the sphere.

The interaction energy between T and T' is given by

$$(22)\qquad U_{TT'} = -r\left[\frac{q_1^2}{G_1}+\frac{q_2^2}{G_2}+\frac{2q_1q_2}{[G_1G_2+r^2R_{12}^2]^{1/2}}\right],$$

$$(23)\qquad G_i = R_i^2 - r^2.$$

The force between T and T' is found to be

$$(24)\quad \mathbf{F}_{TT'} = -r\left\{\frac{q_1^2}{G_1^2}+\frac{q_1q_2G_2}{[G_1G_2+r^2R_{12}^2]^{3/2}}\right\}\mathbf{R}_1 - r\left\{\frac{q_2^2}{G_2^2}+\frac{q_1q_2G_1}{[G_1G_2+r^2R_{12}^2]^{3/2}}\right\}\mathbf{R}_2.$$

When (24) is used to evaluate $\int_C \mathbf{F}_{TT'} \cdot d\lambda$, it is found that the integral is independent of the path C and that, comparing the result with (22), Eq. 12 is satisfied. Finally, we obtain for the electric field at the surface of the sphere produced by T and T'

$$\mathbf{E}_{TT'}(\mathbf{r}) = -\frac{q_1 G_1}{r^2 |\mathbf{r}-\mathbf{R}_1|^3}\mathbf{r} - \frac{q_2 G_2}{r^2 |\mathbf{r}-\mathbf{R}_2|^3}\mathbf{r}. \tag{25}$$

We are now again in a position to write down the integrals arising from Eqs. 6 and 9 for this case and represented by $A_{12}+A_{21}=0$. We obtain from Eq. 6

$$\int \frac{G_2 dS}{r|\mathbf{r}-\mathbf{R}_1| \cdot |\mathbf{r}-\mathbf{R}_2|^3} + \int \frac{G_1 dS}{r|\mathbf{r}-\mathbf{R}_2| \cdot |\mathbf{r}-\mathbf{R}_1|^3} = \frac{8\pi r}{[G_1G_2 + r^2 R_{12}^2]^{\frac{1}{2}}} \tag{26}$$

and from Eq. 9

$$\int\int \frac{G_1 G_2 dS_1 dS_2}{r^2 |\mathbf{r}_1 - \mathbf{R}_1|^3 \cdot |\mathbf{r}_2 - \mathbf{R}_2|^3 r_{12}} \quad \frac{16\pi^2 r}{[G_1G_2 + r^2 R_{12}^2]^{1/2}}. \tag{27}$$

To express these results in a more convenient form, we take $\mathbf{R}_1 = (0, A\sin\alpha, A\cos\alpha)$ and $\mathbf{R}_2 = (0, B\sin\beta, B\cos\beta)$. Also let

$$\begin{aligned} \cos\Theta_{12} &= \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2) \\ \cos\Theta_{\alpha 1} &= \cos\alpha\cos\theta_1 + \sin\alpha\sin\theta_1\cos\phi_1 \\ \cos\Theta_{\beta 2} &= \cos\beta\cos\theta_2 + \sin\beta\sin\theta_2\cos\phi_2. \end{aligned} \tag{28}$$

In terms of these quantities, Eq. 27 may be written

$$\begin{aligned} &\int_0^\pi\int_0^\pi\int_0^{2\pi}\int_0^{2\pi} \frac{\sin\theta_1 \sin\theta_2 \, d\theta_1 d\theta_2 d\phi_1 d\phi_2}{[A^2 - 2Ar\cos\Theta_{\alpha 1} + r^2]^{3/2}[B^2 - 2Br\cos\Theta_{\beta 2} + r^2]^{3/2}\sin\frac{1}{2}\Theta_{12}} \\ &\qquad = \frac{32\pi^2}{(A^2-r^2)(B^2-r^2)[A^2B^2 - 2ABr^2\cos(\alpha-\beta) + r^4]^{1/2}}. \end{aligned} \tag{29}$$

Eq. 26, like its analogue (16) in the case of the plane, is apparently too specialized to be of any great interest. The special case of it which is analogous to Eq. 19 is obtained by taking $A=B$ and $\beta=-\alpha$. The form of the integral may then be simplified by writing $A/\sqrt{A^2+r^2} = \sin\frac{1}{2}\gamma$. The result is found to be

$$\int_0^\pi\int_0^{2\pi} \frac{(\sec\gamma - \cos\alpha\cos\theta)\sin\theta\, d\theta d\phi}{[(\sec\gamma-\cos\alpha\cos\theta)^2 - \sin^2\alpha\sin^2\theta\cos^2\phi]^{3/2}} = \frac{4\pi\cot\gamma}{[\tan^2\gamma + \sin^2\alpha]^{1/2}}. \tag{30}$$

DISCUSSIONS AND NOTES

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The department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

NOTE ON INTEGRATING FACTORS

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The differential equation of the first order which has an integrating factor of the form $x^m y^n$ where m and n are constants can be solved more simply than by some of the special methods given in the elementary textbooks.

If $x^m y^n$ is an integrating factor of

$$f_1(x, y)dx + f_2(x, y)dy = 0, \tag{1}$$

where f_1 and f_2 are functions of x and y, then

$$x^m y^n \frac{\partial f_1}{\partial y} + f_1 \cdot n x^m y^{n-1} = x^m y^n \frac{\partial f_2}{\partial x} + f_2 \cdot m x^{m-1} y^n. \tag{2}$$

Equating coefficients of like terms, m and n can be readily found.

Contrast the simplicity of this method with the tedium of the method of rearranging the terms of the given equation to take the form

$$x^\alpha y^\beta(mydx + nxdy) + x^{\alpha_1} y^{\beta_1}(m_1 ydx + n_1 xdy) = 0, \tag{3}$$

as given in Murray's Differential Equations, page 25. Here, after pointing out that for all values of k and k_1

$$x^{km-1-\alpha} y^{kn-1-\beta} \quad \text{and} \quad x^{k_1 m_1 - 1 - \alpha_1} y^{k_1 n_1 - 1 - \beta_1},$$

are integrating factors of (3), it is stated that these two factors are identical if

$$km - 1 - \alpha = k_1 m_1 - 1 - \alpha_1,$$

and

$$kn - 1 - \beta = k_1 n_1 - 1 - \beta_1,$$

which can be solved for k and k_1.

Two examples will suffice to show the relative effectiveness of the two methods. Let the student solve each example by both methods. Only the proposed method will be used here.

Ex. 1. Solve

$$(xy^2 + y)dx - x \log xdy = 0.$$

Multiplying by $x^m y^n$,

$$(x^{m+1}y^{n+2} + x^m y^{n+1})dx - x^{m+1}\cdot\log x\cdot y^n dy = 0.$$

Differentiating according to (2) above, we have

$$(n+2)x^{m+1}y^{n+1} + (n+1)x^m y^n = -x^{m+1}\cdot\frac{1}{x}\cdot y^n - (m+1)x^m\cdot\log x\cdot y^n.$$

Equating coefficients of like terms,

$$n+2=0,$$
$$n+1=-1,$$
$$m+1=0.$$

Solving we have $m=-1$, $n=-2$. Therefore the integrating factor is $1/xy^2$. Applying this and integrating we have the solution

$$x+\frac{1}{y}\log x = c.$$

Ex. 2. Solve*

$$(y^3-2x^2y)dx + (2xy^2-x^3)dy = 0.$$

Multiplying by $x^m y^n$,

$$(x^m y^{n+3} - 2x^{m+2}y^{n+1})dx + (2x^{m+1}y^{n+2} - x^{m+3}y^n)dy = 0.$$

By (2),

$$(n+3)x^m y^{n+2} - 2(n+1)x^{m+2}y^n = 2(m+1)x^m y^{n+2} - (m+3)x^{m+2}y^n.$$

Equating like terms,

$$n+3 = 2m+2,$$
$$-2n-2 = -m-3.$$

Solving, $m=1$, $n=1$. Therefore the integrating factor is xy. Using this we get

$$x^2y^4/2 - x^4y^2/2 = c_1 \quad \text{or} \quad x^2y^2(y^2-x^2) = c.$$

NOTE ON SEMI-LOGARITHMIC GRAPHS

W. T. LENSER, Brown University

While it is shown in elementary textbooks that data giving a straight line when plotted on semi-logarithmic paper satisfy an equation of the type $y=ae^{bx}$, where y is plotted on the logarithmic and x on the uniform scale, no method is usually given for the determination of the constant b directly from the graph.

* Compare with the solution given in Murray's Differential Equations: Ex. 1, page 26.

The student learns to read off a value for a at the intersection of the graph with the line $x=0$ but is led to believe it is necessary to calculate b by logarithms. Further, if the graph does not intersect the line $x=0$ in the data involved, he must resort to logarithms to determine both a and b. In the method I shall describe, it is possible to compute b from the graph independently of a, and if necessary, to compute a from b. Briefly, the method consists of determining the increment of x required to produce one cycle of y, and has the distinct advantage of being independent of the choice of units made for the axes.

Consider again the equation

$$y = ae^{bx}, \tag{1}$$

and the equation obtained from it

$$\log y = \log a + bx \log e, \tag{2}$$

which may be represented as

$$u = mx + c, \tag{3}$$

where $u=\log y$, $m=b \log e$, and $c=\log a$.

To determine b, simply take one cycle of y from any starting point and read the corresponding values of x (see fig. 1) so that

$$b = \frac{\log y_2 - \log y_1}{(x_2 - x_1)\log e} = \frac{\log y_2/y_1}{\Delta x \log e} = \frac{\log 10}{\Delta x \log e} = \frac{2.3}{\Delta x}\,. \tag{4}$$

In case the graph does not extend over one cycle, determine x for $y=e$ and $y=1$;

$$b = \frac{\log e}{\Delta x \log e} = \frac{1}{\Delta x} \tag{5}$$

which is even simpler (see fig. 2).

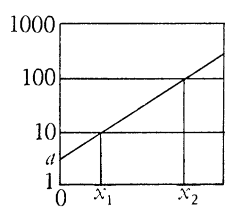

FIG. 1

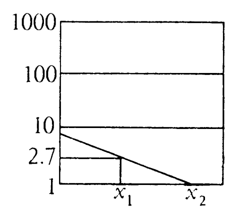

FIG. 2

$$b = \frac{2.30}{(x_2 - x_1)}, \qquad b = -\frac{1}{(x_2 - x_1)}.$$

If an equation of the type $y=a\ 10^{bx}$ is desired, taking $y=10$ as before gives

$$b = \frac{\log 10}{x \log 10} = \frac{1}{x}\,.$$

The sign of b is of course positive if the graph has a positive slope in the ordinary sense and is negative for a negative slope.

If a can not be determined at $x=0$ it can usually be obtained by finding y for some convenient values of x such that $bx=$some integer n, whereupon $a=y/e^n$.

The advantages of this method for problems where at most two significant figures are expected may be summarized as follows: (1) no table of logarithms need be consulted, but a slide rule is convenient; (2) b can always be determined from the graph regardless of a; (3) the method is independent of the scales used for x and y.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR. AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 541. *Proposed by Joseph Rosenbaum, Bloomfield, Conn.*

Given a regular polygon of n sides, $n>4$, design a quadrilateral, Q, such that (1) it shall be possible to fit $2n$ of the Q's to the polygon to form a new regular polygon of n sides, and (2) it shall be possible to fit $2n$ additional Q's to the new polygon to form a still larger third regular polygon of n sides.

E 542. *Proposed by V. Thébault, San Sebastián, Spain*

In what scale of notation (with radix less than a hundred) will the four-digit number 58 58 58 58 be a perfect square?

E 543. *Proposed by N. A. Court, University of Oklahoma*

Find a point whose polar planes for three given spheres (with non-collinear centers) are mutually perpendicular. Show that the problem may have two solutions. When will they be real?

E 544. *Proposed by E. P. Starke, Rutgers University*

Show that it is possible to construct a tetrahedron such that the length of every edge, the area of every face, and the volume all are integers.

E 545. *Proposed by A. H. Stone, Institute for Advanced Study*

Starting with a point P on the side BC of a triangle ABC, mark Q on AB with $BQ=BP$, R on CA with $AR=AQ$, P' on BC with $CP'=CR$, Q' on AB with $BQ'=BP'$, and so on. Prove that the construction closes, *i.e.*, that $CP=CR'$, and that the six points P, Q, R, P', Q', R' are concyclic. (Some obvious restrictions must be placed on the directions on the sides of the triangle in which the intervals BQ, *etc.*, are taken.)

SOLUTIONS

Generating Functions in Statistics

E 504 [1942, 61]. *Proposed by J. F. Kenney, University of Wisconsin at Milwaukee*

If p and q are positive numbers with $p+q=1$, show that

$$\lim_{n\to\infty} (pe^{qt/\sqrt{npq}} + qe^{-pt/\sqrt{npq}})^n = e^{t^2/2}.$$

Solution by Churchill Eisenhart, University of Wisconsin

We shall show that the above limit is uniform in every finite t-interval. To do this we employ the following lemma (which is easily deduced from the usual form of Maclaurin's series):

LEMMA. If $f(x)$ possesses a second derivative which is continuous at $x=0$, then

$$f(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 + o(x^2),$$

where $o(x^2)$ tends to zero faster than x^2 as $x\to 0$.

We define

$$M_n(t) = (pe^{qt/\sqrt{npq}} + qe^{-pt/\sqrt{npq}})^n,$$
$$K_n(t) = \log M_n(t) = n \log (pe^{qt/\sqrt{npq}} + qe^{-pt/\sqrt{npq}}).$$

Expanding the exponentials with the aid of the lemma (together with the fact that $p+q=1$), we see that, as $n\to\infty$,

$$K_n(t) = n \log \left\{1 + \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right\}$$

uniformly in every finite t-interval, $o(1/n)$ denoting a function which tends to zero faster than $1/n$ as $n\to\infty$. Applying the lemma to the expansion of the logarithm, we have (uniformly in every finite t-interval as $n\to\infty$)

$$K_n(t) = n\left\{\frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right\} = \frac{t^2}{2} + o(1),$$

where $o(1)$ denotes a function which tends to zero as $n\to\infty$.

Therefore, in every finite t-interval, $K_n(t)$ tends uniformly to $t^2/2$, and $M_n(t)$ to $e^{t^2/2}$.

Discussion. If x is a random variable, such that the probability $x=X$ is given by

$$\binom{n}{X} q^{n-X} p^X$$

for $X=0, 1, 2, \cdots, n$, then $M_n(t)$ is the moment-generating function of the distribution of the variable

$$y = (x - np)/\sqrt{npq}.$$

In other words, the rth moment of y (*i.e.*, the average value of y^r) is given by

$$\left[\left(\frac{d}{dt}\right)^r M_n(t)\right]_{t=0}.$$

Moreover, $K_n(t)$ is the semi-invariant-generating function of the distribution of y, as its rth derivative, evaluated at the point $t=0$, gives the rth semi-invariant of y. Since $e^{t^2/2}$ is known to be the moment-generating function of the normal distribution of zero mean and unit standard deviation, the implication of the above limit is that for large values of n the variable x is approximately normally distributed about a mean of np with standard deviation $\sqrt{npq}$. This latter result is often known as Laplace's Theorem, being given by him in his *Théorie analytique des probabilités*, 1812. But it has recently been found that De Moivre obtained the same result in his *Miscellanea Analytica, Supplementum*, 1733, bringing it before the probability-minded public in the second edition of his *Doctrine of Chances*, 1738.

Also solved by G. A. Baker, Paul Brock, Albert Furman, J. A. Greenwood, H. D. Larsen, E. P. Starke, and the proposer.

For the properties of generating functions (also called characteristic functions), see Curtiss, *Generating functions in the theory of statistics*, this MONTHLY, vol. 48, pp. 383–385.

Ternary Repeaters

E 505 [1942, 61]. *Proposed by H. T. R. Aude, Colgate University*

How many different proper fractions when written in the ternary scale will be repeaters with not more than three digits in their repetends?

Solution by E. P. Starke, Rutgers University

Assume the desired fractions to be "pure" repeaters, the first occurrence of the repetend being immediately at the point, for otherwise there would be an unlimited number of such fractions. The problem is easy to solve by commencing at either end.

(A) Since the available digits are 0, 1, 2, repeaters with three-digit repe-

tends can be written down in 3^3 ways, from which we must exclude the 3 cases in which the digits are alike. Similarly there are 3^2-3 repeaters with two-digit repetends; but with a single digit in the repetend there is only $.\dot{1}$. Thus there are $24+6+1=31$ such fractions altogether.

(B) If k is the smallest integer such that 3^k-1 is divisible by n, then $1/n$ is a repeater with k digits in the repetend. For $k=1, 2, 3$, we have $n=2, 8, 26$, respectively. Thus we have $1/2$, $1/8$, $1/26$, and such integral multiples as are proper fractions. Finally there are $1+7+25$ such fractions, less 2, because $4/8$ and $13/26$ are duplicates of $1/2$. There are then 31 fractions, as in (A).

Also solved by the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4056. *Proposed by J. R. Musselman, Western Reserve University*

Let the line of images of any point T on the circumcircle of triangle $A_1A_2A_3$ cut the sides A_jA_k in the points A_i'. The perpendiculars to the sides A_jA_k at A_i' form the triangle $B_1B_2B_3$; show that the straight lines A_iB_i meet in T.

4057. *Proposed by J. R. Musselman, Western Reserve University*

Let B_1, B_2, B_3 be the points symmetric to the vertices of triangle $A_1A_2A_3$ in its circumcenter O, and let C_1, C_2, C_3 be the reflections of A_i in the perpendicular bisector of the sides of $A_1A_2A_3$. It is known that the circles OB_1C_1, OB_2C_2, OB_3C_3 meet at a point P. Show that P lies on the Euler line of $A_1A_2A_3$ and that O is the midpoint of PD, where D is the inverse in the circumcircle of the orthocenter H of $A_1A_2A_3$.

4058. *Proposed by R. P. Agnew, Cornell University*

Give an example of a sequence $f_n(x)$ of real continuous functions, defined over $-\infty<x<\infty$ and vanishing outside the interval $-1\leq x\leq 1$, such that the dominating function $F(x)$ defined by

$$F(x) = \underset{n=1,2,3,\cdots}{\text{l.u.b.}} \left| f_n(x) \right| \tag{1}$$

and the inferior and superior limit functions

$$\liminf_{n\to\infty} f_n(x), \qquad \limsup_{n\to\infty} f_n(x) \tag{2}$$

are all continuous and such that the members of the inequality (in which integration is over $-\infty < x < \infty$)

$$\begin{aligned} -\int F(x)dx &\leqq \int \liminf_{n\to\infty} f_n(x)dx \leqq \liminf_{n\to\infty} \int f_n(x)dx \\ &\leqq \limsup_{n\to\infty} \int f_n(x)dx \leqq \int \limsup_{n\to\infty} f_n(x)dx \leqq \int F(x)dx \end{aligned} \tag{3}$$

assume in order the values in the arithmetic progression -5, -3, -1, 1, 3, 5.

Remark: An elegant theorem on "integration of sequences" states that if $f_n(x)$ is a sequence of functions measurable over E_m (Euclidean space of m dimensions) and the dominating function (1) is integrable (Lebesgue), then the functions (2) are integrable and (3) holds. See Carathéodory, *Vorlesungen über Reele Funktionen*, Leipzig, 1927, p. 444. An example meeting the conditions of the problem shows that, even when all functions involved are continuous, the differences between successive members of (3) may all be equal to the constant 2.

4059. *Proposed by V. Thébault, San Sebastián, Spain*

Let D, E, F be the points of contact of the inscribed circle (I) with the sides BC, CA, AB of triangle ABC, and A', B', C' the feet of its altitudes. Show that the distances of the points of intersection of the pairs of straight lines such as $B'C'$, EF from the radical axis of (I) and the nine point circle of triangle ABC are inversely proportional to the distances of the Feuerbach point from the feet of the altitudes.

4060. *Proposed by V. Thébault, San Sebastián, Spain*

If a point P is the orthopole of the three sides of a triangle $A_1B_1C_1$ with respect to another triangle $A_2B_2C_2$ inscribed in the same circle as the first, the product of its distances from the sides of the first triangle is equal to the similar product for the second.

SOLUTIONS

Circumcenters of Associated Tetrahedrons

4001 [1941, 409]. *Proposed by V. Thébault, San Sebastian, Spain*

The spheres (O_1'), (O_2'), (O_3'), (O_4'), symmetric to the circumsphere (O) of the tetrahedron $A_1A_2A_3A_4$ with respect to its faces, intersect in sets of three in the points A_1', A_2', A_3', A_4' distinct from the vertices; and the spheres described on the circumcircles (O_1), (O_2), (O_3), (O_4) of the triangles of the faces as great circles intersect in sets of three in A_1'', A_2'', A_3'', A_4''. Show that: (1) the tetrahedrons $A_1'A_2'A_3'A_4'$ and $A_1''A_2''A_3''A_4''$ are homothetic; (2) the centers of the circumspheres of the tetrahedrons $A_1A_2A_3A_4$, $O_1O_2O_3O_4$, $O_1'O_2'O_3'O_4'$, $A_1'A_2'A_3'A_4'$, $A_1''A_2''A_3''A_4''$ are collinear.

Editorial Note. The proposer gave indications of a solution as follows: The radical center Ω of the four spheres (O_1'), (O_2'), (O_3'), (O_4') coincides with the radical center of the second set of spheres on (O_1), (O_2), (O_3), (O_4); and Ω is the isogonal conjugate of the circumcenter O of $T=A_1A_2A_3A_4$ with respect to T. Also the points A_i' and A_i'' are the inverses of A_i in two inversions with the same pole Ω. From this it follows that the two tetrahedrons $T'=A_1'A_2'A_3'A_4'$ $T''=A_1''A_2''A_3''A_4''$ are homothetic, the center being Ω. (2). The circumcenters of T, T', $O_1'O_2'O_3'O_4'$ are collinear, *etc.* see V. Thébault, *Mathesis*, 1922, p. 363.

The proofs of these statements follow easily from theorems in Court's *Modern Pure Solid Geometry*, p. 246, 756, and p. 244, 752. Thus the three spheres, (O_2'), (O_3'), (O_4') are orthogonal to a sphere (Ω, r_1) with center Ω and radius r_1, intersect in A_1 and A_1' which are inverses with respect to this sphere, and hence $\Omega A_1 \cdot \Omega A_1' = r_1^2$. Similarly, $\Omega A_1 \cdot \Omega A_1'' = r_2^2$, and then $\Omega A_1'/\Omega A_1'' = r_1^2/r_2^2$, and the proof of (1) follows.

Since the circumspheres (T'), (T) are inverses in the first inversion, and (T''), (T) are inverses in the second, the three centers are collinear with Ω. We show next that the circumcenters of the tetrahedrons $O_1O_2O_3O_4$, $O_1'O_2'O_3'O_4'$ lie also on $O\Omega$ and this will conclude the proof of (2). Let N be the midpoint of $O\Omega$, and O_i' the center of (O_i'). Then $OO_i' = 2OO_i$, and N is the center of the common pedal sphere of the isogonal conjugates O and Ω, with respect to T, with the radius NO_i; and Ω is the center of the sphere $(O_1'O_2'O_3'O_4')$ with the radius $\Omega O_i' = 2NO_i$.

A Number Theory Function

4002 [1941, 483]. *Proposed by F. A. Lewis, University of Alabama*

Give an interpretation to the function that results from the Euler ϕ-function when the minus signs are changed to plus, namely $f(n) = n(1+1/p_1)(1+1/p_2) \cdots (1+1/p_k)$.

I. *Solution by E. P. Starke, Rutgers University*

Let $p_1, p_2, \cdots, p_k$ be the distinct prime divisors of n and set $m = n/p_1p_2 \cdots p_k$, where a_i is the exponent of p_i in n. Then $f(n)$ is the sum of all numbers which are simultaneously multiples of m and divisors of n. In other words: Let S be any divisor of n which is not divisible by a square, and let T be its complementary factor $(S \cdot T = n)$; then $f(n)$ is the sum of all numbers T. Of course, if $a_1 = a_2 = \cdots = a_k = 1$, $f(n)$ is the sum of all divisors of n. These statements follow immediately from two evident facts:

$$f(p_i^{a_i}) = p_i^{a_i} + p_i^{a_i-1}, \tag{1}$$

$$f(xy) = f(x) \cdot f(y), \tag{2}$$

where x, y are relatively prime.

II. *Solution by the Proposer*

The required function is formed when $\phi_2(n)$ is divided by $\phi(n)$. Since $\phi_2(n)$

represents the number of elements of period n in an Abelian group of order n^2 and type (1, 1), the function formed represents the number of cyclic subgroups of order n in an Abelian group of order n^2 and type (1, 1).

Note by D. M. Seward, University of Tenn.

Referring to Dickson's *History of the Theory of Numbers*, Vol. 1, p. 123, we find: "R. Dedekind proved that, if n be decomposed in every way into a product ad, and if e is the g.c.d. of a, d, then

$$\sum a/e\phi(e) = n\prod (1 + 1/p) = f(n),$$

where a ranges over all divisors of n, and p over the prime divisors of n."

We shall use Jordan's generalization of Euler's ϕ-function, $J_k(n)$, p. 147. By definition, $J_k(n)$ is the number of different sets of k (equal or distinct) positive integers $\leqq n$, whose g.c.d. is prime to n. The formula is given

$$J_k(n) = n^k(1 - 1/p_1^k) \cdots (1 - 1/p_q^k).$$

We note that $f(n) = J_2(n)/\phi(n)$, $J_2(n) = \phi(n)\sum a/e\phi(e)$.

Letting $f_k(n) = n^k(1+1/p_1^k) \cdots (1+1/p_q^k)$, we have $f_k(n) = J_{2k}(n)/J_k(n)$.

It is noted (p. 150) that J. Vályi used $f(n)$ in his enumeration of the n-fold perspective polygons of n sides inscribed in a cubic curve.

Editorial Note. After the appearance of this problem in print the proposer discovered that his interpretation is given in Fricke's *Die Elliptischen Functionen*, vol. 2, p. 120. In the above mentioned volume of Dickson it is also stated on page 155 that G. A. Miller evaluated $J_k(m)$ by noting that it is the number of operators of period m in the abelian group with k independent generators of period m. This leads to the proposer's interpretation for $k=2$.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to C. O. Oakley, Haverford College, Haverford, Pennsylvania.

Dr. H. N. Wright, formerly professor of mathematics at the College of the City of New York, was installed as president of the college on September 30, 1942. Professor H. F. MacNeish represented the Mathematical Association on this occasion.

Professor Virgil Snyder of Cornell University was one of four graduates of Iowa State College to receive the Chicago Merit Award in June 1942. The awards are presented by Chicago alumni of the college.

One of the most difficult tasks in library reconstruction after the first World War was that of completing foreign institutional sets of American scholarly, scientific, and technical periodicals. The attempt to avoid a duplication of that situation is now the concern of the Committee on Aid to Libraries in War Areas, headed by J. R. Russell, the Librarian of the University of Rochester. In a recent mimeographed statement the Committee appeals to subscribers to scientific journals to save old periodicals. Questions concerning the project should be sent to W. M. Hartwell, Executive Assistant to the Committee, University of Rochester.

The National Research Council, Washington, D. C., has issued the fourth edition of a *Handbook of Scientific and Technical Societies and Institutions of the United States and Canada* (N.R.C. Bulletin No. 106, January, 1942; 389 pp).

Illinois Institute of Technology has made the following appointments: Professor Max Dehn of the University of Idaho to a visiting lectureship, Dr. M. H. Heins to an assistant professorship, and Dr. W. B. Caton and Dr. G. W. Mackey to instructorships. Associate Professor Rufus Oldenburger has been promoted to a professorship.

At Louisiana State University: Appointments to assistant professorships include Dr. O. G. Harrold, Jr.; and Dr. P. A. White. The following members of the staff are on leave: Associate Professor R. C. Yates, U. S. Military Academy, Captain; Assistant Professor P. C. Scott, Army Air Corps, First Lieutenant; Dr. Nelson Robinson, U. S. Navy, Lieutenant (j.g.); Dr. A. B. Carson, Army Air Corps, Second Lieutenant.

From the University of New Mexico: Dr. C. B. Barker has been made an assistant professor. Assistant Professor H. D. Larsen has been promoted to an associate professorship. Dr. Arthur Rosenthal, lecturer during the second semester of the past year, has been given the status of professor.

Dr. J. M. Dobbie and Dr. J. W. Givens of Northwestern University have been promoted to assistant professorships. Dr. T. F. Holgate, who retired from active deanship in 1937, has accepted a lectureship because of the war emergency and is teaching during the fall quarter at Northwestern University.

From Oklahoma Agricultural and Mechanical College: Professor A. H. Diamond, Associate Professor H. S. Mendenhall and Assistant Professor C. E. Marshall have entered the Army Air Forces as second lieutenants. Associate Professor O. H. Hamilton is a lieutenant in the Naval Reserve. Assistant Professor Herbert Scholz and Dr. Paul E. Lewis are civilian instructors in the local Naval Radar Training School. Mr. Patrick Butler has been appointed to a teaching fellowship at the University of Texas. Associate Professor J. H. Zant has been promoted to a professorship and made acting head of the department.

From Park College, Missouri, come the following items: Dr. H. E. Crull,

chairman of the department of mathematics and astronomy, has been granted leave for the duration and is a Lieutenant (j.g.) in the U. S. Naval Reserve. Dr. C. A. Messick of Oakland City College, Indiana, has been appointed in Lt. Crull's place. Associate Professor L. A. Robbins has been granted leave of absence and is now with Pratt-Whitney Aircraft Corporation. Helen K. Milleson of Buena Vista College has been appointed to an assistant professorship.

From Pennsylvania State College: Professor C. H. Graves is on leave of absence serving as assistant educational statistician in the Office of Education in the Federal Security Agency, Washington, D. C. Professor H. B. Curry is on leave serving as Associate Mathematician at the Frankford Arsenal, Philadelphia. Dr. R. H. Cook has been appointed an assistant professor at the South Dakota State School of Mines.

Kaj Nielson of Louisiana State University, and Alben Nordling and C. H. W. Sedgwick of the University of Connecticut, worked in the research engineering department of Pratt and Whitney Aircraft the past summer. Professor F. C. Jonah of Western Reserve University also spent the summer in this group and is now on leave from the University as a research engineer for the Vought-Sikorsky division of United Aircraft.

Dr. O. P. Akers, professor of mathematics and surveying at Allegheny College, has retired after thirty-seven years of teaching there.

Dr. Fred Assadourian of New York University has been appointed an associate professor at the Texas Technological College.

O. E. Bennett has been appointed an assistant professor at Washington College, Chestertown, Maryland.

Professor R. E. Bruce, chairman of the department of mathematics, Boston University, became professor emeritus on July 1 after thirty-nine years of service. Because of the present emergency Professor Bruce will continue in active service.

C. L. Buxton of Clarkson College of Technology has been promoted to an assistant professorship.

W. B. Campbell, who has been serving as engineer in the Philadelphia Ordnance District, has been appointed an associate professor of engineering at the Pennsylvania Military College, Chester.

Dr. J. L. Dorroh of Ouachita College has been appointed an assistant Professor at Louisiana State University.

Professor H. J. Ettlinger of the University of Texas represented the Mathematical Association at the inauguration of President J. N. R. Score at Southwestern University on October 6.

H. W. Eves of the mathematics staff of the Tennessee Valley Authority has been appointed assistant professor of applied mathematics in the College of Applied Sciences of Syracuse University.

Professor Emeritus J. G. Hardy of Williams College is teaching for two semesters at Reed College.

Associate Professor C. T. Hazard of Purdue University has been promoted to a professorship.

Dr. C. C. Hurd has been granted leave of absence from Michigan State College to teach in the U. S. Coast Guard Academy.

Assistant Professor H. S. Kaltenborn of Louisiana Polytechnic Institute has been promoted to an associate professorship.

Associate Professor E. C. Kennedy of Texas College of Arts and Industries is now a first lieutenant in the Army Air Corps.

Dr. D. E. Kibbey of Michigan State College has accepted a commission to teach at West Point.

Associate Professor V. S. Lawrence of Cornell University is now instructing in mathematics at the U. S. Military Academy, West Point, with the rank of major.

Dr. H. L. Lee of the University of Tennessee has been promoted to an assistant professorship.

At the University of Oregon, Ingo Maddaus, Jr., has been made an assistant professor.

Assistant Professor F. H. Miller of the Cooper Union School of Engineering has been promoted to an associate professorship.

Dr. E. N. Nilson has been appointed an assistant professor at Mount Holyoke College.

Dr. M. G. Pawley of the Colorado School of Mines is now a radio engineer in the Naval Research Laboratory at Washington.

Dr. D. R. Shreve of Purdue University has been appointed a stress analyst at the McDonnell Aircraft Corporation, St. Louis.

Dr. M. M. Slotnick has been appointed supervisor at the Bureau of Ordnance Navy Department, Washington.

Associate Professor D. E. South of the University of Kentucky is on leave of absence.

Dr. E. R. Stabler has been appointed an assistant professor at Hofstra College.

J. A. Straw of the University of Louisville has been appointed an assistant professor at Rose Polytechnic Institute.

Dr. S. Helen Taylor of Ashland, Kentucky, Junior College has been appointed adjunct professor at the University of South Carolina, the first woman to be appointed to any position in the department of mathematics since the founding of the University over 141 years ago.

Dr. D. L. Webb of Georgia School of Technology has been appointed an assistant professor at the Texas Technological College.

Dr. André Weil of the New School for Social Research and Haverford College has been appointed an assistant professor at Lehigh University.

Professor Evelyn P. Wiggin of Randolph-Macon Woman's College is on leave of absence and has been appointed to a lectureship at Wellesley.

Dr. Clyde Wolfe has been appointed Civilian Instructor, Academic Training Department, at the Santa Ana, California, Army Air Base.

The following appointments to instructorships are announced:

A. and M. College of Texas: A. R. Wapple

University of Arkansas: R. V. Simpson

Case School of Applied Science: Dr. L. J. Green

Cornell University: R. L. Beinert, C. D. Firestone, R. L. Hull, Dr. Fred Kiokemeister, Dr. L. J. Savage, A. R. Turquette

University of Delaware: G. L. Walker

Hamilton College: L. B. Williams

Iowa State College: J. W. Beach

John Carroll University: Dr. P. H. Anderson

Louisiana State University: Dr. K. L. Nielsen, J. E. Pryor, J. C. Stewart.

Miami University : Alberta Wolfe

Michigan College of Mining and Technology: J. C. Butler, T. R. Richards, Earl Roberts

Michigan State College: Dr. W. L. Mitchell, Dr. L. V. Toralballa, Elaine Van Aken, K. C. Walters.

Michigan State Normal College: Dr. Edith Schnedkenburger

Muskingum College: L. C. Knight, Jr.

Northwestern University: Mrs. Louise Marvel Sagen, R. S. Wolfe and on part-time: Dr. Walter Cerf, Mary M. Handel, R. H. Lence

Rutgers University, College of Engineering: R. B. Kleinschmidt

Smith College: Anne Frances O'Neill

Texas Technological College: Mrs. Annie N. Rowland

The name of Professor J. F. Barnhill was, through an oversight, omitted from the necrology of members in the March issue of the MONTHLY. Professor Barnhill, who died May 5, 1941, became a member of the Association in 1920 and had taught at Michigan State Normal College since 1922.

Eleanor C. Doak, professor emeritus at Mount Holyoke College, died on August 27, 1942 at the age of seventy-two. She had been a member of the Mathematical Association since 1916.

Professor Emeritus W. A. Harshbarger of Washburn Municipal University of Topeka died on July 17, 1942. He had been connected with the University for fifty years, and was a charter member of the Mathematical Association.

Professor Byron Ingold of Culver-Stockton College died March 23, 1942. He had been a member of the Mathematical Association since 1927.

Professor Emeritus D. A. Lehman of Goshen College died September 8, 1942. He was a charter member of the Mathematical Association and had been professor at Goshen since 1906.

Dr. E. W. Miller, associate professor of mathematics at the University of Michigan, died on July 23, 1942, at the age of thirty-seven.

D. T. Petty, head of the department of mathematics in the F. W. Parker School, a member of the Mathematical Association, died April 16, 1942.

Dr. T. M. Putnam, professor emeritus and dean of undergraduates at the University of California, died September 22, 1942. He was a charter member of the Mathematical Association.

Professor W. B. Robinson, since 1913 head of the department of mathematics at the American International College, Springfield, Mass., died July 19, 1942, at the age of fifty-three.

OCCUPATIONAL CLASSIFICATION OF MATHEMATICIANS

General Lewis B. Hershey, Director of Selective Service, has issued Occupational Bulletin No. 23 which certifies that the educational services of instructors and professors of mathematics in colleges, universities, or professional schools are activities essential to the war effort. This Bulletin supplements Occupational Bulletin No. 10 which was distributed to department chairmen by the office of the Secretary of the American Mathematical Society on June 30, 1942. Bulletin No. 10 and Bulletin No. 23 are given at the end of this note.

This memorandum is published with the realization that the whole situation with respect to our colleges and universities will undoubtedly be fundamentally altered by the legislation concerned with the lowering of the draft age. Nevertheless, we do believe that plans will be developed whereby the government will use the machinery already existing in our colleges and universities in order to train young men in the mathematics and science which are the necessary equipment of Army and Navy officers. The need for instructors in mathematics is bound to increase as the tempo of the war is increased.

The Secretary of War has stated that teachers in fields that are essential to the war effort are doing the job that the country wants them to do and are performing their full duty in the war effort. With this in mind, every effort should be made to conserve the supply of mathematicians; this supply is already greatly depleted. Mathematicians should not be discouraged from assuming positions in fields other than teaching, if these offer greater opportunities for service to the nation in the war effort. On the other hand, mathematics teachers should not feel that their services will shortly be unnecessary and consequently rush into positions where their contribution is less vital than that of teaching men who are preparing for commissions in the armed forces. Mathematics at the college level will have to be taught to prospective officers. We hope that this teaching will be done in our colleges and universities. Whether it is done there or in schools set up by the government, mathematicians will be needed and the supply should be conserved.

1. Requests for occupational classification. In view of the recent Bulletin, we wish to offer the following suggestions for securing the proper classification of mathematicians:

(a) *For those who have completed their training, i.e., instructors and men of professorial grade.* Requests for continued occupational deferment or for reclassification should be based on Occupational Bulletin No. 23 in which the Manpower Commission has certified as essential to the war effort professors and instructors engaged in full-time instruction and research in mathematics. This Bulletin should be attached to every request. In addition, Occupational Bulletin No. 10, which certifies that there is a serious shortage of mathematicians, should be attached. Local boards must be supplied with all the necessary information; this should include evidence that the registrant cannot be replaced. The request for reclassification should be made on Form 42A which can be obtained from the registrant's local board.

(b) *For those in training, i.e., graduate assistants and undergraduate majors in mathematics.* The officer requesting the deferment should file Form 42A directly with the local board with the ordinary questionnaire filled out by all registrants. Reference should be made to Occupational Bulletin No. 10. It is urged that a copy of this Bulletin be attached to Form 42A when it is filed.

2. Appeals. Having made a request for occupational deferment, the person making the request should be prepared to exert every effort to secure that deferment, including the making of an appeal. The appeal may be made by the registrant, his employer, or the government appeal agent. When it is clear that the end of negotiations with the local board has been reached, without having modified the registrant's IA classification, the appeal should be carried through. *Such an appeal must be made within ten days of the date when the local board mailed to the registrant a notice of his classification.* The form for appeals appears at the end of the questionnaire.

3. Additional Steps.

National Roster of Scientific and Specialized Personnel,
10th and U Streets, N.W., Washington, D. C.

Any mathematician who has not already done so should register immediately with the National Roster. This may be done by writing a letter to the Roster at the above address, and asking for appropriate forms in mathematics. In this communication, the writer should indicate whether he is subject to the Selective Training and Service Act of 1940.

The National Roster may take appropriate action in the case of professional mathematicians and graduate students in mathematics. Under a cooperative plan with the Selective Service System, the Roster may send to the National Headquarters of the Selective Service System appropriate information about professionally trained men of military age, and that office in turn may then forward letters about these men through the various State Directors to the individual local boards, in order to assist in proper classification.

It is essential that a man (or his employing institution) should *not* wait until he is placed in Class IA before writing to the Roster. The Roster should be kept informed as to any changes in his Selective Service status or in his employment. In communicating with the Roster, a man should give the following information: (1) the name, number, and address of his Selective Service local board; (2) his own order number; (3) a description of his specific duties, especially as related to the war effort.

Office of Scientific Personnel, National Research Council,
2101 Constitution Avenue, Washington, D. C.

Dean Homer L. Dodge is now Director of the Office of Scientific Personnel. In cases where an appeal has been made and developments indicate that a proper decision may not be reached, it is advisable to contact Dean Dodge. He must be informed as to the applicant's order number and the name, number, and address of the local board under which he is registered. Dean Dodge has offered to consult with the State Director concerned or with Selective Service Headquarters as the urgency of the case demands.

However, the Office of Scientific Personnel must start where the person making the original request leaves off and can do little to make up for deficiencies in the presentation of the case by the employer. Unless there is presented evidence that a genuine, but unsuccessful, attempt has been made to secure a replacement for the particular registrant in question, it will be impossible for the Office of Scientific Personnel to be of assistance.

Copies of Occupational Bulletin No. 10 and of Occupational Bulletin No. 23 will be furnished, on request, by the office of the Secretary of the American Mathematical Society.

MARSTON MORSE, *President American Mathematical Society,*
J. R. KLINE, *Secretary.*

Philadelphia, Pa.
November 2, 1942.

OCCUPATIONAL BULLETIN NO. 10

SUBJECT: SCIENTIFIC AND SPECIALIZED PERSONNEL June 18, 1942

PART I

1. There are certain persons trained, qualified, or skilled in scientific and specialized fields who, if engaged in the practice of their respective professions, are in a position to perform a vital service in activities necessary to war production and in activities essential to the support of the war effort.

PART II

1. The National Roster of Scientific and Specialized Personnel has certified to the Director of Selective Service that in activities necessary to war production and in activities essential to the support of the war effort, there are certain "critical occupations" which for the proper discharge of the duties involved require a high degree of training, qualification, or skill in scientific and specialized fields. The critical occupations in these scientific and specialized fields, as certified to the Director of Selective Service, are attached to this bulletin.

2. All of these critical occupations, as listed, require highly specialized periods of training of two years or more. The critical occupations on the attached list exist within the provisions of Part V, Memorandum to All State Directors (I-405).

PART III

1. The National Roster of Scientific and Specialized Personnel has certified to the Director of Selective Service that there are serious shortages of persons trained, qualified, or skilled to engage in these critical occupations in activities necessary to war production and in activities essential to the support of the war effort. These shortages exist within the provisions of Part VII, Memorandum to All State Directors (I-405), and accordingly careful consideration for occupational classification should be given to all persons trained, qualified, or skilled in these critical occupations and who are engaged in activities necessary to war production or essential to the support of the war effort.

PART IV

1. There are many registrants who are in training and preparation to acquire the qualification or skill to engage in these critical occupations. Normally the period of training and preparation to acquire the necessary qualification or skill in these scientific and specialized fields extends over a period of four academic years in a recognized academic, professional, or technical college or university. In many instances, however, it is necessary for persons to have additional study in a recognized academic, professional, or technical college or university in order to acquire the more highly specialized qualification or skill necessary for the performance of particular services in activities necessary to war production or essential to the support of the war effort, Persons engaging in further studies in addition to the four academic years normally required are referred to as graduate or postgraduate students.

2. A registrant who is in training and preparation for one of these scientific and specialized fields may not be considered for occupational classification until the close, or approximately the close, of his second or sophomore year in a recognized college or university.

3. A registrant who is in training and preparation for one of these scientific and specialized fields may be considered for occupational deferment at the close, or approximately at the close, of his second or sophomore year in a recognized college or university if he is pursuing a course of study upon the successful completion of which he will have acquired the necessary training, qualification, or skill, and if he gives promise of continuing and will be acceptable for continuing such course of study and will undertake actual further classroom work within a period of not to exceed four months from the close of his second year.

4. A registrant who is in training and preparation for one of these scientific and specialized fields shall be considered for occupational classification during his third and fourth years in a recognized college or university, provided that he gives promise of the successful completion of such course of study and the acquiring of the necessary degree of training, qualification, or skill.

5. A graduate or postgraduate student who is undertaking further studies for these scientific and specialized fields, following the completion of the normal four academic years, may be considered for occupational classification if, in addition to pursuing the additional studies, he is also acting as "graduate assistant" in a recognized college or university or is engaged in scientific research related to the war effort and which is supervised by a recognized Federal agency. A graduate assistant is a student in postgraduate studies who, in addition, is engaged in the teaching and instruction of undergraduate students in these scientific and specialized fields.

6. When a registrant has completed his training and preparation in a recognized college or university and has acquired a high degree of training, qualification, or skill in one of these scientific and specialized fields, such registrant should then be given the opportunity to become engaged in the practice of his profession in an activity necessary to war production or essential to the support of the war effort. In many instances following graduation from a recognized college or university, a certain period of time will be required in the placing of trained, qualified, or skilled personnel in an essential activity. When a registrant has been deferred as a necessary man in order to complete his training and preparation, it is only logical that his deferment should continue until he has an opportunity to use his scientific and specialized training to the best interest of the nation. Accordingly, following graduation from a recognized college or university in any of these scientific and specialized fields, a registrant should be considered for further occupational classification for a period of not to exceed 60 days in order that he may have an opportunity to engage in a critical occupation in an activity necessary to war production or essential to the support of the war effort, provided that during such period the registrant is making an honest and diligent effort to become so engaged.

LEWIS B. HERSHEY
Director of Selective Service

21st Street and C Street, N.W.,
Washington, D. C.

CRITICAL OCCUPATIONS.—SCIENTIFIC AND SPECIALIZED PERSONNEL

- Accountants
- Chemists
- Economists
- Engineers:
  - Aeronautical Engineers
  - Automotive Engineers
  - Chemical Engineers
  - Civil Engineers
  - Electrical Engineers
  - Heating, Ventilating, Refrigerating, and Air Conditioning Engineers
  - Marine Engineers
  - Mechanical Engineers
  - Mining and Metallurgical Engineers, including Mineral Technologists
  - Radio Engineers
  - Safety Engineers
  - Transportation Engineers—Air, Highway, Railroad, Water
- Geophysicists
- Industrial Managers
- Mathematicians
- Meteorologists
- Naval Architects
- Personnel Administrators
- Physicists, including Astronomers
- Psychologists
- Statisticians

OCCUPATIONAL BULLETIN NO. 23

SUBJECT: EDUCATIONAL SERVICES September 30, 1942

1. The War Manpower Commission has certified that educational services are essential to the support of the war effort.

2. This bulletin covers the following essential activities which are considered as included within the list attached to Local Board Release No. 115, as amended:

(a) *Educational services*: Public and private industrial vocational training; elementary, secondary and preparatory schools; junior colleges, colleges, universities and professional schools; educational and scientific research agencies; and the production of technical and vocational training films.

3. In considering registrants engaged in educational services there must be taken into consideration the following:

(a) The kind of institution in which the registrant is engaged;

(b) the occupation of the registrant in that institution; and

(c) the classroom studies under the registrant's instruction, supervision, or administration jurisdiction. Attached is a list of occupations by institutions and classroom studies in educational services which require a resonable degree of training, qualification, or skill to perform the duties involved. It is the purpose of this list to set forth by institutions and classroom studies the important occupations in educational services which must be filled by persons capable of performing the duties involved in order that the essential portions of the activity may be maintained. Item 4 of the list does not include classroom studies but occupations which shall be considered in the same manner as any other occupations. The entire list is confined to occupations which require more than six months of training and preparation.

4. In classifying registrants employed in these activities, consideration should be given to the following:

(a) The training, qualification, or skill required for the proper discharge of the duties involved in his occupation;

(b) the training, qualification, or skill of the registrant to engage in his occupation; and

(c) the availability of persons with his qualifications or skill, or who can be trained to his qualification, to replace the registrant and the time in which such replacement can be made.

LEWIS B. HERSHEY, *Director*

CRITICAL OCCUPATIONS—EDUCATIONAL SERVICES

1. ELEMENTARY, SECONDARY AND PREPARATORY SCHOOLS

(a) Superintendents of elementary, secondary and preparatory school systems; and

(b) teachers who are engaged in full-time instruction in one or more of the following subjects:

Aeronautics	Mathematics
Biology	Physics
Chemistry	Radio

2. JUNIOR COLLEGES, COLLEGES, UNIVERSITIES AND PROFESSIONAL SCHOOLS, EDUCATIONAL AND SCIENTIFIC RESEARCH AGENCIES

(a) Presidents, Deans, and Registrars in junior colleges, colleges, universities and professional schools; and

(b) professors and instructors engaged in full-time instruction and research in one or more of the following subjects:

Agricultural Sciences	Engineering Sciences	Navigation, Aerial and Marine
Architecture, Naval	Geology	Oceanography
Astronomy	Industrial Management	Pharmacy
Bacteriology	Mathematics	Physics
Biology	Medicine and Surgery	Physiology
Chemistry	Metallurgy	Veterinary Sciences
Dentistry	Meteorology	

3. Public and Private Industrial Vocational Training

(a) Superintendents of public and private industrial vocational training; and
(b) teachers who are engaged in full-time instruction in one or more of the following subjects designated to prepare students for war activities:
Trade, Vocational and Agricultural subjects (such as, Machine Shop practice, Aircraft, Sheetmetal Work, and similar subjects) and in Vocational Rehabilitation.

4. Production of Technical and Vocational Training Films

(a) Persons engaged full-time and exclusively in the production of technical and vocational training films for the Army, Navy and war production industries.

Animator	Project Supervisor
Cameraman	Technical Consultant
Cutter	Technical Writer
Film Editor	Sound Engineer

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-seventh Annual Meeting, New York, N. Y., December 30–31, 1942.

The following is a list of the Sections of the Associations, with dates of future meetings so far as they have been reported to the Secretary.

Allegheny Mountain
Illinois, Notre Dame, Ind., April 9–10, 1943
Indiana, Notre Dame, April 9–10, 1943
Iowa
Kansas
Kentucky
Louisiana-Mississippi, Ruston, La., 1943
Maryland-District of Columbia-Virginia, Baltimore, Dec. 5, 1942
Metropolitan New York
Michigan, Notre Dame, Ind., April 9–10, 1943
Minnesota
Missouri, Kansas City, Dec. 4, 1942
Nebraska
Northern California, San Francisco, Jan. 30, 1943
Ohio, Columbus, April 1, 1943
Oklahoma
Philadelphia, Philadelphia, Nov. 28, 1942
Rocky Mountain
Southeastern
Southern California, Los Angeles, March 13, 1943
Southwestern
Texas, Lubbock, April, 1943
Upper New York State
Wisconsin, Milwaukee, May 7, 1943

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THE AMERICAN MATHEMATICAL MONTHLY

By W. M. DAVIS, Cornell College

REPORTS OF THE MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Edited by J. R. MUSSELMAN, Western Reserve University

MEETINGS OF THE ASSOCIATION

MEETINGS OF ITS SECTIONS

GENERAL MATHEMATICAL PAPERS

DISCUSSIONS AND NOTES

Edited by R. J. WALKER, Cornell University

RECENT PUBLICATIONS

Edited by VIRGIL SNYDER, Cornell University

NEW BOOKS RECEIVED

REVIEWS

Names of authors are in ordinary type, those of reviewers in capitals.

CLUBS AND ALLIED ACTIVITIES

Edited by E. H. C. HILDEBRANDT, New Jersey State Teachers College and J. S. FRAME, Allegheny College

TOPICS

ACTIVITIES

PROBLEMS AND SOLUTIONS

Edited by OTTO DUNKEL, Washington University, H. S. M. COXETER, University of Toronto, and ORRIN FRINK, JR., Pennsylvania State College

AUTHORS

Numbers refer to pages, black face type indicating a problem solved and solution published; italics, a problem solved, but the solution not published; ordinary type, a problem proposed.

SOLUTIONS

Numbers in black face type refer to problems; those in light face to pages.

NEWS AND NOTICES

Edited by C. O. OAKLEY, Haverford College

PERSONAL MENTION

This section contains the names of persons taking an active part in meetings, newly elected members, officers of the Association and of the various Sections, and persons mentioned in the department of News and Notices.

NECROLOGY

THE AMERICAN

MATHEMATICAL MONTHLY

DEVOTED TO THE INTERESTS OF

COLLEGIATE MATHEMATICS

VOLUME 49 NUMBER 10

CONTENTS

DECEMBER 1942

The AMERICAN MATHEMATICAL MONTHLY

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MATHEMATICAL ASSOCIATION OF AMERICA, INC.

THIS MONTHLY WAS FOUNDED IN 1894 BY BENJAMIN F. FINKEL

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, L. R. FORD, Illinois Institute of Technology, 3300 Federal St., Chicago, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association (for 1943), W. B. CARVER, 450 Ahnaip Street, Menasha, Wisconsin, or McGraw Hall, Cornell University, Ithaca, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER (for 1943), W. B. CARVER, McGraw Hall, Cornell University, Ithaca, N. Y., to reach him five weeks before the change becomes effective.

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DERIVATIVES IN THE CALCULUS

A. E. TAYLOR, University of California, Los Angeles

1. Introduction. The use of derivatives, especially those of the first and second order, finds an important place in our calculus books in connection with the study of curve tracing. Such topics as extreme values, the sense of concavity of a curve, and points of inflection, are usually emphasized at an early stage of the work. These matters make relatively light demands upon the student's knowledge of limiting processes, especially if he is content to rely upon an intuitive understanding of the following proposition: *a function is increasing if its derivative is positive, and decreasing if its derivative is negative.* Without either adequate definitions or proofs, the foregoing proposition may be used to make highly plausible the usual tests, involving first and second derivatives, for extreme values, sense of concavity, and points of inflection on a curve $y=f(x)$. But most students, and, I suspect, a good many teachers, would experience considerable difficulty if they were pressed to give precise statements and proofs for the traditional theorems in this part of the calculus. There are several stumbling blocks. Just what is the meaning of the statement that a curve is concave upwards? How is a point of inflection defined? To what extent is it necessary for first and second derivatives to be continuous, or even to exist? There are some vexing questions about tangents parallel to the y-axis, and the implications about the derivative.

An elaborate treatment of these matters in elementary calculus texts would be out of place. The substance of this paper is not put forward with the suggestion that it be incorporated with a first course in calculus, but rather with the idea that instructors would welcome a rigorous presentation of the material which they are obliged to teach with a minimum of formal proofs. It is my hope that it will be of help in illuminating the possibilities of a simple and precise discussion of the problems mentioned above.

The theorems and proofs here set forth are, for the most part, not new; many of them, however, are not well known. So far as I have been able to determine, theorems 5.3 and 5.4 have not appeared elsewhere.

Throughout the paper f denotes a single-valued, real function of the real variable x. For the open interval $a<x<b$ we use the notation (a, b). The corresponding closed interval is denoted by $[a, b]$. By a neighborhood of $x=a$ is meant an open interval containing $x=a$. The statement "$f'(a)$ exists" means that the derivative exists as a finite limit. If

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = +\infty,$$

we say that f has the infinite derivative $f'(a)=+\infty$ at $x=a$. A similar definition is laid down to give meaning to the equation $f'(a)=-\infty$, When the symbols $+\infty$, $-\infty$ are used in inequalities, we observe the usual custom of writing $-\infty<x<+\infty$ for any real number x.

The law of the mean will be a fundamental tool in many parts of the paper; we shall therefore state it formally here, without proof.

1.1 (THE LAW OF THE MEAN). *Let f be defined and continuous on $[a, b]$. Let f possess a finite or infinite derivative at each point of (a, b). Then there exists a number X such that $a < X < b$ and*

$$f(b) - f(a) = (b - a)f'(X). \tag{1.11}$$

One of the important consequences of this theorem is that a function is constant throughout any interval on which its derivative is identically zero.

2. Relative extrema. Increasing functions.

DEFINITION 2.1. Let f be defined in a neighborhood of $x = a$, and suppose that, within a sufficiently small neighborhood of $x = a$, $f(x) - f(a)$ does not change sign. Then f is said to have a relative extreme at $x = a$. In particular, f is said to have a relative maximum if $f(x) \leqq f(a)$, and a relative minimum if $f(x) \geqq f(a)$.

In calculus we are usually concerned with relative extrema such that $f(x) < f(a)$ or $f(x) > f(a)$ except when $x = a$. When the situation $f(x) = f(a)$, $x \neq a$, is thus excluded, the extreme is said to be *proper*.

We shall develop tests for relative extrema, using the first derivative. We being by exploring the implications of a non-vanishing derivative.

2.11. *Let f be defined in a neighborhood of $x = a$, and suppose that f has a finite or infinite derivative at $x = a$, such that $f'(a) \neq 0$. Then, within a sufficiently small neighborhood of $x = a$, the inequalities $x_1 < a < x_2$ imply the inequalities*

$$f(x_1) < f(a) < f(x_2) \quad \text{if} \quad f'(a) > 0,$$
$$f(x_1) > f(a) > f(x_2) \quad \text{if} \quad f'(a) < 0.$$

The proof follows from the observation that the difference quotient

$$\frac{f(a + h) - f(a)}{h}$$

has the same sign as $f'(a)$, provided that $|h|$ is sufficiently small.

As a consequence of theorem 2.11 we have:

2.12. *Let f be defined in a neighborhood of $x = a$, and suppose that f has a finite or infinite derivative at $x = a$. Then $f'(a) = 0$ is a necessary condition for f to have a relative extreme at $x = a$.*

DEFINITION 2.2. Let f be defined on an interval, open or closed, of the x-axis. If $x_1 < x_2$ implies that $f(x_1) < f(x_2)$, f is said to be increasing on the interval. If $x_1 < x_2$ implies that $f(x_1) > f(x_2)$, f is said to be decreasing on the interval.

It should be noted that $-f$ is increasing if and only if f is decreasing.

It may be inferred immediately from theorem 2.11 that an increasing function cannot have a negative derivative.

2.21. *Suppose that f is defined and continuous on $[a, b]$, and that it possesses a finite or infinite derivative at each point of (a, b). Furthermore, suppose that in (a, b) $f'(x) \geqq 0$, and that $f'(x)$ does not vanish identically over any subinterval. Then f is increasing on $[a, b]$.*

It suffices to prove that $f(a) < f(b)$, for the same argument can then be applied to any subinterval. If $a < x < b$, the law of the mean, together with the hypothesis on f', implies the existence of values x_1, x_2 such that $a < x_1 < x < x_2 < b$, and

$$f(x) - f(a) = (x - a)f'(x_1) \geqq 0,$$
$$f(b) - f(x) = (b - x)f'(x_2) \geqq 0.$$

Therefore $f(a) \leqq f(x) \leqq f(b)$. Necessarily, then, $f(a) < f(b)$, for otherwise $f(x)$ would be constant, and f' would vanish identically.*

This theorem (although not necessarily in this very general form) is crucial for a number of later tests. It is unfortunate that its proof depends upon the law of the mean, for the latter theorem does not ordinarily come into a course in calculus early enough to be of use in this connection. However, students will be convinced on the basis of theorem 2.11 that f is increasing in any interval through which $f'(x) > 0$, and it may be well to omit a formal proof unless it is demanded. The next theorem, in conjunction with theorem 2.11, supplies such a proof without recourse to the law of the mean. It is not suited to the needs of a beginner, however.

2.22. *Let f be defined and continuous on (a, b), and suppose that to each point x_0 of the interval there corresponds a neighborhood within which $x_1 < x_0 < x_2$ implies that $f(x_1) < f(x_0) < f(x_2)$. Then f is increasing on (a, b).*

For lack of space, the proof of this theorem is omitted. A good student in a first course in the theory of functions should be able to construct a proof by an argument depending upon the concept of the greatest lower bound of a set of real numbers.

The next theorem furnishes a sufficient, though not necessary, condition for a proper relative extreme.

2.3 *Let f be continuous in a neighborhood of $x = a$, and suppose that f has a finite or infinite derivative at each point of the neighborhood, except possibly at $x = a$ itself. Further, suppose that within the given neighborhood $f'(x)$ is negative on one side of $x = a$, and positive on the other. Then f has a proper relative minimum or a proper relative maximum at $x = a$ according as $f'(x) < 0$ or $f'(x) > 0$ when $x < a$.*

We give the proof for the case of a minimum, using the method of contradiction. Suppose that, for some x_1 in the neighborhood, $f(x_1) \leqq f(a)$ and $x_1 \neq a$. If

* This method of proof is taken from Ch. J. de la Vallée Poussin, Cours d' Analyse Infinitésimale, Paris, 1914, I, p. 95.

necessary we can rechoose x_1, still nearer $x=a$, so that $f(x_1)<f(a)$, for f is decreasing on the left of $x=a$, and increasing on the right, by theorem 2.21 (or by 2.11 and 2.22). Now, since f is continuous, we can choose x_2 between x_1 and a in such a way that $f(x_1)<f(x_2)$. This contradicts the fact that f is decreasing on the left of $x=a$, and increasing on the right. The proof is therefore complete.

That the conditions of the theorem are not necessary for an extreme, even when f' is continuous, was pointed out long ago.*

3. Concavity. Points of inflection. In this section we shall discuss curves defined in rectangular coordinates by the equation $y=f(x)$, where f is a continuous function.

DEFINITION 3.11. Let f be defined and let its derivative exist at each point of (a, b). If the curve $y=f(x)$, throughout (a, b), lies entirely above its tangent at each point of (a, b), (the point of contact of course excepted, we say that the curve is concave upward in (a, b). The analytical requirement is that the following inequality hold when $x_1 \neq x_2$:

$$f(x_1) - f(x_2) - (x_1 - x_2)f'(x_2) > 0. \tag{3.12}$$

If the reversed inequality holds we say that the curve is concave downward. In that case the curve $y=-f(x)$ is concave upward.

This definition is closely related to the notion of a convex function, as we shall point out in detail later on, in §5.

3.21. *The curve $y=f(x)$ is concave upward on (a, b) if and only if the derivative f' exists at each point and is an increasing function on (a, b).*

We suppose first that the curve is concave upward. Then, if $x_1<x_2$, it follows from (3.12) and the inequality which results upon interchanging x_1 and x_2 that

$$f'(x_1) < \frac{f(x_2) - f(x_1)}{x_2 - x_1} < f'(x_2). \tag{3.22}$$

Hence f' is increasing.

Suppose now that f' is increasing, and that $x_1 \neq x_2$. Then, by the law of the mean,

$$f(x_1) - f(x_2) - (x_1 - x_2)f'(x_2) = (x_1 - x_2)[f'(\xi) - f'(x_2)],$$

where ξ lies between x_1 and x_2. Since f' is increasing, $f'(\xi)-f'(x_2)$ has the same sign as x_1-x_2; therefore (3.12) must hold, and the curve is concave upward.

3.3 *Suppose that f' exists and is continuous on (a, b), and that f' has a finite or infinite derivative f'' at each point of (a, b). Then the curve $y=f(x)$ is concave upward on (a, b) if and only if $f''(x) \geqq 0$ throughout (a, b) and the equality sign does not hold throughout any sub-interval.*

* E. R. Hedrick, On a function which occurs in the law of the mean, Annals of Mathematics, Series 2, vol. 7, 1906, pp. 190–192.

If the conditions are fulfilled, it follows from theorem 2.21 that f' is increasing on (a, b); the curve is then concave upward, by theorem 3.21. Conversely, if f' is increasing on (a, b), $f''(x) \geqq 0$. If the equality were to hold throughout a subinterval, the curve would be a line segment, and hence not concave upward.

Theorem 3.3 is applicable to prove that the curve $y = x^{4/3}$ is concave upward. Here $f''(0) = +\infty$. It is also applicable to the curve $y = x^4$, for which $f''(0) = 0$.

DEFINITION 3.4. If the curve $y = f(x)$ is continuous in the neighborhood of $x = a$, if it is concave upward on one side of $x = a$, and concave downward on the other, and if f has a finite or infinite derivative at $x = a$, we say that the curve has a point of inflection at $x = a$.

Before stating a test for points of inflection we observe the following very useful fact about derivatives.

3.5. *Let f be continuous in the interval $[a, a+\delta]$,* where $\delta > 0$, *and let the derivative f' exist in $(a, a+\delta)$. Suppose that $f'(x)$ approaches a finite or infinite limit A as $x \to a$ from the right. Then f has a finite or infinite right-hand derivative at $x = a$, given by $f'(a) = A$. A similar theorem holds for limits from the left.*

The proof is a direct consequence of the law of the mean, for if $0 < h < \delta$

$$\frac{f(a + h) - f(a)}{h} = f'(a + \theta h), \qquad 0 < \theta < 1,$$

and the result follows when we let h approach zero.

We can now state a necessary condition for a point of inflection, under certain conditions.

3.61. *Let f' exist in a neighborhood of $x = a$; let f' have a finite or infinite derivative at $x = a$, and let the curve $y = f(x)$ have a point of inflection at $x = a$. Then $f''(a) = 0$.*

It will suffice to assume that the curve is concave upward if $x < a$, and concave downward if $a < x$, since the other case is reduced to this by considering the function $-f$. Now f' is increasing if $x < a$, and it therefore approaches a finite or infinite limit as $x \to a$. This limit must be finite, and equal to $f'(a)$, by theorem 3.5. A similar argument when $a < x$ leads to the conclusion that f' is continuous at $x = a$ (and, in fact, throughout the neighborhood). It is now clear that f' has a relative maximum at $x = a$; therefore $f''(a) = 0$, by theorem 2.11.

If $f'(a) = 0$, any sufficient condition for the curve $y = f(x)$ to be concave upward in the neighborhood of $x = a$ is also a sufficient condition for a proper relative minimum of f at $x = a$. Such a condition is furnished by theorem 3.3. However, a test can be given involving the second derivative at $x = a$ only, without any implications as to concavity.

3.7. *Let f' exist in the neighborhood of $x = a$, and let $f'(a) = 0$. Suppose that $f''(a) > 0$, where $f''(a)$ may be either finite or infinite. Then f has a proper relative minimum at $x = a$.*

The proof is simple. Since

$$0 < f''(a) = \lim_{h\to 0} \frac{f'(a+h)}{h},$$

$f'(a+h)$ has the sign of h when $|h|$ is small. The result now follows by theorem 2.3. A direct proof may be given, using the law of the mean, for if $|h|$ is small and positive, $f(a+h)-f(a)=hf'(a+\theta h)$, $0<\theta<1$, and the right member of this equality is positive.

We leave it for the reader to devise a simple proof that the condition $f''(a)\geqq 0$ is necessary for a relative minimum. Some interesting examples of functions with minima not detectable by any of the usual sufficiency tests have been given by Hedrick, in the reference cited earlier in this paper.

4. Tangents. We shall discuss plane curves C of the type $y=f(x)$, where x and y are rectangular coordinates, and f is defined and continuous on some interval. If P_i is a point of C with abscissa x_i, let the inequality $P_1<P_2$ mean that $x_1<x_2$.

DEFINITION 4.11. Let P_0 be a point of C. A half line L^+ emanating from P^0 is said to be the forward tangent to C at P_0 if, given any angle of aperture less than 180°, with vertex at P_0, and L^+ within the angle, there exists a circle with center at P_0 such that if P is on C and inside this circle, and $P_0<P$, then the directed half line P_0P is also within the angle. We write $\lim P_0P=L^+$ as $P\to P_0{}^+$.

The backward tangent L^- at P_0 is defined in a similar manner.

It will be obvious to anyone familiar with topology that the limits L^+, L^- may be regarded as limits of functions on one topological space to another. In each case the independent variable is P, and the dependent variable is the directed half line P_0P. The limits are necessarily unique, if they exist.

DEFINITION 4.12. The curve C is said to have a tangent at P_0 if both the backward and forward tangent to C exist at P_0, and if they lie on the same straight line through P_0.

It is possible that L^+ and L^- may coincide. For example, in the case of $y=x^{2/3}$, when $P_0=0$, both L^+ and L^- coincide with the positive y-axis.

DEFINITION 4.13. If L^+ and L^- exist at P_0, and if together they form an entire straight line through P_0, this line L is called an *ordinary* tangent of C at P_0.

This definition permits us to distinguish between tangents at "ordinary" points and cuspidal tangents, such as the one mentioned in the foregoing example.

In order to discuss tangents it is convenient to use the notations $f'_+(a)$, $f'_-(a)$ for the right-hand and left-hand derivatives of f at $x=a$. Our conventions about such statements as "$f'_+(a)$ exists, $f'_-(a)=-\infty$," etc., are the same as for the derivative f' (see §1). If $f'_+(a)$ and $f'_-(a)$ exist and are equal, their common value is the derivative $f'(a)$.

4.21. *Let f be continuous in a neighborhood of $x=a$. Then the curve $y=f(x)$ has a tangent at $x=a$ if and only if either*

(1) *$f'_-(a)$ and $f'_+(a)$ are finite and equal,*

or

(2) *$f'_-(a)$ and $f'_+(a)$ are both infinite (but not necessarily of the same sign). In the first case the tangent is an ordinary tangent not parallel to the y-axis. In the second case the tangent is parallel to the y-axis, but it need not be ordinary.*

4.22. *Under the hypothesis of theorem* 4.21 *the curve $y=f(x)$ has an ordinary tangent at $x=a$ if and only if f has a finite or infinite derivative at $x=a$.*

The proofs of theorems 4.21 and 4.22 are left to the reader.

There are several theorems about derivatives which are important for the light they shed on the behavior of the tangent to a curve. These theorems are not new, but they are comparatively unknown. One of them, theorem 3.5, has already been mentioned. From it we conclude the following:

4.31. *Suppose that f' exists (as a finite limit) in (a, b). Then it is continuous in (a, b) except at those points where $f'(x)$ fails to approach a limit from at least one side.*

As a corollary we have:

4.32. *If f' exists and is monotonic on $[a, b]$, it is continuous there.*

It follows from this theorem that if the curve $y=f(x)$ is concave upward (or downward) on (a, b), it possesses a continuously turning ordinary tangent.

The next theorem is acribed to Darboux by de la Vallée-Poussin.*

4.33. *Let f be continuous, and possess a finite or infinite derivative at each point of $[a, b]$. Let k be a number between $f'(a)$ and $f'(b)$. Then there is a number X, $a<X<b$, such that $f'(X)=k$.*

The hypothesis implies that $f'(a)$ and $f'(b)$ are either (1) finite and distinct, (2) one finite and the other infinite, or (3) one $+\infty$ and the other $-\infty$. To prove the theorem let us first suppose that $f'(a)$ and $f'(b)$ are of opposite sign, and prove that f' vanishes at least once between a and b. Suppose, for example, that $f'(a)>0$ and $f'(b)<0$. Now f will have a maximum value on $[a, b]$, and we see from theorem 2.11 that it cannot occur at a or b. Suppose that the maximum occurs at X, where $a<X<b$. Then $f'(X)=0$, by theorem 2.12. The proof of the general case is reduced to the case just considered by defining $\phi(x)=f(x)-kx$ and observing that $\phi'(a)$ and $\phi'(b)$ are of opposite sign.

It is worth emphasizing at this point that in Darboux's theorem and the law of the mean the derivative is not assumed to be continuous, or even always finite. Many calculus texts insist upon the continuity of the derivative. One

* Ch. J. de la Vallée-Poussin, loc. cit. p. 97. For a discussion of an issue raised by this theorem see H. Lebesgue, Leçons sur l'Integration, Paris, 1928, p. 97. Lebesgue cites a paper by Darboux: Sur les fonctions discontinues, Annales de l'École Normale, 1875.

recent book even cites a function such as $f(x) = x^{2/3}$ to show that Rolle's theorem may fail when the derivative is discontinuous!

The assumption of continuity of f cannot be dropped from Darboux's theorem. This is shown by the following counter-example. Define $f(x) = x^{-1}$ if $x \neq 0$, and $f(0) = 0$. Then $f'(0) = +\infty$, $f'(1) = -1$, but $f'(x) < 0$ if $0 < x \leqq 1$. Here is another counter-example, in which both $f'(a)$ and $f'(b)$ are finite: define $f(x) = x^{-1}$ if $x < 0$, $f(x) = x^{1/2}$ if $x \geqq 0$. Then $f'(-1) = -1$, $f'(0) = +\infty$, $f'(1) = \frac{1}{2}$, but $f'(x) \neq 0$ if $-1 \leqq x \leqq 1$.

Darboux's theorem affords an alternative method of proving theorem 4.32; we leave the details to the reader.

The next two theorems deal with tangents parallel to the y-axis.

4.41. *Let f be continuous in a neighborhood of $x = a$, and let the curve $y = f(x)$ have an ordinary tangent at each point of this neighborhood, except at $x = a$. At $x = a$ let there be a tangent parallel to the y-axis, but not necessarily ordinary. Then $f'(x)$ is unbounded in every neighborhood of $x = a$, but it is not necessarily true that $|f'(x)| \to \infty$ as $x \to a$.*

We apply the law of the mean to f, obtaining

$$\left| \frac{f(a + h) - f(a)}{h} \right| = | f'(a + \theta h) |, \qquad 0 < \theta < 1,$$

where $|h|$ may be as small as we please. The quantity on the left becomes infinite at $h \to 0$, by theorem 4.21. Therefore $f'(x)$ cannot be bounded near $x = a$. The last assertion of the theorem is provided for by the following counter-example. We define

$$f(x) = x^{1/2} + x \sin (1/x) \quad \text{if} \quad x > 0,$$
$$f(0) = 0, \qquad f(x) = -f(-x) \quad \text{if} \quad x < 0.$$

This function is continuous, and $f'(0) = +\infty$. If $x > 0$ we have

$$f'(x) = x^{-1/2}/2 + \sin (1/x) - (1/x) \cos (1/x).$$

If $x_n = 2(1+4n)^{-1}/\pi$ and $u_n = (2n\pi)^{-1}$, it is found that $f'(x_n) \to +\infty$ and $f'(u_n) \to -\infty$. Therefore, when n is sufficiently large, it follows by Darboux's theorem that f' vanishes somewhere between x_n and u_n. It is then false that $|f'(x)| \to \infty$ as $x \to 0$.

4.42. *Let f be continuous in a neighborhood of $x = a$, and let the curve $y = f(x)$ have an ordinary tangent at each point of the neighborhood distinct from $x = a$. Finally, suppose that $|f'(x)| \to \infty$ as $x \to a$. Then the curve has a tangent parallel to the y-axis at $x = a$.*

The proof depends on Darboux's theorem. We may suppose the neighborhood so small that $f'(x) \neq 0$ when $x \neq a$. Then, within this neighborhood, $f'(x)$ is of constant sign when $a < x$, since otherwise it would vanish, by Darboux's theorem. The situation is similar when $x < a$. It is now clear that $f'(x)$ becomes

infinite (with definite sign) when x approaches a from one side. It follows from theorem 3.5 that both $f'_+(a)$ and $f'_-(a)$ are infinite. This completes the proof.

5. Convex functions. The meaning of concavity was discussed in §3, where a definition suitable for use in elementary calculus was introduced. We shall now consider the relation of this notion of upward concavity to the more general concept of a convex function. The latter concept is of great importance in analysis; we hope that a brief discussion of the concept will prove to be of interest in the present situation.*

DEFINITION 5.11. A function f is said to be convex on (a, b) if, for every two points on the graph of $y=f(x)$, the graph between these points lies entirely below the corresponding chord.

An analytic formulation of the condition for convexity, convenient for our purposes, may be made as follows: if $a<x_1<x<x_2<b$, then

$$\frac{f(x)-f(x_1)}{x-x_1}<\frac{f(x_2)-f(x)}{x_2-x}. \tag{5.12}$$

From the very definition it may be verified that the slope of a chord is an increasing function of the abscissa of either of its end points. Now suppose that in (5.12) we allow x_1 and x_2 to approach x. It is evident that the unilateral derivatives $f'_-(x)$ and $f'_+(x)$ must exist. Furthermore, if $x_1<x<x_2$, a little reflection will reveal that

$$\frac{f(x)-f(x_1)}{x-x_1}<f'_-(x)\leqq f'_+(x)<\frac{f(x_2)-f(x)}{x_2-x}. \tag{5.13}$$

The existence of the unilateral derivatives implies that f is continuous. The unilateral derivatives are increasing functions. For, if $x_1<x_2<x_3$, it follows from (5.13) and (5.12) that

$$f'_-(x_1)<\frac{f(x_2)-f(x_1)}{x_2-x_1}<\frac{f(x_3)-f(x_2)}{x_3-x_2}<f'_-(x_3).$$

That $f'_+(x)$ is increasing is proved similarly.

5.21. *If the function f is convex on (a, b) it is continuous there. It possesses right-hand and left-hand derivatives at each point of the interval; these derivatives are increasing functions. They are different on a set of points which is at most enumerable.*

The proof of most of this theorem has already been given. For the proof of the last assertion in the theorem we observed from (5.13) that if $x_1<x_2$

* The reader interested in the literature of convex functions may profitably consult the following references: G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge, 1934, Chapter III; E. C. Titchmarsh, Theory of Functions, Oxford, 1932, pp. 172–174; O. Haupt and G. Aumann, Differential- und Integralrechnung, Berlin, 1938, I, pp. 106–112, II, pp. 57–63.

$$f'_+(x_1) < \frac{f(x_2) - f(x_1)}{x_2 - x_1} < f'_-(x_2). \tag{5.22}$$

Hence, if $x_1 < x_2$,

$$f'_-(x_1) \leqq f'_+(x_1) < f'_-(x_2).$$

Suppose that f'_- is continuous on the right at x_1. Then, if we make x_2 approach x_1 in the above inequality, we see that $f'_+(x_1) = f'_-(x_1)$. Thus the derivative f' exists at x_1. Now f'_-, being monotonic, is continuous except possibly on an enumerable set. Hence the derivative f' exists except perhaps on such a set.

5.23. *If f is such that f' exists on (a, b), the assertion that f is convex is equivalent to the assertion that the curve $y = f(x)$ is concave upward in the sense of definition 3.11 (or, what is equivalent, to the assertion that f' is increasing).*

It follows from theorem 5.21 that if f is convex and f' exists, it is increasing. On the other hand, if f' exists and is increasing, the inequalities (3.22) hold, and from them we conclude, if $x_1 < x < x_2$, that

$$\frac{f(x) - f(x_1)}{x - x_1} < f'(x) < \frac{f(x_2) - f(x)}{x_2 - x},$$

so that f satisfies the convexity condition (5.12).

Next we prove that a convex function is the integral of either of its unilateral derivatives.*

5.3 *If f is convex on the interval (a, b), and if c, d are points of the interval,*

$$f(d) - f(c) = \int_c^d f'_+(t)dt = \int_c^d f'_-(t)dt.$$

Suppose that $[c, d]$ is divided into n subintervals by points x_i, where $c = x_0 < x_1 < \cdots < x_n = d$. From the inequalities (5.13) and (5.22) it may be seen that if $\Delta x_i = x_i - x_{i-1}$,

$$\begin{aligned}\sum_{i=1}^n f'_-(x_{i-1})\Delta x_i \leqq \sum_{i=1}^n f'_+(x_{i-1})\Delta x_i < f(d) - f(c) = \sum_{i=1}^n [f(x_i) - f(x_{i-1})] \\ < \sum_{i=1}^n f'_-(x_i)\Delta x_i \leqq \sum_{i=1}^n f'_+(x_i)\Delta x_i.\end{aligned}$$

Since the increasing functions f'_-, f'_+ are integrable, the proof follows from these inequalities.

Some of the important properties of a convex function were described in theorem 5.21. We now show that these properties are sufficient to characterize

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* The proof indicated here was suggested by the referee. It is much simpler than the one originally included in the paper.

convexity. Let us first recall that with any function there are associated four derivates; if we write

$$Q(x, h) = \frac{f(x + h) - f(x)}{h},$$

the upper and lower derivates on the right are, respectively $D^+f(x) = \limsup_{h\to 0+} Q(x, h)$, $D_+f(x) = \liminf_{h\to 0+} Q(x, h)$, while the upper and lower derivates on the left are $D^-f(x) = \limsup_{h\to 0-} Q(x, h)$, $D_-f(x) = \liminf_{h\to 0-} Q(x, h)$. Some or all of these derivates may be infinite. It is true that $D_-f(x) \leqq D^-f(x)$; if the equality holds the common value is $f'_-(x)$. Similar remarks apply to the right-hand derivates.

5.4 *Let f be continuous on (a, b), and suppose that one of its derivates is increasing on (a, b). Then f is convex on (a, b).**

The proof depends upon a lemma which very closely resembles the law of the mean.

5.41. *If f is continuous on $[a, b]$, there exist numbers x_1, x_2 in $[a, b]$ such that*

$$Df(x_1) \leqq \frac{f(b) - f(a)}{b - a} \leqq Df(x_2),$$

where $Df(x)$ denotes any one of the derivates of f.

The assertion is obviously true if f is linear; hence we dismiss this case, and define

$$\phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

The function $\phi(x)$ is continuous on $[a, b]$. It vanishes at a and b, but does not vanish identically. Hence it has either a positive maximum or a negative minimum at some point X, where $a < X < b$. We shall consider the case of a positive maximum, and leave the other case to the readers. We may without loss of generality suppose that $\phi(x) > 0$ when $a < x < b$. Then it it is easily seen that for a right-hand derivate one has $D\phi(X) \leqq 0 \leqq D\phi(a)$, while for a left-hand derivate $D\phi(b) \leqq 0 \leqq D\phi(X)$. Since

$$D\phi(x) = Df(x) - \frac{f(b) - f(a)}{b - a},$$

the proof of the lemma is complete.

The proof of theorem 5.4 may now be given in three steps, as follows:

(a) The curve $y = f(x)$ contains no line segment, because one of the derivates of f is increasing, whereas the derivative of a linear function is constant.

(b) If $x_1 < x < x_2$, it follows from lemma 5.41 that there exist number x_3 and x_4 such that, for the increasing derivate $Df(x)$,

* This theorem, for the case of the particular derivate $D_+f(x)$, was proposed and proved, in a slightly different way, by my colleague, Professor Zorn.

$$\frac{f(x) - f(x_1)}{x - x_1} \leqq Df(x_3), \qquad x_1 \leqq x_3 \leqq x,$$

$$Df(x_4) \leqq \frac{f(x_2) - f(x)}{x_2 - x}, \qquad x \leqq x_4 \leqq x_2.$$

But then $x_3 \leqq x_4$, and so, by hypothesis, $Df(x_3) \leqq Df(x_4)$. Therefore

$$\frac{f(x) - f(x_1)}{x - x_1} \leqq \frac{f(x_2) - f(x)}{x_2 - x}. \tag{5.42}$$

This means that no point of an arc of the curve $y=f(x)$ lies above the corresponding chord

(c) Finally, the equality cannot hold in (5.42) at any point x between x_1 and x_2. This would mean that points P_1, P, P_2 of the curve all lie on the same straight line. By (a) and (b) there exists a point P_3 of the curve, with abscissa x_3 between x and x_2, such that P_3 lies below the chord PP_2. If P_1PP_2 were a straight line, P would lie above the chord $P_1 P_3$,contrary to (b). This completes the proof of the theorem.

THE SIXTH ANNUAL MEETING OF THE SOUTHWESTERN SECTION

The sixth annual meeting of the Southwestern Section of the Mathematical Association of America was held at New Mexico State College of Agriculture and Mechanic Arts, State College, New Mexico, on April 28, 1942, in conjunction with the annual meeting of the Southwestern Division of the American Association for the Advancement of Science. Professor Roy MacKay,* chairman of the Section, presided over the Tuesday morning session.

The attendance was twenty-six, including the following eleven members of the Association: J. W. Branson, Eupha A. Buck, E. A. Hazlewood, W. P. Heinzman, H. D. Larsen, Roy MacKay, C. V. Newsom, Arthur Rosenthal, Nathan Schwid, P. M. Swingle, W. W. Wallis.

At the business meeting the following officers were elected for next year: Chairman, E. A. Hazlewood, Texas Technological College; Vice-Chairman, P. M. Swingle, New Mexico State College; Secretary, H. D. Larsen, University of New Mexico. It was voted to hold the 1943 meeting at a time and place selected by the officers of the Section. A vote of appreciation was extended to New Mexico State College for its generous hospitality.

The annual luncheon which followed the morning session was addressed by Professor C. V. Newsom of the University of New Mexico. His subject was, "The transition between elementary and advanced mathematics." The meeting

* Deceased, May 12, 1942.

was climaxed by an excursion on Tuesday afternoon to the White Sands National Monument.

The following papers were presented:

1. "Topology in biology, physics, and logic" by Professor P. M. Swingle, New Mexico State College.

2. "A vertical-curve sight-distance nomograph chart for use in highway construction" by Professor J. J. McKinley, New Mexico State College, introduced by the Secretary.

3. "The asymptotic expansion of the generalized hypergeometric functions" by Professor C. V. Newsom, University of New Mexico.

4. "On the synthesis of some mathematical tools" by J. A. Joseph, New Mexico State College, introduced by Professor MacKay.

5. "A note on the extraction of square roots" by Professor H. D. Larsen, University of New Mexico.

6. "A theorem involving average functions" by Dr. C. B. Barker, University of New Mexico, introduced by the Secretary.

7. "On interval-functions and associated set-functions" by Dr. Arthur Rosenthal, University of New Mexico.

8. "Characteristic functions of hypernumbers" by Professor J. B. Shaw, University of Illinois, introduced by the Secretary. (Read by title)

Abstracts of these papers follow:

1. Professor Swingle considered an abstract definition of limit, which he modified through Aristotle's four traditional types of propositions. He modified other definitions of topology in a similar manner, and sought interpretations in various sciences. He raised questions concerning the nature of abstract topologies, as well as of other topics based upon the concept of limit.

2. Professor McKinley discussed a nomograph which he constructed, and which permits the rapid estimate of the length of vertical curve required for a given grade break and sight distance, and the sight distance for a given grade break and curve length.

3. Professor Newsom recently proved a lemma which stated that certain entire functions given in the form of power series could be represented asymptotically by a sum of integrals of a comparatively simple type. He now showed that a series which includes many forms of the hypergeometric function satisfies the condition of the lemma. Consequently, a very convenient form of the asymptotic representation of such a function was obtained, valid in a large sector of the complex plane.

4. Mr. Joseph pointed out that now, more than ever before, the conservation of time and energy is a big factor in the solving of engineering problems; but that many of the standard mathematical methods are unnecessarily long and accurate for the engineer's purpose. He gave several examples showing how various problems can be solved quickly by graphical methods.

5. Professor Larsen discussed the well-known theorem: If the square root of a number consists of $2n+1$ figures, when the first $n+1$ of these have been

obtained by the ordinary method, the remaining n may be obtained by division, if the remainder arising from the division be neglected. He pointed out that the theorem is true only if the given number is a perfect square and otherwise the theorem as stated leads to a doubt in the last digit retained in the square root.

6. Dr. Barker defined a function of two variables which possessed certain desired properties relative to a given set of functions of one variable.

7. Dr. Rosenthal reported on a section of his elaboration of the second volume of H. Hahn's "Real Functions," which he is preparing on the basis of manuscripts left by Hahn. A set-function ϕ shall be called associated to a given interval function χ with respect to a given totally additive set-function, ψ, if $\chi(S_\nu)\rightarrow\phi(M)$ for the sequences $((S_\nu))$ of systems of intervals which converge to the set M in a certain sense with respect to ψ. Necessary and sufficient conditions for the existence of such an associated set-function ϕ were given. If ψ is continuous with respect to the Lebesgue measure μ_n, these conditions consist of the ψ-continuity of χ, the differentiability of χ with respect to ψ (almost everywhere), and the ψ-summability of this derivative. If ψ is not μ_n-continuous, other conditions have to be added.

8. Professor Shaw pointed out that in any algebra, associative or nonassociative, there is a characteristic function (lowest degree) for each hypernumber, the terms being homogeneous in the hypernumber if we remember that the scalar coefficients have degrees. If we substitute for the hypernumber ρ_0 the hypernumber $\rho_0+mx\rho_1+m(m-1)/1\cdot 2\ x^2\rho_2+\cdots$, the resulting form can be obtained by a device commonly used in Quaternions. The substituted form is indicated by $(1+x\delta)^m\rho_0$ where $\delta\rho_0=\rho_1$, $\delta\rho_1=\rho_2, \cdots$. Then by tagging δ, the form F becomes $(1+x\delta_1)^m(1+x\delta_2)^m\cdots(1+x\delta_a)^m\ F(\rho_0', \rho_0'', \cdots \rho_0^{(a)})$ where δ_1 acts only on ρ_0', δ_2 on ρ_0'', etc. The operator in front may be expanded with no trouble. Professor Shaw considered the various cases arising by letting the coefficients of the powers of x vanish.

H. D. LARSEN, *Secretary*

THE TENTH ANNUAL MEETING OF THE WISCONSIN SECTION

The tenth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at State Teachers College, Oshkosh, on Saturday, May 2, 1942. Miss Irene Price, chairman of the Section, presided.

There were fifty-five present, including the following nineteen members of the Association: R. H. Bardell, Ethelwynn R. Beckwith, May M. Beenken, A. C. Berry, W. W. Bigelow, M. Mirabella Boehmer, Margaret C. Eide, H. P. Evans, Fannie Hopkins, R. C. Huffer, J. F. Kenney, R. E. Langer, Morris Marden, Mary Felice, Irene Price, E. A. Nordhaus, P. L. Trump, B. R. Ullsvik, J. I. Vass.

Sessions were held in the morning and afternoon, with a noon luncheon at Hotel Raulf, and a tea, served by the mathematics department of Oshkosh State Teachers College, following the afternoon session.

The following officers were elected for the coming year: Chairman, R. H. Bardell, University of Wisconsin in Milwaukee; Secretary-Treasurer, P. L. Trump, University of Wisconsin; Program Committee, Ethelwynn R. Beckwith, Milwaukee-Downer, A. C. Berry, Lawrence College. It was voted to hold the next meeting at the University of Wisconsin in Milwaukee on May 7, 1943. Invitations to meet at Eau Claire State Teachers College and at Milwaukee-Downer were acknowledged and filed.

At the morning meeting the following papers were presented:

1. "The normal law in probability" by Professor A. C. Berry, Lawrence College.

2. "The Lemoine and Brocard points of the triangle" by Sister Mary Felice, Mount Mary College.

3. "The application of mathematics in the field artillery" by Professor W. E. Roth, University of Wisconsin in Milwaukee, introduced by Professor Bardell.

The afternoon session was devoted to a panel discussion relating to certain problems involved in the war emergency. Miss May Beenken presided, and the leaders in the discussion with their topics were:

1. Professor H. P. Evans, University of Wisconsin, "What the University of Wisconsin is doing by way of special courses to meet the national emergency."

2. Professor J. F. Duncan, Oshkosh State Teachers College, introduced by Miss Beenken, "Government sponsored courses in this emergency."

3. A. R. Luebke, Fairbanks-Morse Company, introduced by Professor Trump, "Mathematical training necessary for people entering industry."

4. Professor Glen Eye, Principal of Wisconsin High School, Madison, introduced by Professor Trump, "The place of mathematics in the high school curriculum, both for students entering college and those entering industry."

5. Professor Merlin Hayes, University of Wisconsin Extension Division, introduced by Professor Bardell, "Opportunities in the military service for college and university students."

The sessions were well attended by high school teachers, and there was a spirited discussion in the afternoon session.

Abstracts of the morning papers follow:

1. Professor Berry discussed the justification of the use of the Normal Law as an approximation simplifying the computations arising in Probability and Statistics. A simple numerical estimate for the error introduced by making this approximation was announced. This makes it possible to measure the "degree of abnormality" of a given sample whether the sample be "large" or "small." The measure applies to n-dimensional distributions whose third order moments are finite, and to the calculations of probabilities associated with convex regions. A particular and basic case treats the reliability of the Pearson chi-square test.

2. Sister Mary Felice pointed out that besides being called the minimum point, and the symmedian point of the triangle, this point has been called by German writers, Grebe's point, and by French and English writers, Lemoine's point. In neither case was there a question of priority of discovery. Grebe announced no properties of the point which had not been known before his time, but he related this point to the old problem the locus of a point the sum of the squares of whose distances from the sides of a triangle is constant. Lemoine gave an entirely new definition for the point, discovered a number of properties that had not before been noted, and recognized that it was the same point which had been before his time known in an isolated way by one or other of its properties. Some of these properties which are not usually in the ordinary text on College Geometry are of considerable interest and were briefly reviewed.

3. Professor Roth indicated that the principal reason for the failure of young men to obtain commissions in an infantry division was their inability to show proof that they had completed a course of trigonometry in high school or college. The artilleryman in particular must have a very substantial knowledge of mathematics. This knowledge is used in survey operations; determination of locations of guns and of targets on maps or improvised grids; in the use of firing tables, making corrections upon range for wind, temperature of air, barometric pressure, temperature of powder, height of the target above or below the gun, the variations in weight of projectiles, etc.; in the preparation of visibility charts, and of dead space charts; in the determination of changes in range and direction to place rounds on the target after rounds have been observed.

P. L. TRUMP, *Secretary*

ANNUAL MEETING OF THE NEBRASKA SECTION

The fourteenth regular meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Omaha on Saturday, May 2, 1942. Professor J. M. Earl, chairman of the Section, presided.

The attendance was thirty-one including the following fourteen members of the Association: A. K. Bettinger, W. C. Brenke, A. R. Congdon, H. M. Cox, D. M. Dribin, J. M. Earl, W. C. Foreman, M. G. Gaba, F. S. Harper, F. E. Marrin, Lulu L. Runge, E. Marie Hove, W. T. Stratton, and A. E. White.

The following officers were elected for the coming year: Chairman, M. A. Basoco, University of Nebraska; Secretary, Lulu L. Runge, University of Nebraska; member of Executive Committee, E. M. Berry, State Teachers College, Chadron.

As Part I of the program, the following four papers were read:

1. "A test for a stabilization technique" by Professor W. A. Dwyer, Creighton University, introduced by Professor Earl.

2. "Generalized Laguerre polynomials" by Professor W. C. Brenke, University of Nebraska.

3. "Some metric properties of a vector associated with the tangent to a general curve on an analytic surface" by W. C. Foreman, Municipal University of Omaha.

4. "Pre-study examinations in mathematics" by Professor H. M. Cox, University of Nebraska.

Part II of the program consisted of a joint meeting with the Nebraska Section of the National Council of Teachers of Mathematics. The demonstrations presented were as follows:

5. "Mechanical apparatus and elementary nomograms for the construction and illustration of harmonic curves and stereographs associated with them" by L. E. Smith and A. B. Tussel of South High School, Omaha, introduced by Professor Gaba.

6. "Demonstrations by means of oscillograms of Lissajou and other curves of value in certain ultrafrequency work used by the Signal Corps" by T. T. Smith, University of Nebraska, introduced by Professor Gaba.

7. "Precision construction of the curves demonstrated above by means of the curve tracing machine" by Professor M. G. Gaba, University of Nebraska.

Part III of the program consisted of a joint luncheon with the National Council. The two papers there presented were:

8. "High lights of the convention of the National Council of Teachers of Mathematics at San Francisco" by Professor A. R. Congdon, University of Nebraska.

9. "Mathematics in the war program" by Professor W. C. Brenke, University of Nebraska.

Abstracts of some of the papers follow:

2. Professor Brenke discussed the generalization of $L_n(x)$ of Laguerre to $L_n^{(\alpha)}(x)$ of Szegö and to a still more general type $L_n^{(\alpha,\beta)}(x)$. He made a comparison of the difference and differential equations, the generating functions, and other characteristic properties.

3. Mr. Foreman used the notation of P. O. Bell in the *Bull. Am. Math. Soc.*, vol. 47. Some of the theorems demonstrated were: (1) The unique parametric net for which the directions λ and $\bar{\lambda}$ are mutually perpendicular at a point of a non-ruled analytic surface S for every choice of λ consists of the mean orthogonal curves. (2) A necessary and sufficient condition that the directions λ and $\bar{\lambda}$ be conjugate directions at a point of S for every choice of λ is that S be referred parametrically to a conjugate system of curves. (3) If the minimal curves are parametric, the R_λ-correspondent is self corresponding only along a mean orthogonal curve; and conversely.

4. In the development of a program of student guidance at the University of Nebraska, the mathematics classification examination has been integrated into an all-university measurement program. Professor Cox explained how the extended use of the examination has made necessary certain changes in the form and scope of the examination. Correlational and statistical analyses have justi-

fied the changes on a technical basis, and have, in turn, suggested instructional problems and problems of curriculum.

5. Mr. Smith and Mr. Trussell demonstrated the drawing of harmonic curves by the use of two simple machines which were equipped with cones instead of gears. These allowed the continuous passage from a given curve to others involving different frequencies. A small mirror was so connected with moving parts of the second machine that a spot of light was caused to trace a curve with such great speed that a line of light in the form of the curve appeared on the screen. Stereoscopic effects were produced by use of lights of two different colors from suitably oriented directions. The development of cylindrical curves, whose projections are Lissajou curves, was demonstrated by using transparent celluloid sheets.

6, 7. Professor Smith demonstrated, by means of oscillograms, the Lissajou and epitrochoidal curves which have been found to be of great value in certain ultrafrequency work used by the Signal Corps. All of the curves were constructed by Professor Gaba, using the University of Nebraska curve tracing machine.

LULU L. RUNGE, *Secretary*

WHAT IS A STOCHASTIC PROCESS?

J. L. DOOB, University of Illinois

1. Introduction. A stochastic process is simply a probability process; that is, any process in nature whose evolution we can analyze successfully in terms of probability. We shall not attempt an exhaustive description. On the empirical side, a discussion of the nature of probability would take us too far afield (and might sidetrack us into philosophy), and on the mathematical side the definitions would require too much high-powered mathematics. We shall limit ourselves to a description of a stochastic process in simple terms, followed by a discussion of a few important particular types. A stochastic process was described above as an empirical entity. On the mathematical side, certain concepts have characteristics in close correspondence with those of these processes, and will be called mathematical stochastic processes. The point is that although at an elementary level probability courses frequently deal with urns, dice and events, a strictly mathematical treatment is possible, with no immediate empirical flavor.

There is a popular prejudice that probability is the subject which deals with gambling games, and perhaps with life insurance and statistics, but is otherwise useless. Moreover its principal method is the calculation of permutations and combinations, than which nothing could be more boring. This calculation looms large in elementary courses, however, only because calculations need little mathematical preparation, either for student or teacher. It is considered easier

to perform long calculations than to develop a general point of view, bolstered by theory. This may be good pedagogy, up to a certain point, but gives students a false picture of the subject. The basic reason for the applicability of permutation-combination theory to the study of probability will be noted below. It will be seen, however, that considerably more than the counting of favorable and unfavorable events by such methods is necessary to deal with the problems to be discussed.

2. Concept of a chance variable. Perhaps the basic difference between the older mathematical probability and that of the last fifteen years lies in the stress now put on the concept of a chance variable, and in the development of the whole subject in terms of that concept. In this spirit, we shall develop the idea of a stochastic process from that of a chance variable.

A chance variable, sometimes called a stochastic variable, or a random variable, or (especially by statisticians) a variate, is (non-mathematically speaking) the numerical result of an experiment to which probability analysis is to be applied. There are various possible results, with varying probabilities assigned to them. Thus let N_t be the number of telephone subscribers who will initiate a call tomorrow in Chicago between 4 P.M. and t minutes after 4. Then, fixing t, the practical determination of the necessary number of central telephone facilities needed to prevent more than a given percentage of lost calls is based on assumptions about N_t. For each t, certain probabilities suggested by theoretical considerations are assigned to the various possible N_t values $0, 1, \cdots$. The character of the chance variable N_t depends on t. For example, the more probable values of N_t increase as t does. In many cases the chance variable y under consideration can take on any one of a continuous set of values (for example let y be the value of t when the first subscriber to initiate a call after 4 does so). In such cases individual values of y may each have 0 probability, and intervals of y-values are assigned probabilities. The probability characteristics of any chance variable y are determined by the specification of every probability of the type. Prob. $\{y<k\}$, where k is any number. Thence the probability that y satisfy other types of condition, for example Prob. $\{a\leqq y\leqq b\}$, is determined. The function

$$F(k) = \text{Prob. } \{y < k\}$$

is called the distribution function of y.

Now suppose $y_1, \cdots, y_n$ are n chance variables. To discuss them together, not only the probability relations of each y_j must be given (that is, its distribution function), but we need also the combined distribution: If $k_1, \cdots, k_n$ are any numbers, the probability $F(k_1, \cdots, k_n)$ that $y_j<k_j$ simultaneously for all j is supposed known. All the probability relations of $y_1, \cdots, y_n$ are determined by the knowledge of the "distribution function" F.

Any set of values of n chance variables $y_1, \cdots, y_n$ determines a point P in n-dimensional space. A condition imposed on the y_j can be interpreted as a condition that P lie in some portion of this space. If the probability distribution

of $y_1, \cdots, y_n$ is determined by setting the probability that P lie in a given region R proportional to the integral of e raised to a second degree polynomial, integrated over R, then $y_1, \cdots, y_n$ are said to be normally distributed, or to have a Gaussian (n-variate) distribution. The normal distributions are the most common distributions of theoretical statistics because they really do arise frequently in practice, and also because their common statistical parameters are easy to compute.

3. Concept of a stochastic process. Having finished all these preliminaries, we finally come to the concept of a stochastic process. As we have said, a stochastic process in practice is any process whose evolution we find it possible to follow and predict in terms of probability. Now the evolution of such a process is measured by a number (or can be expressed in terms of such measurements) registered in some way, perhaps continuously (at each instant of time), perhaps at discrete times (say every hour), depending on the nature of the process and our analysis of it. Each measurement is by hypothesis a chance variable. From this point of view, a stochastic process is,—and this is the usual definition,—merely a family of chance variables $\{y_t\}$ where t represents the time. Here t may run through all numbers, or only through the integers, or more generally through any set of numbers. If the process consists of a recurring experimental procedure, it is frequently convenient to think of the y_t with t (an integer) ranging to ∞ in both directions even though physically we cannot make (and cannot have made) these infinitely many measurements.

Throughout this exposition, we have not introduced any pure mathematical analysis. For example our "definition" of a chance variable was merely a description of a certain type of experimental procedure, and not a very explicit one at that. This is not the place to go into the proper mathematical definitions. We can only say that the mathematical chance variable is a certain type of function (a measurable function), so that a mathematical stochastic process is a family of (measurable) functions. The analytical study of probabilities is the study of measurable functions. Some of the simple examples we shall discuss below are far simpler to describe physically than to analyze mathematically. What makes an elementary treatment of probability possible is that in many problems the chance variables concerned can only assume a finite number of values, and the time parameter t is allowed to run through only finitely many values. Because of these two characteristics, such problems can always be solved by combinatorial methods. Any finite repetition of a gambling game has these simple properties; and this is one reason for the central place of gambling games in elementary treatments of probability. In our discussion here we are considering the stochastic processes of nature, and carefully evading a proper mathematical analysis (which incidentally dates back less than ten years).

4. Examples. We conclude with several examples of stochastic processes. First consider repeated coin tossing. Here the family of chance variables is $y_1, y_2, \cdots$, with $y_j=1$ (heads) or $y_j=0$ (tails), with respective probabilities

p and $1-p$. The chance variable y_j corresponds to the jth toss. If the coin is evenly balanced, $p=1/2$. It is supposed as usual that the probability of a set of n tosses containing m heads and $n-m$ tails in some given order is $p^m(1-p)^{n-m}$. It is convenient in the mathematical analysis, as we have remarked above, to think of the tosses as continuing in both directions in time, so that we have $\cdots, y_0, y_1, \cdots$. Physically this is impossible, conceptually it is easy to swallow (but even if one gags, the results are easy to translate in terms of finitely many y_j), and the mathematical basis is not very difficult. The translation just noted is of course essential if the theorems are to make any empirical sense at all. A more important interpretation of the same mathematical setup would be the succession of male and female births in a community, or simple examples of Mendel's laws of heredity.

Of this stochastic process we note first that there is no change in the probability relations with time; that is, the probability relations of any $y_m, \cdots, y_n$ are the same as those of $y_{m+1}, \cdots, y_{n+1}$. Such a process is called stationary, or temporally homogeneous. Now what are proper questions that can be asked of such a process? There are various well known ones. For example, the number of heads in n throws divided by n approaches p, the probability of heads in a single throw: formally, if according to the usual rules the probability that

$$\lim_{n\to\infty} \frac{y_1 + \cdots + y_n}{n} = p$$

is calculated, the answer is 1. Although the tosses are independent, there is thus still an average character, a fact which has caused a considerable amount of head shaking, and induced peculiar remarks in well known books. Knowing that $s_n/n \to p$ $(s_n = y_1 + \cdots + y_n)$ one might ask how quickly $\sigma_n = s_n/n - p$ goes to 0, or whether the probability distribution of the chance variables $\sigma_1, \sigma_2, \cdots$ has some limiting form, as n increases. These questions have been answered by the law of iterated logarithm, which states that $\lim_{n\to\infty} \sup \sqrt{n}\,\sigma_n/\sqrt{2pq \log \log n} = 1$ with probability 1, and the central limit theorem, which states that $\sqrt{n}\,\sigma_n$ has more and more nearly a normal distribution as $n \to \infty$.

In the above example, the y_j were mutually independent. The next most simple type of connection is that of a Markoff process. Consider the chance variable $s_1, s_2, \cdots$ in the previous example. Evidently if $n > m$, knowing $s_1, \cdots, s_m$; that is, knowing the number of heads obtained in each toss up through the mth, considerably restricts s_n, the number of heads obtained in n tosses. For example, $s_n \geqq s_m$. It is easy to compute the probability of the various values of s_n if $s_1, \cdots, s_m$ have preassigned values. The answers will, of course, depend on these preassigned values. Now actually it is evident, since $s_n = s_m + y_{m+1} + \cdots + y_n$, where $y_{m+1}, \cdots, y_n$ are independent of $s_1, \cdots, s_m$, that the assigned value of s_m is all that is relevant to the computation. The conditional probabilities depend only on the value assigned to s_m, not on those assigned to $s_1, \cdots, s_{m-1}$. Such a process is called a Markoff process; formally the hypothesis is that whenever $t_1 < t_2$, the probability distribution of y_{t_2} calculated

under the assumption that y_t has been assigned values for all $t \leqq t_1$ depends only on the value assigned to y_{t_1}.

Let y_t be the size of a population under statistical analysis. Whenever the population's change depends on the state of the population at a given moment, but, this being known, is independent of how the present state was attained, the y_t determine a Markoff process. It is then evident that Markoff processes will have general application in statistical studies of the population trends of a species and in epidemiology (if the number of infected individuals depends only on those already infected). Important questions to be asked here are: what is the asymptotic character of y_t for large t, is there a limiting value, any kind of periodicity, etc.?

Let y_t be a chance variable depending on the time t, which we assume runs through all values. Suppose that the increments of y_t: $(y_{t+h}-y_t)$ over non-overlapping time intervals are independent, that is if $t_1 < \cdots < t_n$ we suppose that $y_{t_2}-y_{t_1}, \cdots, y_{t_n}-y_{t_{n-1}}$ are mutually independent. The stochastic process is then called a process with independent increments, or a differential process. These arise in many connections. For example let $y_t = N_t$ where N_t was defined above in the telephone example. Then the y_t determine a differential process: the number of subscribers initating a phone call in any time interval can under ordinary circumstances be supposed independent of the calls offered before this time interval (or later). Or, to take an example from physics, let y_t be the number of radioactive disintegrations of a given substance by time t. Again we have a differential process. Or let y_t be the amount of money to be paid out by an insurance company to its claimants between time 0 and time t. Again the process can frequently be assumed a differential process.

In the examples of differential processes considered above, it is clear that y_t, considered as a function of t has as graph a set of horizontal lines, no matter how the process turns out. The proof that in general the y_t of a differential stochastic process are continuous in t except for non-oscillatory discontinuities (jumps)* is quite complicated, and is less than ten years old. Aside from simple changes of scale, there is only one differential process in which the functions y_t are actually continuous in t (with probability 1)—that in which the increments $y_{t+h}-y_t$ have normal distributions. The best known application of this process is to the haphazard Brownian movements of small particles immersed in a liquid. About 35 years ago it was shown that if y_t is a coördinate value of such a particle at time t, the movement can be analyzed fairly accurately by considering the y_t as the determining family of chance variables of a differential stochastic process whose increments have normal distributions. Thus the probability analysis predicts continuous curves as the particle trajectories. That is certainly desirable, but it was noted at once that this same probability analysis predicts infinite

* The meaning of this statement is the following. The probability can be computed that y_t as a function of t will be continuous at a point, everywhere continuous, etc. The probability that the y_t of a differential process will have any non-oscillatory discontinuity is 0 (if very minor restrictions are imposed on the process to eliminate degenerate cases).

velocities for the particles, and that the length of a trajectory between any two of its points is infinite. This may seem remarkable physically, but man can predict anything after it happens, and in fact physicists have asserted that the particle movements as observed seem to have these peculiar properties.

We go finally to a simple but very important type of stationary stochastic process. Suppose that the distribution of y_t is normal, and even that for any $t_1, \cdots, t_n$ $y_{t_1}, \cdots, y_{t_n}$ have an n-variate normal distribution (for all n). We shall call such a process a stationary normal process. It is easy to see that apart from a scaling factor, and a centering constant, such a process is completely determined by a single function $\rho(h)$, the correlation coefficient of the two chance variables y_t, y_{t+h}, which measures the connection between these chance variables. (This is independent of t since the process is stationary.) Necessary and sufficient conditions are known that a function be such a correlation function, but many questions are still unanswered about these processes, and many more will remain unformulated until the need arises. The simplest case is the degenerate case $\rho(h)\equiv 1$, when y_t is independent of t: the process never changes from its initial state, and the theory of probability is quite superfluous. At the other extreme is the case $\rho(h)=0$ for all $t\neq 0$, when y_s, y_t are independent of each other if $s\neq t$. The first non-trivial class of stationary normal stochastic processes to investigate, is certainly the subclass of Markoff processes, of which the above two cases are degenerate special cases. For a Markoff process, aside from these two cases, the connection between y_t and y_{t+h} goes down exponentially, as $h\to\infty$: $\rho(h)=e^{-c|h|}$ (c a positive constant). In work on the Brownian movement dating back about nine years, and giving a better approximation than the earlier work, it was found that the particles had finite velocities after all (but infinite accelerations) and in fact that each component of the velocity of a particle at time t can be considered as a chance variable y_t, where the y_t determine one of the Markoff processes just described.

In this case of a Markoff process, with $\rho(h)=e^{-c|h|}$, it is known that the y_t are continuous functions of t, with probability 1, but it is not known generally which correlation functions $\rho(h)$ determine processes with continuous y_t.

Another example of a stationary normal process arises in electricity. The spontaneous thermal movements of the electrons in any wire cause current fluctuations in all electric circuits. If y_t is the current in a given wire at time t, y_t determines a stationary normal stochastic process. The correlation function depends on the particular circuit, and on the resistances, capacities, and inductances. This phenomenon is important in radio since the current fluctuations cause receiver noises which cannot be entirely eliminated. These current fluctuations are a disturbing influence in making any kind of delicate electrical measurements. A somewhat similar phenomenon occurs in radio tubes, known as the shot effect. The study (not yet completed) of these electrical disturbances is thus the study of certain types of stochastic processes.

GELFOND'S SOLUTION OF HILBERT'S SEVENTH PROBLEM

EINAR HILLE, Yale University

1. In 1900 D. Hilbert presented his famous list of twenty-three problems which he regarded as the outstanding questions awaiting solution by mathematicians of the future. [1] The seventh problem is concerned with irrational and transcendental numbers and in particular with the following question. *If* $\omega \neq 0, 1$ *is algebraic and* θ *is algebraic but not rational, is* ω^θ *transcendental or at least not rational?* As specific examples he mentioned $2^{\sqrt{2}}$ and $e^\pi = i^{-2i}$.

The first contribution to this problem was given by the Russian mathematician A. Gelfond [2] in 1929. He proved that e^π is actually transcendental and indicated how his method could be used to prove transcendentality whenever θ belongs to an imaginary quadratic field. The extension to real quadratic fields was given by C. L. Siegel (unpublished) and R. A. Kuzmin [3] in 1930.

Gelfond returned to the question later and in 1934 he could give a complete solution [4] of Hilbert's problem which was followed in 1935 by a more elementary solution obtained independently by Th. Schneider [5], one of Siegel's pupils. The proof of Gelfond, though more advanced, has quite a simple basis and gives a beautiful example of teamwork between algebraical and analytical ideas. For this reason it deserves to be much better known. The original paper is not easily accessible and is full of disturbing misprints, especially in the French text. Though there is an excellent review by K. Mahler in *Zentralblatt für Mathematik*, vol. 9, pp. 53–54, a detailed exposition of the ideas of the proof would still seem to be of some value to the mathematical community. The present version is based upon an analysis presented to a seminar at Stanford University in November 1941 and is a fairly free account.

2. We start by recalling some algebraical concepts. An *algebraic number* α of degree m over the rational field $\mathbf{R}$ is the root of an algebraic equation of degree m,

$$A_0 x^m + A_1 x^{m-1} + \cdots + A_m = 0, \tag{2.1}$$

with rational integral coefficients, irreducible over $\mathbf{R}$. The other roots are known as the *conjugates* of α. $N(\alpha) = A_m/A_0$ is the *norm* of α. If $A_0 = 1$, α is called an *algebraic integer*. We can always find a rational integer A such that $A\alpha$ is an algebraic integer. It suffices to take $A = A_0$. If the roots of (2.1) are $\alpha_1 = \alpha$, $\alpha_2, \cdots, \alpha_m$, we denote the largest of the numbers $|\alpha_j|$ by $\mu(\alpha)$.

The number α generates an *algebraic number field* $\mathbf{K}(\alpha)$ of degree m over $\mathbf{R}$ consisting of all polynomials in α of degree $\leqq m-1$ with coefficients in $\mathbf{R}$. Every element γ of $\mathbf{K}(\alpha)$ is an algebraic number of degree d over $\mathbf{R}$, where d is a divisor of m, and the conjugates of γ are obtained by replacing α by its conjugates in the defining polynomial. If β in an algebraic number of degree n over $\mathbf{K}(\alpha)$, *i.e.*, the root of an equation of degree n with coefficients in $\mathbf{K}(\alpha)$ and irreducible over $\mathbf{K}(\alpha)$, then the *extension field* $\mathbf{K}(\alpha, \beta)$ consists of all polynomials in β of degree $\leqq n-1$ with coefficients in $\mathbf{K}(\alpha)$ and the elements are of degree $\leqq mn$ over $\mathbf{R}$. Similarly for further extensions. The following property of algebraic numbers is

fundamental in all work involving transcendentality.

LEMMA 1. *If γ lies in an algebraic field of degree m over R and if A is a rational integer such that $A\gamma$ is an algebraic integer, then either $\gamma=0$ or*

$$|\gamma| \geqq A^{-d}[\mu(\gamma)]^{1-d} \geqq A^{-m}[\mu(\gamma)]^{1-m}. \tag{2.2}$$

The Lemma merely expresses the fact that $|N(A\gamma)| \geqq 1$.

3. In addition to the algebraic notions we shall need two results of analytical nature. The first is a less common form of Kronecker's theorem on Diophantine approximations [6] and the second is the theorem of Jensen [7].

LEMMA 2. *Given a matrix (a_{ij}) with m rows and n columns, $n>2m$, where the a_{ij} are complex numbers and $|a_{ij}| \leqq A$. Let P be a given positive number. Then there exist n rational integers $N_1, N_2, \cdots, N_n$, such that*

$$\left| \sum_{i=1}^{n} N_i a_{ij} \right| \leqq 1/P, \qquad j = 1, 2, \cdots, m, \tag{1}$$

$$\sum_{i=1}^{n} |N_i| \geqq 1, \tag{2}$$

$$| N_i | \leqq [2^{3/2}nAP]^{2m/(n-2m)}, \qquad i = 1, 2, \cdots, n. \tag{3}$$

For the proof, let $\nu_1, \nu_2, \cdots, \nu_n$, be a set of nonnegative integers each less than or equal to an integer B which will be disposed of later. There are $(B+1)^n$ possible distinct choices of such sets. Put

$$\sum_{i=1}^{n} \nu_i a_{ij} = a_j = c_j + id_j, \qquad i = 1, 2, \cdots, m.$$

In a space of $2m$ dimensions we mark the points

$$(c_1, d_1, c_2, d_2, \cdots, c_m, d_m).$$

These points are located in a $2m$-dimensional cube of side $\leqq 2nAB$. If we lay off the length $2^{-1/2}P^{-1}$ along the side, we get at most $2^{3/2}nABP+1$ intervals which in turn determine a subdivision of the big cube into at most $(2^{3/2}nABP+1)^{2m}$ subcells, cubes or parallellepipeda, no side of which can exceed $2^{-1/2}P^{-1}$. If now B is so chosen that

$$(B + 1)^n > (2^{3/2}nABP + 1)^{2m},$$

there will be at least one of the $2m$-dimensional subcells which contains two points of our set, P_1 and P_2 say. Suppose $(\nu_{11}, \nu_{12}, \cdots, \nu_{1n})$ and $(\nu_{21}, \nu_{22}, \cdots, \nu_{2n})$ are the corresponding integers. We put $N_i=\nu_{1i}-\nu_{2i}$, $i=1, 2, \cdots, n$. There is obviously at least one $N_i \neq 0$. The points P_1 and P_2 being in the same subcell, their coördinates in homologous position differ by at most $2^{-1/2}P^{-1}$. A simple computation shows that (1) holds, (2) has already been verified. Now $(B+1)^n$ is certainly greater than $(C+1)^{2m}$ if $B^n \geqq C^{2m}$. This gives in our case the value of B occurring in (3).

LEMMA 3. *Let $G(z)$ be holomorphic in $|z| \leqq R$ and have the zeros $z_1, z_2, \cdots, z_n$ in $|z| < R$. Then*

$$|G(0)| \leqq R^{-n} |z_1 z_2 \cdots z_n| \max_{0 \leqq \theta < 2\pi} |G(Re^{i\theta})|.$$

The inequality is trivial if $G(0) = 0$, so we can assume $G(0) \neq 0$. For each zero z_j we form the function

$$B_j(z) = \frac{R(z - z_j)}{R^2 - z\bar{z}_j},$$

which vanishes at $z = z_j$, has a simple pole at $z = R^2/(\bar{z}_j)$ outside the circle $|z| = R$, and is of absolute value one on the circle. Put

$$F(z) = G(z)\left[\prod_{j=1}^{n} B_j(z)\right]^{-1}.$$

This function is holomorphic in $|z| \leqq R$ and

$$\max |F(z)| = \max |G(z)| \quad \text{on} \quad |z| = R.$$

But

$$F(0) = \frac{1}{2\pi i}\int_{|t|=R} \frac{F(t)}{t}\, dt,$$

so that

$$|F(0)| \leqq \max |F(z)| = \max |G(z)|, \qquad |z| = R,$$

and

$$G(0) = (-R)^{-n}\left[\prod_{j=1}^{n} z_j\right] F(0).$$

Combining, we get the desired inequality. It is obvious, and it was observed already by Jensen, that the proof does not call for a complete knowledge of the zeros of $G(z)$ in $|z| < R$, but that the more zeros we can locate, the smaller will $|G(0)|$ have to be. This observation will be used repeatedly below.

4. We are now ready for the proof of Gelfond's theorem which he formulated as follows:

If α and β, different from zero and one, are algebraic while $\eta = (\log \alpha)/(\log \beta)$ is irrational, then η is transcendental.

The theorem asserts that no irrational algebraic power of an algebraic number can be algebraic, with obvious and trivial exceptions. This is obviously equivalent to Hilbert's problem.

Let us begin by giving a brief outline of underlying ideas of the proof. By assumption, α and β are algebraic and determine an algebraic field $\mathbf{K}(\alpha, \beta)$. Suppose now that η is also algebraic. We can then form the extension field $\mathbf{K}(\alpha, \beta, \eta)$.

We then form an auxiliary function $f(x)$, having values in $\mathbf{K}(\alpha, \beta)$ for every rational integral value of x. The simplest such function would be a polynomial in x with coefficients in $\mathbf{K}(\alpha, \beta)$, but such a function would not enable us to bring in η by any obvious process. Instead, Gelfond uses the old stand-by, the exponential function. He forms an exponential polynomial,

$$f(x) = \sum_{k=-q}^{q} \sum_{l=-q}^{q} C_{kl}\alpha^{kx}\beta^{lx}, \tag{4.1}$$

where the C_{kl} are rational integers and q is a positive integer, which will be disposed of later. Here $\alpha^{kx} = \exp [kx \log \alpha]$ with some definite but arbitrary definition of the logarithm, and similarly for β^{lx}. This function obviously has values in $\mathbf{K}(\alpha, \beta)$ when x is a rational integer. But more can be asserted. We have

$$f_s(x) \equiv f^{(s)}(x)(\log \beta)^{-s} = \sum_{k=-q}^{q} \sum_{l=-q}^{q} C_{kl}(k\eta + l)^s\alpha^{kx}\beta^{lx}, \tag{4.2}$$

so that $f_s(x)$ has values in $\mathbf{K}(\alpha, \beta, \eta)$ for every integral x and $s = 0, 1, 2, \cdots$.

The function $f(x)$ contains $(2q+1)^2$ parameters, the C_{kl}, which together with the value of q are at our disposal. In particular, we can take q as large as we please. By virtue of Lemma 2 the C_{kl} can be determined in such a manner that a fairly large set of the algebraic numbers $f_s(j)$ are very small in absolute value, so small in fact that condition (2.2) is violated and they, therefore, have to be zero. In other words, we can impose a certain number of zeros on $f(x)$ at prescribed integers while keeping the coefficients as integers of limited size. Gelfond distributes the zeros between several points, but it is simpler to place them all at the origin. This accumulation of zeros makes itself felt in a fairsized neighborhood of the origin and Jensen's theorem shows that $|f(x)|$ has to be quite small, say for $|x| < q^{2/3}$. By Cauchy's formula for the derivatives, this extends also to $|f^{(s)}(x)|$ for values of s not too large. But this in turn makes a still larger set of the algebraic numbers $f_s(j)$ so small that they have to be zero by Lemma 1. Thus we get a large number of additional zeros of large multiplicity in a neighborhood of the origin and by Jensen's theorem this forces $|f(x)|$ to be still smaller in a still larger neighborhood. By alternate use of Lemmas 1 and 3 we could show that $f(x)$ and all its derivatives vanish at all integers which of course implies $f(x) \equiv 0$. Actually, two applications of Jensen's theorem give us enough conditional equations on the C_{kl} to conclude that they are either all zero or η is a rational number. The first possibility is excluded by the construction of $f(x)$ which ensures $\sum\sum |C_{kl}| \geqq 1$, while the second violates the hypothesis of the theorem. We conclude that η cannot be algebraic.

5. The crux of the proof is right at the start in the application of Lemma 2, in particular the choice of m and P. We shall try to elucidate this point somewhat. It is desired to choose m, P, and integers C_{kl} with $\sum\sum |C_{kl}| \geqq 1$, such that

$$|f_s(0)| \equiv \left| \sum_{k=-q}^{q} \sum_{l=-q}^{q} C_{kl}(k\eta + l)^s \right| \leqq 1/P, \qquad s = 0, 1, \cdots, m-1, \tag{5.1}$$

while condition (2.2) is violated. Lemmas 1 and 2 assert that this can be done provided certain inequalities hold.

Suppose that η is algebraic of degree h over $\mathbf{R}$, and that $E\eta$ is an algebraic integer, E being a rational integer. Then $E^s f_s(0)$ is also an algebraic integer of degree $\leqq h$ over $\mathbf{R}$ if the C_{kl} are rational integers. If $|C_{kl}| \leqq B$, we have

$$\mu[f_s(0)] \leqq B(2q+1)^2\{q[1+\mu(\eta)]\}^s.$$

In order to violate (2.2) it is thus sufficient that

$$P > E^{sh}\{B(2q+1)^2 q^s[1+\mu(\eta)]^s\}^{h-1}, \tag{5.2}$$

and this is certainly satisfied for large m if*

$$P > \{B(E_1 q)^m\}^h, \tag{5.3}$$

where E_1 is a fixed integer exceeding $2E[1+\mu(\eta)]$.

In the other direction we have to take into account the conditions imposed by Lemma 2. First we have $n=(2q+1)^2$, $m<2n$, $a_{k,l;s}=(k\eta+l)^s$, $-q\leqq k$, $l\leqq q$, $s=0, 1, \cdots, m-1$, so we can take $A=\{q[1+\mu(\eta)]\}^{m-1}$. It is clear that P must be much larger than B if (5.3) is to hold. This requires $m=o(q^2)$, but on the other hand it is advantageous to have m as large as possible. Condition (3) of Lemma 2 will be satisfied if

$$B > \{P(E_1 q)^m\}^{(m/q)^2} \tag{5.4}$$

where E_1 may be chosen as the same integer as in (5.3).

These inequalities still give considerable leeway for the quantities involved. They show that P must be larger than both B and q^m and that $B>q^{(m/q)^2}$, but they do not show which should be the larger, B or q^m. The later application of Jensen's theorem requires, however, that q^m is dominant.

We can satisfy all the requirements by Gelfond's choice,†

$$B = 3^{q^2}, \qquad m = \left[q^2 \frac{\log\log q}{\log q}\right], \qquad P = \exp\left\{\gamma q^2 \frac{\log q}{\log\log q}\right\}, \tag{5.5}$$

where γ depends only upon the field $\mathbf{K}(\alpha, \beta, \eta)$. It is of course supposed that q is a large number.

We know now that it is possible to choose integers C_{kl}, not all zero, such that $|C_{kl}| \leqq 3^{q^2}$, and $x=0$ is a zero of the function $f(x)$ defined by (4.1), the multiplicity of the zero being at least m. Further

$$|f(x)| \leqq \sum\sum |C_{kl}|\,|\alpha^{kx}\beta^{lx}| < 3^{q^2}(2q+1)^2 e^{\delta q|x|}$$

or

$$|f(x)| \leqq \exp\,[2q^2+\delta q|x|], \qquad q \geqq 2. \tag{5.6}$$

where δ depends only upon α and β.

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* It is enough that $m \geqq h-1$ and $2q \geqq \mu(\eta)$. The estimates required here and below are, up to a certain point, extremely crude.

† $[u]$ is the largest integer $\leqq u$.

6. We come now to the first application of Lemma 3. Let us put $G(z)=f(a+z)$ where a is any point on the circle $|x|=q^{2/3}$ (any power of q less than the first would do). Consider a circle with center at $z=0$ and radius q and substitute in Lemma 3. $G(z)$ is known to have a zero of multiplicity m at $z=-a$. Hence

$$\begin{aligned} |G(0)| = |f(a)| &\leqq \left(\frac{q^{2/3}}{q}\right)^m \exp\left[2q^2 + \delta q(q+q^{2/3})\right] \\ &= q^{-m/3} \exp\left[(2+\delta)q^2 + \delta q^{5/3}\right] \\ &< \exp\left[-\tfrac{1}{3}q^2 \log\log q + 2(1+\delta)q^2\right]. \end{aligned}$$

Hence by the principle of the maximum*

$$|f(x)| < \exp\left[-\tfrac{1}{6}q^2 \log\log q\right], \qquad |x| \leqq q^{2/3}, \tag{6.1}$$

if q is sufficiently large.†

We now apply Cauchy's formula

$$f^{(s)}(x) = \frac{s!}{2\pi i}\int \frac{f(t)dt}{(t-x)^{s+1}}$$

to the circle $|t|=q^{2/3}$ and choose $|x|\leqq\frac{1}{2}q^{2/3}$. Replacing $s!$ by s^s, we get

$$|f^{(s)}(x)| < 2(2sq^{-2/3})^s \exp\left[-\tfrac{1}{6}q^2 \log\log q\right]$$

and the estimate

$$|f^{(s)}(x)| < \exp\left[-\tfrac{1}{12}q^2 \log\log q\right], \quad |x| \leqq \tfrac{1}{2}q^{2/3}, \quad 0 \leqq s \leqq q^2/(\log q) \tag{6.2}$$

for all large q. Since $f_s(j)=f^{(s)}(j)\ (\log\beta)^{-s}$, this implies

$$|f_s(j)| < \exp\left[-\tfrac{1}{24}q^2 \log\log q\right] \tag{6.3}$$

for $j=0\pm1,\ \pm2, \cdots, \pm[\frac{1}{2}q^{2/3}]$, $s=0,\ 1,\ 2, \cdots, [q^2/(\log q)]$, provided q is sufficiently large.

We show next that (6.3) implies that $f_s(j)=0$ by virtue of Lemma 1. We have

$$f_s(j) = \sum_{k=-q}^{q}\sum_{l=-q}^{q} C_{kl}(k\eta + l)^s \alpha^{kj}\beta^{lj}.$$

Choosing rational integers A, B, C, D, and E such that $A\alpha$, $B\beta$, C/α, D/β, and $E\eta$ are algebraic integers, we see that $(ABCD)^{jq}E^s f_s(j)$ is an integer of the field

* The maximum of the absolute value of an analytic function is reached on the boundary of the domain under consideration.

† The integer q is subject to several conditions which assign lower bounds for it. Such bounds involving η where noted in connection with formula (5.3). The choice of m and P in (5.5) requires $\log\log q>0$ or $q>e$. (6.1) calls for $\log\log q\geqq 12(1+\delta)$, a bound depending only upon α and β. (6.2) requires a lower bound which is an absolute constant while (6.3) gives a bound depending upon β. In (6.4) we again have a lower bound for q depending upon δ, but a closer investigation shows that this does not impose any new restrictions on q. (7.1) imposes the last restriction which again depends upon β.

$\mathbf{K}(\alpha, \beta, \eta)$. Suppose that the degree of the latter is H which is at most equal to the product of the degrees of α, β, and η over $\mathbf{R}$. For the maximum of the conjugates of $f_s(j)$ we have the estimate

$$\mu[f_s(j)] \leqq 3^{q^2}(2q+1)^2\{q[1+\mu(\eta)]\}^s\tau^{2jq},$$

where $\tau = \max [\mu(\alpha), \mu(\beta), \mu(1/\alpha), \mu(1/\beta)]$. Hence if (2.2) is to hold we must have

$$|f_s(j)| \geqq [(ABCD)^{jq}E^s]^{-H}\{\mu[f_s(j)]\}^{1-H} > \exp[-\sigma q^2],$$

where σ is a fixed constant depending only upon the field, but not upon q. This contradicts (6.3), however, for large q and, therefore, we must have $f_s(j) = 0$.

Thus $f(x)$ has a zero at each integer j with $|j| \leqq \frac{1}{2}q^{2/3}$ and every zero has a multiplicity of at least $[q^2/(\log q)]$. We now resort to Jensen's theorem once more. We put $G(z) = f(a+z)$, $|a| = q^{4/3}$, $R = q^{3/2}$. Every known zero within the circle $|z| = R$ satisfies the inequality $|z_\nu| < 2q^{4/3}$. Hence using (5.6)

$$|f(a)| < \exp\left\{\frac{q^{8/3}}{\log q}\log(2q^{-1/6}) + 2(1+\delta)q^{5/2}\right\}$$

so that

$$(6.4) \qquad |f(x)| < \exp[-\tfrac{1}{12}q^{8/3}], \qquad |x| \leqq q^{4/3},$$

for all large q.

It is clear that this process can be continued indefinitely and would grind out more and more zeros of $f(x)$ at the rational integers and would thus ultimately lead to the conclusion that $f(x) \equiv 0$.

7. For our purposes it is enough to observe, however, that using Cauchy's formula for $f^{(s)}(0)$ together with (6.4) we obtain

$$(7.1) \qquad |f^{(s)}(0)(\log \beta)^{-s}| < \exp\{-\tfrac{1}{24}q^{8/3}\}, \qquad 0 \leqq s \leqq q^{5/2}.$$

We compare this estimate with formulas (5.1) and (5.2). Taking $1/P$ equal to the right hand side of (7.1), we see that (5.2) is satisfied, *i.e.*, (2.2) is violated and, therefore, $f^{(s)}(0) = 0$ for $s \leqq q^{5/2}$.

But this gives a system of $[q^{5/2}]$ linear equations

$$(7.2) \qquad \sum_{k=-q}^{q}\sum_{l=-q}^{q}(k\eta + l)^s C_{kl} = 0$$

to be satisfied by the $(2q+1)^2$ unknowns C_{kl}. Since the latter are known to be not all zero, any $(2q+1)^2$-rowed determinant must be zero. In particular,

$$(7.3) \qquad \det |(k\eta + l)^s| = 0, \qquad -q \leqq k, l \leqq q, \; 0 \leqq s \leqq 4q(q+1).$$

But this is a Vandermonde determinant which vanishes if and only if two columns are equal. This implies that η is rational against our assumptions. This completes the proof of Gelfond's theorem.

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THE INSTANTANEOUS MOTION OF A RIGID BODY

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1. Combination of instantaneous rigid motions. The assertion that a rigid body is rotating about the x-axis with a certain angular velocity, and rotating at the same time with another angular velocity about the y-axis, puts a strain on the imagination of a student meeting this form of statement for the first time. He is relieved to find that the statement is not one that needs to be interpreted literally, being merely a somewhat irresponsible substitute for a clear formulation in mathematical terms. What is meant is merely that if the body is regarded as a continuous distribution of matter, the vector velocity of each of its points is the geometric resultant of the velocity which *would be* associated with that point by the first rotation and the one which it would have in the second rotation.

The analysis of the most general instantaneous motion of a rigid body can be carried through in terms of ideas of corresponding simplicity. (The phrase "instantaneous motion" is understood for the purposes of this paper to be concerned throughout with the velocities of the points considered, not with their accelerations, which would present a more complex problem.) The notions involved are of course essentially vectorial. In particular, the theory offers notable concrete or semi-concrete exemplification of the significance of the distributive law for vector multiplication. The main features of the theory are presented below from this point of view.* The presentation lays no claim to novelty; its purpose is merely to give one possible arrangement of the details in order for consecutive reading.

* For a formulation in quite different language see for example R. S. Ball, The Theory of Screws, Dublin, 1876, pp. xix–xxiv; for further comparison see W. F. Osgood, Mechanics, New York, 1937, Chapter V.

The problem is that of characterizing the motion itself, without reference to the forces which produce or control it. The characterization is independent also of the size and shape of the body considered. If its motion or that of any three-dimensional portion of it is specified, the velocity which a particle at any point of space would possess if rigidly attached to it is determined. It is to be supposed for the purposes of the present study that *each point of space* has a definite vector velocity assigned to it, in a manner consistent with the *condition of rigidity* presently to be laid down. Any such distribution of velocities, or velocity field, will be called for brevity a *rigid motion*. For similar purposes of abbreviation any set of velocities which there is occasion to consider will be called a *motion*, whether subjected explicitly to the hypothesis of rigidity or not.

It will be seen that in a sense to be carefully defined *the most general rigid motion is either a translation or a rotation or the resultant of a translation and a rotation.*

The *condition of rigidity* is that *for every pair of points P_1, P_2, the vector velocity of P_1 and the vector velocity of P_2 have equal components along the line P_1P_2.* This expresses for instantaneous motion the property that the distance between any two points of the body remains invariable. As between the two opposite directions along the line, it is naturally to be understood that the equal components agree in direction as well as in magnitude.

The discussion here, as already remarked, is concerned exclusively with the motion itself, not with the conditions by which it is produced. It is of no consequence whether any material body to which the conclusions may be applied is *capable* of deformation or not, provided that the velocities which its particles actually possess under specified circumstances are such that the condition of rigidity is fulfilled.

If any two "motions," *i.e.* sets of velocities defined for the points of space, are denoted by M' and M'', their *resultant*, represented symbolically by $M'+M''$, is the motion in which the velocity of each point is the resultant or vector sum of the velocities assigned to that point by M' and M'' separately. It is an immediate consequence of the definitions that *the resultant of any two rigid motions is a rigid motion*, since for each pair of points P_1, P_2 the components of the resultant velocities along P_1P_2 are obtained by algebraic addition of components which are separately equal for the two points.

2. Simple rigid motions: translation and rotation. A *translation* is a motion in which all points have equal vector velocities; that is to say, in less technical but colloquially* more descriptive language, the velocities of all points are equal and parallel. It follows from the definition that a translation is a rigid motion.

Another fundamental type of motion, called a *rotation*, can be described as follows:

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* In an endeavor to minimize technicality of expression, the word *velocity* will be used interchangeably for the vector velocity and for its magnitude, when no misunderstanding seems possible. If the reader desires to have the distinction appear in the record he can of course accomplish this with brevity by using the word *speed* on occasion for the magnitude of the vector velocity.

a) There is a straight line, called the *axis* of the rotation, all of whose points have zero velocity.

b) The velocity of any point not on the axis is perpendicular to the plane containing the point and the axis.

c) All points at equal distances from the axis have equal velocities (the word "equal" being used again as an abbreviation for "equal in magnitude").

d) Points at different distances from the axis have velocities proportional to those distances.

e) All velocities correspond to the same "sense" or direction of turning about the axis.

It will be seen presently that the content of all this verbal description can be condensed into a simple vector formula.

A rotation is "obviously" a rigid motion, in the sense that intuition recognizes it as a kind of motion possible for a rigid body. It is another matter to show formally that it satisfies the definition of a rigid motion in terms of the velocities of an arbitrary pair of points. To give such a proof by the methods of elementary geometry would be a somewhat substantial exercise. It can be done in a few lines by means of vector algebra, for the reason that the geometric relations involved are precisely those which vector algebra recognizes as fundamental. Another reason for stressing this proof is that it appears in some way to form the backbone of the entire theory; simple as it is, all the other proofs to be given subsequently are so much simpler as to be scarcely more than a succession of "remarks."

By the specifications describing a rotation above, the ratio of the velocity of an arbitrary point P to its distance from the axis is the same for all points of space. This ratio is the *angular velocity* of the rotation. If the same unit of length is used for measuring velocity and for measuring distance, the angular velocity is measured in radians per unit of time. All the essential characteristics identifying a particular rotation are conveniently represented by a vector ω lying in the axis, with magnitude numerically equal to the angular velocity just defined, in the adopted scale of measurement, and pointing in the direction in which the rotation would carry a right-handed screw. This ω is called the *vector velocity of rotation*. Like a vector representing a force in the dynamics of a rigid body, it is to be thought of as lying in a definite line, but may be laid off from any point of that line as initial point.

Let O be an arbitrary point of the axis, and let ρ be the vector from O to an arbitrary point P of space. By a check of magnitude and direction it is seen at once that *the velocity of P is represented by the vector product $\omega\times\rho$.* For any specified ω this product defines a set of vectors throughout space having the characteristics of a rotation.

A rotation being given with ω as its vector representation, and a fixed point of reference O on its axis, let ρ_1 and ρ_2 be the vectors from O to a pair of arbitrary points P_1, P_2 anywhere in space. The vector velocities of P_1 and P_2 are $\omega\times\rho_1$ and $\omega\times\rho_2$. Their components in the direction from P_1 toward P_2, if the distance

P_1P_2 is denoted by D, are $D^{-1}\omega\times\rho_1\cdot(\rho_2-\rho_1)$ and $D^{-1}\omega\times\rho_2\cdot(\rho_2-\rho_1)$, and D times the difference between these components is

$$(\omega\times\rho_2-\omega\times\rho_1)\cdot(\rho_2-\rho_1)=\omega\times(\rho_2-\rho_1)\cdot(\rho_2-\rho_1)=0,$$

because of the distributive law for the scalar product, the distributive law for the vector product, and the fact that $\omega\times(\rho_2-\rho_1)$ is perpendicular to $\rho_2-\rho_1$ (or, as an alternative formulation for the last step, the fact that a scalar triple product is zero if two of its factors are alike). The vanishing of the difference means that the condition of rigidity is fulfilled.

As a trivial special case, the assignment of zero velocities to all points will be regarded alternatively as defining a translation of zero magnitude or a rotation of zero magnitude about an arbitrary axis, or will be referred to simply as a zero motion.

3. Analysis of general rigid motion. A description of the most general rigid motion is now obtained through the following sequence of observations.

I. *A rigid motion in which three non-collinear points have zero velocities is a zero motion.* Let O_1, O_2, O_3 be three such points. Let P be any point outside their plane. By the condition of rigidity P can not have any velocity component along any of the lines O_1P, O_2P, O_3P. That is to say, all three lines must be perpendicular to the velocity of P, *if any*. Since they do not lie in one plane, this is impossible, and the words "if any" indicate a condition contrary to fact; the velocity of P must be zero. As for points in the plane $O_1O_2O_3$, let Q be any such point, and let O_4 be a point outside the plane. By the preceding proof the velocity of O_4 is zero, and repetition of the argument with O_4 in place of one of the three points originally given shows that Q has zero velocity.

II. *A rigid motion in which two distinct points have zero velocities is a rotation about the line of these points as axis.* Let O_1, O_2 be the given points. Let P_0 be a point outside their line. If P_0 has zero velocity, all points have zero velocity by the preceding paragraph, and the motion can be regarded as a zero rotation. If P_0 has a velocity, this velocity can have no component along O_1P_0 or O_2P_0, by the condition of rigidity, and so must be perpendicular to both lines and to the plane of the three points. Let v_0 be the magnitude of the velocity of P_0, and r the perpendicular distance of P_0 from the line O_1O_2. Let M denote the given rigid motion; of the two opposite rotations about O_1O_2 with angular velocity v_0/r, let R denote the one that gives to P_0 the velocity which it possesses in M, and $-R$ the other. Then the resultant of M and $-R$, for brevity $M-R$, is a rigid motion in which O_1, O_2, and P_0 have zero velocities, and so all points have zero velocities, by reference to the preceding paragraph once more. That is to say, M as a set of velocities for the points of space is identical with R.

III. *A rigid motion in which a point O has zero velocity is a rotation about an axis through O.* The assertion that if there is a point at rest there must be a whole line of points at rest recalls the similarly striking fact in solid geometry that if two planes have a point in common they must have a whole line in common. It will be seen that one fact is a consequence of the other. In the formulation

of the proof, trivial specializations which have the effect merely of bringing back the conditions of the preceding paragraphs will not be explicitly enumerated.

Let P_1 be a point distinct from O, and ϕ_1 its vector velocity. Since ϕ_1 must be perpendicular to OP_1, by the condition of rigidity, the plane through P_1 perpendicular to ϕ_1 contains O. Let p_1 denote this plane. Let P_1' be any point of p_1 outside the line OP_1. The velocity of P_1' must be perpendicular to OP_1' and to P_1P_1', and so perpendicular to p_1, since P_1 has no component of velocity along P_1P_1' and O has no velocity at all. A selected reference point in p_1 outside OP_1 may then be used to extend the conclusion to points of this line, in analogy with the final step in the proof of I. *All* points of p_1 have their velocities, if any, perpendicular to p_1.

Let P_2 be a point outside p_1, ϕ_2 its velocity, and p_2 the plane through P_2 perpendicular to ϕ_2. By reasoning similar to that just presented, p_2 passes through O, and the velocities of all points of p_2 are perpendicular to p_2.

Since p_1 and p_2 have the point O in common, they intersect in a line. If O_1 is another point of the line of intersection its velocity, if any, must be perpendicular to both planes. As this is impossible, O_1 has zero velocity, and reference to II completes the proof.

IV. *If O is an arbitrarily chosen point, the most general rigid motion is resultant of a rotation about an axis through O and a suitable translation.* (It is understood naturally that either the rotation or the translation may in particular be zero.) Let M be the given rigid motion, let ϕ be the vector velocity of O, and let T be the translation in which all points have this vector velocity, while $-T$ is the opposite translation, with velocity $-\phi$. Then $M-T$, interpreted as $M+(-T)$, is a rigid motion in which O has no velocity, and so by III is a rotation about an axis through O. If this rotation is denoted by R, M is the resultant of R and T.

It is to be noted that only a single point of the axis, not the whole axis, is arbitrary.

V. *The resultant of a rotation and a translation perpendicular to the axis of the rotation is a rotation of equal angular velocity about a parallel axis.* Let R denote the rotation, with ω as the vector representation of its angular velocity, and T the translation, with velocity ϕ; the hypothesis requires that $\omega\cdot\phi=0$.

To demonstrate formally a fact which is obvious to geometric intuition, namely that there is a line of points to which R assigns the velocity $-\phi$, let $\psi=\omega\times\phi/\omega^2$, where ω^2 denotes the square of the magnitude of ω, let O be an arbitrary point of the axis of R, and let O' be the corresponding point such that the vector OO' is ψ. Then the velocity $\omega\times\psi$ which R gives to O' is in fact $-\phi$, as may be seen either by application of the rule for evaluating a vector triple product or by elementary interpretation of the successive operations of simple vector multiplication.

As O describes the axis of R, the point O' describes a parallel line, and the rigid motion $R+T$, giving zero velocity to all points of this line, is a rotation R' about it. The equality of the angular velocities is recognized by comparing

the vector velocities of a pair of corresponding points O' and O in the respective rotations, the line joining them being perpendicular to both axes.

VI. *The most general rigid motion is resultant of a rotation and a translation parallel to the axis of the rotation.** On the basis of this analysis the motion is called a *screw motion*. Let O be an arbitrary point, and let the given motion M be expressed by IV as a rotation R about an axis through O plus a translation T. Let T be resolved into component translations T_0 and T_1, parallel and perpendicular to the axis of R. By V, R and T_1 can be combined into a rotation R_0 about a parallel axis, and M is then the sum of R_0 and T_0.

With the conclusions IV and VI the theory attains a certain stage of completeness. Some additional facts are nevertheless deserving of emphasis.

Of these perhaps the most striking relates to the combination of rotations about intersecting axes. It follows at once from III that the resultant of two such rotations is a rotation about an axis through the point of intersection, since this point has zero velocity in the resultant motion. More specifically, let O be the intersection, let ω_1 and ω_2 be the vectors representing the given rotations, let P be an arbitrary point of space, and let ρ be the vector OP. The resultant of the velocities given to P by the two rotations separately is

$$\omega_1 \times \rho + \omega_2 \times \rho = (\omega_1 + \omega_2) \times \rho,$$

which is the same as the velocity corresponding to a single rotation represented by the vector $\omega_1+\omega_2$. *Instantaneous rotations about intersecting axes can be added vectorially, as an immediate consequence of the distributive law for vector multiplication.*

This fact is the basis for the resolution of an instantaneous rotation into component rotations about a set of coordinate axes.

In the combination of a rotation with a *non-vanishing* translation parallel to its axis, the velocity given to any point by the rotation, having no component parallel to the axis, can not cancel the velocity due to the translation; there is no point with zero velocity, and the resultant motion is not equivalent to any single rotation alone.

Suppose a given rigid motion is resolved in any way into a rotation R_1 and a translation T_1, and again into a rotation R_2 and a translation T_2. Then one may write the equations.

$$R_1 + T_1 = R_2 + T_2, \qquad R_2 = R_1 + T_1 - T_2.$$

(Such manipulation does not involve the setting up of any new type of algebra; it is merely a symbolism for representing comprehensively the corresponding elementary combinations of the vector velocities for the various points of space.) Let the translation $T_1 - T_2$ be resolved into components T', T'' perpendicular and parallel to the axis of R_1. By V, the resultant of the rotation R_1 and the

* Discovery of this theorem is ascribed to G. Mozzi, 1763; see Encyklopädie der mathematischen Wissenschaften, vol. 4: 1, article IV 2, H. E. Timerding, Geometrische Grundlegung der Mechanik eines starren Körpers, pp. 125–189; p. 143.

translation T' is a rotation R' with equal angular velocity about a parallel axis. In the equation

$$R_2 = R' + T''$$

it follows from the preceding paragraph that T'' must vanish. *The rotations* R_1 *and* R_2 *have equal angular velocities about parallel axes; the translations* T_1 *and* T_2 *have equal components in the common direction of these axes.* To restate a part of this conclusion in different words, *a given rigid motion has a vectorial angular velocity* ω *which is determined in magnitude and direction by the rigid motion itself, and is independent of any choice of a particular point of reference as origin.*

If in particular it is supposed that neither T_1 nor T_2 has any transverse component, T' vanishes, R_2 is the same as R_1, and T_2 is the same as T_1; *the resolution given by* VI *is uniquely determined.*

4. Supplementary notes. By way of additional comment, attention may be directed to certain facts with regard to translations which were not needed in the main body of the discussion. The obvious fact that a translation satisfies the condition of rigidity was noted at an early stage. It is almost as easy to see that a rigid motion in which all the velocities are parallel is necessarily a translation. For if P_1, P_2 are any two points such that the line joining them is not perpendicular to the common direction of the velocities, equality of the components along P_1P_2 implies that the total velocities of P_1 and P_2 are equal; from an equation $v_1 \cos\theta = v_2 \cos\theta$ it follows that $v_1 = v_2$, if $\cos\theta \neq 0$. If the line P_1P_2 is perpendicular to the direction of translation, the velocities of P_1 and P_2 are equal to that of any third point P_3 outside the plane through P_1 and P_2 perpendicular to that direction, and so again equal to each other.

Less obvious perhaps at the outset, but easily recognized when the general theorems have been established, is the proposition that if the velocities in a rigid motion are all equal they must be parallel, and the motion is a translation once more. For if the motion is resultant of a translation and a rotation in which the latter is not zero, the magnitudes of the velocities are certainly not all equal.

In summary, of the properties of rigidity, parallelism, and equality of magnitude, any two imply the third.

The reader may be interested to show as an "exercise" that *the most general rigid motion is either a rotation or* (in an infinite variety of ways) *resultant of two rotations about non-intersecting axes.*

A more extensive exercise, which is of importance in itself and will serve to throw much light in retrospect on the theory that has been developed, is to carry through a corresponding discussion in two dimensions, that is to say, with consideration only of points in one plane, and with the assumption that all velocities lie in that plane. A noteworthy difference between two dimensions and three is found in connection with the theorem numbered VI: *Every plane rigid motion is either a translation or a rotation.*

A PAIR OF GENERATORS FOR THE SIMPLE GROUP $LF(3,3)$

JAMES SINGER, Brooklyn College

The linear fractional group $LF(3, 3)$ is simple and of order 5616. We wish to prove the

THEOREM. *The group $LF(3, 3)$ can be generated by two elements of orders 13 and 8, respectively.*

The group $LF(3, 3)$ is simply isomorphic with the collineation group G of the finite plane projective geometry $PG(2, 3)$. If $P_0, P_1, \cdots, P_{12}$, are the thirteen points of the $PG(2, 3)$, the group G can be exhibited as a permutation group of degree 13 on the letters $P_0, P_1, \cdots, P_{12}$.*

For the sake of brevity, we shall designate a point P_i by its subscript i. The $PG(2, 3)$ is then given by the array

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
3	4	5	6	7	8	9	10	11	12	0	1	2
9	10	11	12	0	1	2	3	4	5	6	7	8

The thirteen lines $l_0, l_1, \cdots, l_{12}$ of the $PG(2, 3)$ are the thirteen columns of the array reading from left to right. An element of the group G is then a permutation on the subscripts of the P's which has the property that it sends lines into lines. We shall prove the theorem by showing that every collineation of the $PG(2, 3)$ can be written as a product of the collineations

$$a = (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12),$$
$$b = (1\ 5\ 11\ 8\ 3\ 4\ 10\ 12)(2\ 9\ 7\ 6),$$

of orders 13 and 8, respectively. We can verify by direct trial that a and b are indeed collineations.

We need for the sequel the collineations

$$c = a^{11}ba^{12} = (1\ 6\ 2\ 5\ 9\ 11\ 12\ 4)(3\ 10\ 8\ 7),$$
$$d = a^5ba^2 = (1\ 4\ 9\ 7\ 3\ 5)(2\ 8)(6\ 10\ 11),$$
$$e = aba^8 = (1\ 4\ 6)(2\ 12\ 8)(3\ 5\ 10)(7\ 11\ 9),$$
$$f = a^4ba^3 = (2\ 5\ 10\ 8\ 4\ 6)(3\ 9)(7\ 11\ 12),$$
$$g = a^{10}ba = (1\ 9\ 3)(4\ 6\ 5\ 10\ 7\ 11)(8\ 12).$$

The collineations

* For the necessary definitions, etc., see L. E. Dickson, Linear Groups; O. Veblen and W. H. Bussey, Finite projective geometries, Trans. A. M. S., vol. 7, pp. 241–259, April 1906; R. D. Carmichael, Introduction to the Theory of Groups of Finite Order.

$$c^2,\ b^4,\ b^5,\ b,\ c,\ d^3,\ b^3,\ c^4,\ b^6,\ b^2,\ b^7, \tag{1}$$

leave the point 0 invariant and send the point 1 into the points

$$2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,$$

respectively. The inverses of the collineations (1) leave point 0 invariant and send the points 2, 3, $\cdots$, 12, respectively, into the point 1. Hence there is a collineation

$$A_{ij}, \qquad (i \neq j;\ i, j = 1, \cdots, 12)$$

which leaves point 0 invariant and which sends an arbitrary point i ($\neq 0$) into an arbitrary point j ($\neq 0, i$). The collineations A_{ij}, for all i's and j's, are products of the collineations a and b.

The collineations

$$f^4,\ ed^5f^2g^3,\ ed^5f^2g^3f^4,\ (ed^5f^2g^3f^4)^2,\ ed^5f^2g^3f^2,\ f^2,\ (ed^5f^2g^3)^2,\ ed^5, \tag{2}$$

leave the line $l_0 \equiv 0, 1, 3, 9$, pointwise invariant and send the point 2 into the points

$$4,\ 5,\ 6,\ 7,\ 8,\ 10,\ 11,\ 12,$$

respectively. Hence, as before, there is a collineation

$$B_{ij}, \qquad (i \neq j;\ i, j = 2, 4, 5, 6, 7, 8, 10, 11, 12)$$

which leaves the line l_0 pointwise invariant and sends an arbitrary point i ($\neq 0, 1, 3, 9$) into an arbitrary point j ($\neq 0, 1, 3, 9, i$). The collineations B_{ij}, for all i's and j's, are products of the collineations a and b.

The collineations

$$\begin{aligned}
C_2 &= g^3 = (4\ 10)(6\ 7)(5\ 11)(8\ 12),\\
C_4 &= g^3f^2 = (2\ 10)(5\ 7)(6\ 12)(8\ 11),\\
C_5 &= ed^5f^2 = (2\ 11)(4\ 7)(6\ 8)(10\ 12),\\
C_6 &= ed^5f^4 = (2\ 7)(4\ 12)(5\ 8)(10\ 11),\\
C_7 &= ed^5f^2g^3ed^5 = (2\ 6)(4\ 5)(8\ 10)(11\ 12),\\
C_8 &= ed^5 = (2\ 12)(4\ 11)(5\ 6)(7\ 10),\\
C_{10} &= f^2g^3 = (2\ 4)(5\ 12)(6\ 11)(7\ 8),\\
C_{11} &= ed^5f^2g^3f^4ed^5 = (2\ 5)(4\ 8)(6\ 10)(7\ 12),\\
C_{12} &= ed^5f^2g^3f^2ed^5 = (2\ 8)(4\ 6)(5\ 10)(7\ 11),
\end{aligned} \tag{3}$$

leave the line l_0 pointwise invariant and also leave invariant the points

$$2,\ 4,\ 5,\ 6,\ 7,\ 8,\ 10,\ 11,\ 12,$$

respectively. Furthermore, as we can see from its form, the collineation C_i, for a given i, actually alters all points other than 0, 1, 3, 9, and i. Each of these nine collineations is a product of the collineations a and b.

Now, let w, x, y, z, be an arbitrary set of four independent (that is, no three collinear) points of the $PG(2, 3)$ and let W, X, Y, Z, be another such set of points. Since a collineation of the $PG(2, 3)$ is necessarily a projective collineation, there is just one collineation K which sends w, x, y, z into W, X, Y, Z, respectively. We must show that K can be expressed as a product of the collineations a and b.

The collineation

$$K_1 = a^{W-w}$$

sends the point w into the point W. (The integers $0, 1, \cdots, 12$ now play a double rôle; they are, on the one hand, names of the points and subscripts on the collineations of types A, B, and C. As such, sums and differences of these integers should be reduced modulo 13 to the range $0, 1, \cdots, 12$. On the other hand, the thirteen integers will occur as exponents on the collineations. As such they can, of course, be reduced modulo the order of the collineation.) Let the collineation K_1 send the points x, y, z into the points x', y', z', respectively. The collineation

$$K_2 = a^{-W}A_{x'-W,X-W}a^{W}$$

leaves W invariant, sends x' into X, and sends y' and z' into y'' and z'', say, respectively. Let W and X be on line l_m; the collineation

$$K_3 = a^{-m}B_{y''-m,Y-m}a^{m}$$

leaves W and X invariant, sends y'' into Y, and z'' into z''', say.

The points W, X, Y are not collinear; let W and X be on line l_r, X and Y on line l_s, W and Y on line l_t. No one of the lines l_r, l_s or l_t will contain either z''' or Z. Consider the three collineations

$$k = a^{-r}C_{Y-r}a^{r},$$
$$k' = a^{-s}C_{W-s}a^{s},$$
$$k'' = a^{-t}C_{X-t}a^{t},$$

Each of these collineations leaves the points W, X, Y invariant and hence leaves each of the lines l_r, l_s, l_t invariant. However, k leaves the line l_r pointwise invariant but not the other two, k' leaves l_s pointwise invariant but not the other two, and k'' leaves l_t pointwise invariant but not the other two. Hence k, k', and k'' are distinct collineations. Since each has the form of a collineation of type C and there are just four points in the $PG(2, 3)$ not on l_r, l_s, l_t, one (and only one) of the collineations k, k', k'' sends z''' into Z. Let K_4 be this collineation.

The collineation

$$K = K_1K_2K_3K_4$$

sends the points w, x, y, z into the points W, X, Y, Z, respectively and is a product of the collineations a and b. The theorem follows.

DISCUSSIONS AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Discussions and Notes in the MONTHLY is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

CAN A ROBOT CALCULATE THE TABLE OF LOGARITHMS?

D. F. BARROW, University of Georgia

I think it can, by the following method:* Let r_1 denote the square root of 10, and let subsequent r's be determined by

$$r_{n+1} = \sqrt{r_n}.$$

Table I shows the r's to eight significant figures together with their common logarithms. Eight digits are used because they just fill up the keyboard of the usual calculator, and anyone using it as a teaching device can obtain six or seven accurate digits and thus show how the ordinary five-place table is rounded off. *Any number between* 1 *and* 10 *can be expressed as the product of selected r's, and no r need be used more than once.* This will be obvious after reading the example given below. And so the logarithm of such a number is merely the sum of the logarithms of the selected r's.

Example. Find $\log 235.7 = 2 + \log 2.357$.
First find the largest r in Table I which is not greater than 2.357 (it is r_2), and divide it into this number. The quotient is 1.3254385. Next divide this quotient by the largest r that is not greater than the quotient (it is r_4), and this next quotient is 1.1477824. Next we divide this last quotient by the largest r that is not greater than the quotient (it is r_5), and we obtain 1.0680942.

Now at each step we divide the quotient of the preceding step by the largest r that is not greater than the quotient. *No quotient can be as great as the r that was used to obtain it*; for if it were, we could have divided by the preceding r. Twelve more divisions are required to reach the point where the last quotient, to eight significant figures, is unity. We thus show that 2.357 is equal to the product of $r_2, r_4, r_5, r_6, r_7, r_8, r_{10}, r_{12}, r_{15}, r_{17}, r_{18}, r_{19}, r_{20}, r_{22}, r_{24}$. Hence our mantissa is the sum of the logarithms of these r's, and we have

$$\log 235.7 = 2.37235957.$$

This may sound like a long process, but it takes only about ten minutes on an old fashion hand-crank machine. It is not necessary to write down the successive quotients, but it is necessary to keep up with which r's were used.

* This method is similar to Briggs' original method, Encyclopaedia Britannica, eleventh edition, article on "Logarithm," Vol. 16, p. 875.

THEOREM. *The error in the logarithm of any number, calculated by this method, will not exceed in numerical value the quantity* $16\cdot k\cdot 10^{-k}$, *where k is the number of significant figures used throughout the calculation.*

To prove this we note that in performing any one division the error in the quotient could not exceed one unit in the kth significant figure. Hence, if we let $p(x)$ denote the number of divisions needed to calculate $\log x$, the error in the last quotient could not exceed $p(x)\cdot 10^{-k+1}$. We note further that if n is large (greater than 20), any small increment in r_n, multiplied by $.434\cdots$, will approximate the corresponding increment of $\log r_n$. Consequently the error in $\log x$ due to accumulated errors of division cannot be greater than $4.34\cdot p(x)\cdot 10^{-k}$. To this must be added the error that might accumulate from adding up $p(x)$ values of $\log r$, and this could not exceed $.5\cdot p(x)\cdot 10^{-k}$. *Thus the total error in log x cannot exceed numerically the value* $4.84\cdot p(x)\cdot 10^{-k}$.

Now the greatest value that $p(x)$ could have is not more than the largest value of n in the table, since no r is used more than once. Let N denote this largest value of n, and N will be an integer which approximately satisfies the equation

$$2^{-N} = 10^{-k}.$$

Hence N is approximately equal to $3.32\cdot k$. Substituting this value for $p(x)$ in the formula for maximum error in $\log x$ gives the desired result.

It might be pointed out that usually $p(x)$ is much smaller than N, so that in an isolated calculation, it would be better to use say $5\cdot p(x)\cdot 10^{-k}$ as the upper limit of error instead of the larger value $16\cdot k\cdot 10^{-k}$ which is a uniform upper limit of error for all such calculations.

Example 1. In calculating log 235.7, we note that $p(2.357)$ equals 15. Hence the error in our result cannot exceed $5\cdot 15\cdot 10^{-8}$ or .00000075.

Example 2. It is desired to calculate a 20-place table of logarithms by this method. How many significant figures should be used? Twenty-five digits would suffice since the error will not exceed $16\cdot 25\cdot 10^{-25}$ or $400\cdot 10^{-25}$.

The Robot, since we are building it only in our imagination, might as well be a deluxe affair. It will consist of a dial A, to carry the number whose logarithm is being calculated, a machine B to do division, a machine C to total up the logarithm, and a complicated governor. Dial A carries six digits, and B and C twenty-five digits each; for we are to make out a 20-place table of the mantissas of all numbers from one to one million. Table I must be carried to 25 digits, and will contain about 83 entries; and this table must be permanently recorded and incorporated in the governor. The first number whose logarithm we want is 100,001. We enter this upon dial A, touch a button, and step back.

The governor then enters the number 100,001 on the lower dial of B, enters the number r_1 on the keyboard of B, and initiates the process of automatic division. If the first digit in the quotient is zero (as it will be in this case), B is stopped immediately and r_2 is substituted for r_1. This continues until an r is found which does not give zero for the first digit of the quotient (it will be r_{18}

in this case), and this time B is allowed to complete the division while the logarithm of this r is recorded in C. Next the quotient in the upper dial of B must be transferred to its lower dial. This continues till all the r's have been used. Then the number 100,001 and its logarithm found in machine C are printed. Next, dial A moves up to 100,002, and the process starts over.

It would take quite a while to complete the table. Suppose that each digit in a quotient can be obtained in one second, so that 25 seconds are required to get a quotient of 25 digits. Suppose, furthermore, that 40 divisions are needed and 43 values of r are rejected in calculating one logarithm. At this rate we should secure perhaps two logarithms each hour, and the whole table in 50 years, with no time out for repairs. Of course a hundred robots could divide up the work and finish the job in six months.

Antilogarithms can be found from Table I by reversing the process. It is left to the reader's imagination to conceive another Robot to calculate anti-logarithms.

TABLE I

n	$r_n = 10^{1/2^n}$	$\log_{10} r_n = 1/2^n$
1	3.1622777−	.50000000
2	1.7782794+	.25000000
3	1.3335214+	.12500000
4	1.1547820−	.06250000
5	1.0746078+	.03125000
6	1.0366329+	.01562500
7	1.0181517+	.00781250
8	1.0090350+	.00390625
9	1.0045074−	.00195312+
10	1.0022511+	.00097656+
11	1.0011249+	.00048828+
12	1.0005623+	.00024414+
13	1.0002811+	.00012207+
14	1.0001405+	.00006104−
15	1.0000703−	.00003052−
16	1.0000351+	.00001526−
17	1.0000176−	.00000763−
18	1.0000088−	.00000381+
19	1.0000044−	.00000191−
20	1.0000022−	.00000095+
21	1.0000011−	.00000048−
22	1.0000005+	.00000024−
23	1.0000003−	.00000012−
24	1.0000001+	.00000006−
25	1.0000001−	.00000003−

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Mathematics of Business and Finance. By W. B. Dyess and R. O. Gilmore. New York, McGraw-Hill Book Company, Inc., 1942. 10+221+8+214 pages. $3.50.

The typography and arrangement of material on the pages of this book is excellent. The first 96 pages are devoted to a review of arithmetic, algebra, logarithms, and graphic representation. After the foundation has been made secure the authors take up the usual topics of a course in the mathematics of finance in the next 120 pages. With the text proper is bound the "Compound Interest and Annuity Tables" by F. C. Kent and M. E. Kent. These tables are printed on a paper that minimizes the eyestrain involved in their use.

In the chapter on arithmetic are found some rapid methods of multiplication and division and the methods of checking calculations by casting out nines and elevens. Throughout the book the explanations are clear and illustrative problems are solved in detail by three methods—binomial series, logarithmic computation, and the use of tables of compound interest and annuities. This emphasis on the three procedures which may be used in handling problems involving compound interest is the most admirable feature of the book. In using this text a teacher might well take up the chapter on graphic representation before the chapter on logarithms in order that the class might have the benefit of the discussion of linear interpolation from a graphical standpoint prior to its utilization in the study of logarithms.

The authors have unfortunately used eight or ten place logarithmic tables in many of the computations involved in the illustrative problems (Kent's tables are six place). The results in most, but not all of these problems, have been rounded off to six significant figures, but these answers cannot always be duplicated with the tables in the book. In the chapter on life insurance values for D_x and N_x with ten significant figures are said to be obtained from Kent's tables which give them with six significant figures. A more serious criticism may be made of the author's use of linear interpolation to obtain the value of 1.0325^{20} (page 120) and 1.03125^{-20} (page 125) to ten places of decimals. Three decimals are the best one can hope for in these cases. If logarithmic methods are to be avoided, use should be made of Newton's interpolation formula which is discussed on pages 8–11 in Kent's tables.

The reviewer cannot understand the author's lack of use of the theorem on logarithms of powers. Two examples are cited: In evaluating $S=2{,}478.25(1.0125)^{40}$

on page 120, the text reads, "To find S, we add the logarithms of 2,478.25 and $(1.0125)^{40}$. The value of $(1.0125)^{40}$ is found in the tables on page . . . ." On page 121 the value of $(1.03)^{125}$ is obtained by looking up the values of $(1.03)^{100}$ and $(1.03)^{25}$, finding their logarithms, adding, and then finding the antilogarithm.

In conclusion it may be said that this book will be found to be a very usable text since students pay little attention to the details of illustrative problems.

L. A. DYE

First Year College Mathematics. By C. C. Richtmeyer and J. W. Foust. New York, F. S. Crofts and Co., 1942. xi+461 pages $3.25.

As stated by the authors "This book presents in one volume the essentials of a first year's work in college mathematics. It includes a thorough coverage of algebra, trigonometry, and plane analytics, and an introduction to the fundamental concepts of differentiation and integration."

The notions of the calculus are introduced early, in Chapter III, by means of "average rate of change" and "exact rate of change". Here the reviewer was quite distressed by the authors' complete ignoring of that "fundamental concept of differentiation", the limit. The word does not even appear in the book. This would not be a serious omission if the notion were present. On p. 41, where differentiation is first discussed, appears the phrase "the interval Δx approaches zero" with nothing like an adequate explanation of its meaning. It is unthinkable, to the reviewer, that the Δ-process be presented to a student without mention of the idea of limit. As far as he can see the word "approaches", wherever it is used by the authors, can be and will be replaced by the naïve reader by "equals". Such writing will lead the student to the belief that evaluation of the limit, by whatever name the process may be called, and substitution are identical. Indeed, when we get to p. 126 and a discussion of the differentiation of irrational functions (marked *optional* in the text) the reader is suddenly instructed to rationalize the numerator of the difference-quotient; he must certainly wonder why. Previously the Δx in the numerator just "canceled" the one in the denominator. Here he can see no reason for not obtaining that hardy perennial, $D_x y = 0/0$.

Except for this gap the authors have written a simple, clear, and well-unified text. They have written their book so that the function concept is the unifying principle. A perusal of the chapter headings is indicative of their objectives. The headings are: I. A Review of Elementary Algebra; II. Functions and Graphs; III, Derivatives and Antiderivatives; IV. First Degree Functions and Linear Equations; V. Second Degree Functions and Quadratic Equations; VI. General Polynomial Functions and Polynomial Equations; VII. Fractional Functions and Equations; VIII. Irrational Functions and Equations; IX. Trigonometric Functions of Acute Angles; X. Common Logarithms; XI. Solution of Right Triangles by Logarithms; XII. Trigonometric Functions of General Angles; XIII, Solution of Oblique Triangles; XIV. Radian Measure, Inverse Functions,

Graphs; XV. Systems of Equations; XVI. Introduction to Analytic Geometry; XVII. The Straight Line; XVIII. The Circle; XIX. Parabola, Ellipse, and Hyperbola; XX. Transformation of Coordinates; XXI. Tangents and Normals; XXII. Polar Coordinates; XXIII. Parametric Equations; XXIV. Progressions; XXV. Permutations, Combinations, Probability; XXVI. Mathematical Induction and Binomial Theorem; XXVII. Complex Numbers; XXVIII. Natural Logarithms and Exponentials.

An excellent feature of the book is the series of "Self-tests" which appear at the end of each chapter. These consist of questions which lead to a complete review of the chapter. Answers to these tests are given at the back of the book.

Many of the chapters also have, as concluding sections, optional topics which the teacher with the above-average class will want to include in the course. Some of these are unique in a text of this sort, such as the section on Musical Sounds. There is an abundance of illustrative examples and excellent exercises. Answers are given to the odd-numbered ones.

In the opinion of the reviewer First Year College Mathematics is a good text and merits careful consideration by teachers. L. L. LOWENSTEIN

A Diagnostic Study of Students' Difficulties in Mathematics in First Year College Work. By E. N. Boyd. (Contributions to Education, No. 798.) New York, Bureau of Publications, Teachers College, Columbia University, 1940. 152 pages. $1.85.

A large number of college graduates have told me that the instruction they had in college mathematics was inferior to that which they received in preparatory school. For many years preparatory school teachers have given diagnostic tests to determine a student's difficulties and then proceeded to help the student over these hurdles. On the other hand, most college instructors have taken the attitude—here's the subject, understand it if you can and if you are so dumb you can't, get out! It is indeed a pleasure to find one college instructor who is interested in determining what difficulties her students have. This book deals with tests given freshmen in the general course in mathematics at Hunter College. The tests cover trigonometry, analytic geometry and the elements of differential calculus. It is well worth the while of every teacher of college freshmen to read these tests and study the results, for, by so doing, I am confident he would improve his teaching. F. M. MORGAN

The Reading of Verbal Material in Ninth Grade Algebra. By Margaret Grace McKim. Bureau of Publications, Teachers College. Columbia University, 1941. 8+133 pages. $1.60.

This is a study based on the realization of the importance of the reading problem to the child and the necessity of specific training in a special field like algebra.

There is a minute description of the construction and adaptation of tests

given, in the final experiment, to one hundred twenty New York City pupils of the ninth grade. The tests cover two areas—the reading of explanatory passages and the reading of problems. The tests are reproduced.

The results are then tabulated in juxtaposition with those obtained from the Terman Group Test of Mental Ability, the New Stanford Reading Test, the Orleans Algebra Prognosis Test, and the Mid-Term and Final examinations given in the school.

The significance of the intercorrelations are a problem of statistical measurement. Dr. McKim finds, common to all the tests and to the tests in groups, elements which are significant in determining variation of reading efficiency in passing from non-algebraic to algebraic material and from one kind of algebraic material to another, also in determining relationship between reading ability and achievement in algebra.

MILDRED W. HAFF

Early Military Books in the University of Michigan Libraries. By Thomas M. Spaulding and Louis C. Karpinski. The University of Michigan Press, 1941. 45 pages and 37 plates. $2.00.

This is a compilation of titles of books published before 1800 on military art and science (military history excluded), that can be found in the University of Michigan Library. There is an index of military books by mathematicians and mathematical works with sections on military science.

The plates reproduce the titles pages of some of the books listed.

MARK KAC

NEW BOOKS RECEIVED

Non-euclidean Geometry. By H. S. M. Coxeter, Mathematical Expositions, No. 2, Toronto, University of Toronto Press, 1942. 15+281 pages. $3.25.

Differential and Integral Calculus. By H. M. Bacon. New York and London, McGraw-Hill Book Co., 1942. 7+771 pages. $3.75.

Calculus. By G. E. F. Sherwood and A. E. Taylor. New York, Prentice-Hall, Inc., 1942, 14+503 pages. $3.75.

Analytic Geometry and Calculus. By H. B. Phillips. Cambridge, Mass., Addison-Wesley Press, 1942. 10+490 pages. $5.00.

Metric Methods in Finsler Spaces and in the Foundations of Geometry. By H. Busemann. (Annals of Mathematics Studies, No. 8.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1942. 247 pages. $3.00.

Topics in Topology. By S. Lefschetz. (Annals of Mathematics Studies, No. 10.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1942. 139 pages. $2.00.

Finite Dimensional Vector Spaces. By P. R. Halmos. (Annals of Mathematics Studies, No. 7.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1942. 5+196 pages. $2.35.

Analytic Topology. By G. T. Whyburn. (American Mathematical Society Colloquium Publications, vol. 28.) New York, American Mathematical Society, 1942. 10+278+1 pages. $4.75.

An Outline of Probability and Its Uses. By M. C. Holmes. Minneapolis, Burgess Publishing Co., 1936. 8+119 pages. $1.59.

Mathematics in Human Affairs. By F. W. Kokomoor. New York, Prentice-Hall, Inc., 1942. 9+754 pages. $5.35.

A Mathematics Refresher. By A. Hooper. New York, Henry Holt and Co., 1942. 10+342 pages. $1.90.

The Methodology of Pierre Duhem. By A. Lowinger. New York, Columbia University Press, 1941. 184 pages. $2.25.

Mathematics of Modern Engineering. By E. G. Keller. Volume II. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1942. 12+309 pages. $4.00.

Algebra. By P. L. Evans. New York, Ginn and Company, 1941. 8+126 pages. $1.25.

Plane Trigonometry with Tables. By P. L. Evans. New York, Ginn and Company, 1941. 8+84 pages. $1.25.

Calculus. By P. L. Evans. New York, Ginn and Company, 1941. 8+126 pages. $1.25.

Plane Trigonometry. By R. E. Heinman. New York and London, McGraw-Hill Co., Inc., 12+167 pages. $2.00.

Spherical Trigonometry and Tables. By W. A. Granville, P. A. Smith and J. S. Mikesh. New York, Ginn and Company, 1942. 18+60+4+43 pages. $1.25.

Plane Trigonometry, Solid Geometry and Spherical Trigonometry with Tables. By W. W. Hart and W. L. Hart. Boston, D. C. Heath and Co., 1942. 8+280 +124 pages. $2.60.

Essentials of Plane and Spherical Trigonometry. By A. H. Sprague. New York, Prentice-Hall Inc., 1942. 9+169 pages. $1.35.

Blueprint Reading at Work. By W. W. Rogers and P. L. Welton. New York, Silver, Burdett Co., 1942. 8+136 pages. $1.28.

Shop Mathematics at Work. By P. L. Welton and W. W. Rogers. New York, Silver Burdett Co., 1942. 4+204 pages. $1.56.

Mathematics for Electricians and Radiomen. By N. M. Cooke. New York and London, McGraw-Hill Book Company, Inc., 1942. 8+604 pages. $4.00.

Fundamentals of Radio. By E. C. Jordan, P. H. Nelson, W. C. Osterbrook, F. H. Pumphrey and L. C. Lynne. E. V. Everitt, Editor. New York, Prentice-Hall, Inc., 1942. 12+400 pages. $5.00.

Table for Sine and Cosine Integrals for Arguments from 10 *to* 100. New York, Works Projects Administration, 1942. A. N. Lowan, Technical Director. 32+189 pages. $2.00.

Original Tables to 137 *Decimal Places of Natural Logarithms for Factors of the Form* $1\pm n\cdot 10^{-p}$, *Enhanced by Auxiliary Tables of Logarithms of Small Integers*. By H. S. Uhler. New Haven, Conn., Horace S. Uhler, 1942. 120 pages.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Allegheny College, Meadville, Pa.

REQUESTS FROM CLUBS

In our circular letter of May 20, 1942 we asked "What material, in addition to club reports, do you consider most suitable and valuable for inclusion in the limited publication space of this department of the MONTHLY? Do you know of any papers by students or faculty which might be considered for publication in this department?" Nine replies were received to the first question, but no suggestions were received in answer to the second. Topics with references, and bibliographies were requested by several. They would be welcomed by the editor. Others requested papers suitable for use on mathematics club programs. Another request was for a solicited list of some students who would like to carry on private correspondence on mathematical topics of mutual interest with students from other clubs. Names of such students, with the topics on which they would like to correspond, may be sent to the editor. A request for suggestions of mathematical games for informal parties was also received. We hope the suggestions which follow will prove useful for such occasions.

A MATHEMATICAL PAUL JONES

How to run a *Paul Jones* dance so that no one dances with the same partner twice is easily solved by a mathematician if the number of couples is $p-1$, where p is a prime. Assign numbers from 1 to $p-1$ to the men and also to the ladies. After one round with the partner of the same number, require the gentlemen to double their original numbers, and subtract p if the result exceeds p. Each should then have a new partner. On the third round the gentlemen should multiply their original numbers by three, and on the fourth by four, etc., subtracting multiples of p in each case. Each time the men should find new partners, until each has danced with every lady. If two men claim the same lady at the same time, she should check their arithmetic and pick the correct one.

A variant of the same idea is the following game designed to test speed in multiplication and to provide a little amusement at informal mathematical parties. Let 12 (or $p-1$) persons be seated around a circle on seats which are assigned numbers from 1 to 12 (or $p-1$). The Number One man (who is considered to be at the bottom of the ladder) picks a key number n, less than 13 (or p), and announces it. He then calls out a multiplier m less than 13 (or p). Everyone then stands *except* the one whose seat number s is such that $mn-s$ is divisible by 13 (or p). If that man also stands, he goes to the bottom of the ladder, all those below him move up one seat, and his old seat number s becomes the new key number n. Otherwise everyone sits, and the Number One man tries

another multiplier. After a specified time, the contestant who occupies the highest numbered seat is declared the winner. If the number of contestants is not of the form $p-1$, p prime, certain seats can be turned backwards at the start and left vacant, although they are assigned seat numbers. Then if Number One calls a multiplier m such that mn corresponds to a blank seat, he is declared out of the game, and all seats (including vacant seats) are renumbered, starting with the lowest occupied seat as Number One. The play proceeds until the time is up or until all but one of the contestants are eliminated.

CLUB REPORTS 1941–42

Pi Mu Epsilon, University of California at Los Angeles

Meetings were held once each month during the school year, at which the following papers were presented: *Functions of positive derivatives*, by Dr. A. E. Taylor, *Mathematics in oceanography*, by Mr. Robert Gordon, *Mathematical quotations from nonmathematical sources*, by Dr. Max Zorn, *Statistical methods of weighting examination questions*, by Dr. Paul Hoel, *Summation of m^{th} powers of the first N integers*, by Mr. William Gustin. Six new members were accepted during the year. The pledges for the fall and for the spring were presented at regular meetings. The fall initiation was held at the Helen Mathewson Club on Nov. 14, 1941; the spring initiation was held at the home of Dr. Paul Hoel on April 25, 1942; both in the evening. At each initiation the pledges were required to present a program lasting half an hour. Donald Wall was this year's winner of the ten dollar prize in the annual calculus contest sponsored by the organization. Officers for 1941–42 were as follows. Director, Robert White (Sept. to Feb.), Melvin Henry (Feb. to June); Vice-Director, Dorothy Stanley; Secretary, Jane Zartman; Treasurer, Wendell Mason; Program chairmen, Joel Ginsberg, Clay Perry, Gordon Overholtzer; Scholarship committee, Taffee Tanimoto; Sponsors, Dr. A. E. Taylor and Dr. Ralph Byrne.

Pi Mu Epsilon, University of Georgia

The *Georgia Alpha Chapter of Pi Mu Epsilon* sends greetings and reports a successful year. The chapter held regular meetings twice each month. Included among the topics discussed at the program meetings were *Induction into the army* by Captain H. A. Robinson of Ft. McPherson, *Mathematics in art—the Italian, the Chinese, and the Egyptian methods of projection* by Professor Jean Charlot of the Art Faculty, *Discussion and demonstration of minimum surfaces*, *Extracting cube roots with calculating machines*, *Graphical methods—new applications*, *Solution of practical problems presented to the group*. On April 16 at Memorial Hall the annual initiation banquet was held. Fifteen new members were initiated and the new officers were elected. Officers for 1941–42 and 1942–43 were as follows. Faculty Director, Dr. D. F. Barrow (41–42), Professor Forrest Cumming (42–43); President, J. A. Johnson Jr. (41–42), Gladys Feagin (42–43); Vice-President, Ann H. Davis (41–42), J. R. Mangham (42–43); Secretary-Treasurer, A. T. Harmon (41–42), Robert Collat (42–43); Corresponding secretary, Iris Callaway (41–43).

Pi Mu Epsilon, University of Arkansas

Pi Mu Epsilon held regular meetings during the year with discussions on various topics. Dr. Harrison Hale of the Chemistry Department gave the principal address on *The value of mathematics* at the annual fall banquet. Colonel Neilson gave his views on *The graveness of the war situation* at the spring banquet. There was a total of forty members during the year of whom thirteen were initiated the second semester. Several members bought keys and watch charms. Students who were initiated at the spring banquet were asked to be present at the annual Honors Day program, when the names of the new pledges were announced. The annual spring picnic was a success in spite of the threatening rain. New officers elected for 1942–43 were: Director, V. W. Adkisson; President, Francis Strabola; Secretary, Louise Williams; Treasurer, John Jacks.

Pi Mu Epsilon, University of Pennsylvania

Five talks were given as follows: *Breaking the law, upholding the law* by I. J. Schoenberg, *The Waring problem* by William Turanski, *Lissajou curves* by Fred Orttung, *Double star orbits* by P. M. Witman, *The observational approach to the problem of stellar multiplicity* (with slides) by Peter van de Kamp, director of the Sproul Observatory, Swarthmore College. Prizes were given in a problem contest to William Turanski and Tong Hing. Fred Orttung received a prize as the best student speaker. The initiation banquet was held in the Benjamin Franklin room of Houston Hall in January with Professor C. J. Rees of the University of Delaware as guest speaker. Officers for 1941–42 were President, Morton Brown; Secretaries, Ruth Tobias and Eli Perry.

Mathematics Club, Wellesley College

Meetings this year were informal; no real papers were presented. Rather, students gave short talks on the following topics: *Women in mathematics*; biographical talks on *Sir Isaac Newton, Leibniz, Archimedes*; the applications of mathematics in teaching, architecture, economics, and geology; and impossibilities such as five-sided squares. Source material for these talks consisted of standard reference books such as Bell's *Men of Mathematics.* The club also produced a play by Janet Brown '35, written as a mathematical burlesque of *Romeo and Juliet* and entitled *Ratio and Jacobean, a Trajectory.* It was not very serious, and we had a great deal of fun with it. The club report was written by the retiring secretary, Ruthven Tremain. Officers for 1942–43 are: President June Nesbitt '43, Vice-President Betty Ann Wilson '43, Treasurer Martha Adams '43, Secretary Phyllis Fox '44, Junior Executive Elizabeth Bird '44, Faculty Adviser, Miss Lennie Copeland (Chairman).

Kappa Mu Epsilon, Drake University

The *Mathematics Club of Drake University* became the Iowa Beta Chapter of *Kappa Mu Epsilon* on May 27, 1940. Officers for 1941–42 and 1942–43 were elected as follows. President, Bob Goss (41–42), Bob Lambert (42–43); Vice-President, Norman Landess (41–42), Bob Hansen (42–43); Secretary, Julia Rahm (41–43); Treasurer, Bob Lambert (41–42), Alex Smotkin (42–43); Pledge Master, Earl Carlson (41–42), John McKiernan (42–43); Publicity, Don Johnson (42–43). Professor Floy Woodyard acted as Faculty Adviser. The program for the year included the following topics. *Mathematics for the Navy* by Lieutenant Commander Lauder, *Mathematics in industry and the Telephone Company* by Mr. McLellan of the Telephone Company, *Mathematics in the present emergency and in aviation* by Professor Mehlin.

Mathematics Club, Milwaukee-Downer College

The two papers presented were *Dynamic symmetry* by Professor E. R. Beckwith, and *Taxes and government financing* by Miss Ferol Bosl. The remaining meetings of the year were devoted to instruction in the use of the slide rule, in view of the need for this skill in defense work. Instruction was open to anyone in the college who cared to receive it. Contributions of work and money were made to the Red Cross campaign. Officers for 1942–43 are as follows. President, Dorothy Puelicher; Secretary-Treasurer, Dorothy Rodgers.

Kappa Mu Epsilon, Kansas State Teachers College at Emporia

A total undergraduate membership of forty-five students participated in eight meetings throughout the year, which alternated with meetings of the Mathematics Club. Discussions of the scientific work of the General Motors Corporation, of the importance of mathematics in national defense, of the use of spherical trigonometry in aviation and navigation, and of meteorology were on the program. Officers for the summer were: President Pascal, Eugene Etter; Vice-President Gauss, Kay Wilch; Secretary-Treasurer, Mrs. Ruby Norris. Officers elected for 1941–42 were: President Pascal, Warren Burns; Vice-President Gauss, Alfred Freeman; Secretary Eratosthenes, Frances Breneman; Treasurer Bhaskara, Rosemary Haslouer; Historian Ahmes, Mary Townsend; Corresponding Secretary Descartes, Professor C. B. Tucker; Sponsor Thales, Dr. O. J. Peterson.

Mathematics Club, Kansas State Teachers College at Emporia

The place of mathematics and physics in defense by Dr. Cram, *How to study for finals* by a round table consisting of Lester Meisenheimer, Wilbur Schoof, Carter Sigel, and Frances Breneman, *The light weight champion* (magnesium) *in the present emergency*, by Dr. Blackman, *Prices and free enterprise* by Dr. Pickett, and *Games of chance*—these were the features of a program of eight meetings which wound up with a picnic on May 6. Officers were: President, Mary Alice Anderson; Vice-President, Carter Sigel; Secretary-Treasurer, Frances Peterson.

Kappa Mu Epsilon, Upsala College

The Alpha Chapter of New Jersey closed a very successful year with the annual banquet, initiation of five new members, and the installation of next year's officers. Nine meetings were held during the year, for seven of which the students worked out their own projects and researches. Topics included were: *Fundamentals of drawing from a machinist's standpoint*, by Joseph May, *Sidelights on the development of trigonometry with a note on Mollweide's theorm*, by Lillian Meisel, *The derivation of the normal probability curve*, by Anne Zmurkiewicz, *Mathematical paradoxes with a note on "Geometry on wheels,"* by Marjorie Nicoll, *How mathematics makes money for the worker*, by Bernard Morrow, *Sir Isaac Newton*, by Edward Cohen. Two outside speakers addressed the chapter. Professor Virgil Mallory of Montclair State Teachers College gave an elucidating talk on the *Mathematics of defense* at mid year, and Mr. Edward Molina, research statistician with the Bell Telephone Company, gave a talk on *Probability* at the annual banquet on May 22. Some insight into the history of mathematics was provided at every other meeting by having each officer discuss in turn the life and contribution of his chapter name-sake. Thus Secretary Abel, Vice-president Appollonius, Treasurer Fibonacci, and Secretary Descartes spoke to us from the distant past to the living present. Officers installed on May 22 are: President, Phyllis Gustafson; Vice-President, Edward Cohen; Secretary, Marjorie Nicoll; Treasurer, Lillian Meisel; Historian, the retiring president Anne Zmurkiewicz.

Mathematics Club, Brown University

The club held six monthly meetings through the year, which were scheduled in advance and announced in a printed program before the first meeting. Two students spoke at each of four meetings, after preparing their talks in consultation with a member of the faculty. Their topics were *Short cuts in arithmetic* by Doris Keighley, *The duodecimal system* by R. L. Johnson, *Alignment charts* by R. P. Gosselin, *Areas by machine*, by Dieter Kurath, *Magic squares* by Arline Major, *Chess, caves, and cannibals* by E. J. Bernier, *Linkages* by W. F. Jones, Jr., *Mechanical devices for computation* by Granino Korn. At the January meeting Professor George Polya spoke on *Counting chemical compounds*, and at the March meeting Lieutenant Commander E. T. Goyette spoke on *The mathematical inaccuracy of navigation.* A group picture was taken after this meeting. The club picnic scheduled for May 7 was canceled because of the accelerated program and gasoline rationing Officers for the year were as follows. Committee on program: Professor J. S. Frame, *faculty representative*; Tamara Bachman, R. L. Johnson, Dieter Kurath, Ellen Swanson, Paul Tamarkin. Committee on arrangements: Granino Korn, *chairman*; E. J. Bernier, J. E. Cook, Jr., Ruth Cunningham, W. F. Jones, Jr., Doris Keighley, Arline Major.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 546. *Proposed by W. E. Buker, Pittsburgh Public Schools*

Show that $\frac{1}{5}x^5+\frac{1}{3}x^3+\frac{7}{15}x$ is an integer for every integral value of x.

E 547. *Proposed by V. Thébault, San Sebastián, Spain*

A diameter d of the circumcircle of an equilateral triangle ABC cuts the sides BC, CA, AB in points D, E, F. Prove that the Euler lines of the three triangles AEF, BFD, CDE form a triangle symmetrically equal to ABC, the center of symmetry lying on d.

E 548. *Proposed by R. V. Heath, Wall St., New York City*

Find a perfect square of seven digits with all digits even and positive. Show that the digits of a perfect square (>9) cannot be all *odd*.

E 549. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

The face planes of a proper tetrahedron intersect a circumscribed quadric cone in four parabolas. What conditions are thus imposed on the cone?

E 550. *Proposed by D. H. Browne, Buffalo, N. Y.*

Prove that, for every positive even value of m, the sum of the mth powers of alternate numbers from 1 or 2 up to n is a polynomial function of n (of the same form whether n be odd or even).

SOLUTIONS

An Automorphic Number

E 506 [1942, 120]. *Proposed by R. V. Heath, Wall St., New York.*

Show that, for every positive integer n, the last $n+9$ digits of

$$90625^{2^n}$$

form an automorphic number. [See 1941, 407].

Solution by E. P. Starke, Rutgers University

Since 90625 is divisible by 5^5, the number

$$P = 90625^{2^n}$$

is divisible by $5^{5\cdot 2^n}$ and hence by 5^{9+n} (as $5\cdot 2^n \geqq 9+n$). Furthermore, $P-1$ is divisible by 2^{9+n}, as is easily established by induction from the case when $n=0$, using the relation

$$(k2^{9+n}+1)^2 = (k^2 2^{8+n}+k)2^{10+n}+1.$$

It follows that P^2-P is divisible by both 5^{9+n} and 2^{9+n}, hence by 10^{9+n}. Finally a may be chosen so that the number

$$M = P - a\cdot 10^{9+n}$$

consists of the last $9+n$ digits of P. The difference

$$M^2 - M = P^2 - P - 2a.10^{9+n}P + a^2\cdot 10^{18+2n} + a\cdot 10^{9+n}$$

is seen to be divisible by 10^{9+n}, and thus the last $n+9$ digits of M^2 are the digits of M, as required.

Editorial Note. This problem suggests the following practical rule for successively computing the digits of an automorphic number (from right to left): If N is the odd automorphic number of m digits, then the last $m+1$ digits of N^2 form an automorphic number with one extra digit.

Proof. We have $N^4-N^2=\frac{1}{2}(N+1)\cdot 2N\cdot(N^2-N)$.
But N is an odd multiple of 5, and $N^2-N\equiv 0 \pmod{10^m}$. Hence

$$N^4 - N^2 \equiv 0 \pmod{10^{m+1}}.$$

We can now find both the automorphic numbers. For, if N is the odd one, the even one is 10^m-N+1.

An Extension of E 467 to Three Dimensions

E 507 [1942, 120]. *Proposed by V. Thébault, San Sebastián, Spain*

In an orthocentric tetrahedron with orthocenter H and circumcenter O, show that the radical planes of the circumsphere with the respective spheres whose diameters are the four medians, meet the Euler lines of the corresponding faces in four points lying in a plane perpendicular to OH. (The *medians* of a tetrahedron join its vertices to the centroids of the opposite faces.)

Solution by L. M. Kelly, U. S. Coast Guard Academy

Label the four points P_1, P_2, P_3, P_4. It is evident that if we prove that planes through these points perpendicular to OH all meet OH in the same point Q, the proposition will be established. We first recall the following known properties of an orthocentric tetrahedron $A_1A_2A_3A_4$:

(1) the four altitudes are concurrent,

(2) the centroid G is the midpoint of OH,

(3) the product of the segments into which the orthocenter divides the altitudes is the same for all four altitudes,

(4) if G_1 is the centroid of the face $A_2A_3A_4$, then $3GG_1=GA_1$.

Now let H_1 and O_1 be the orthocenter and circumcenter of $A_2A_3A_4$, and let M_1 be the midpoint of A_1G_1. In the plane OHA_1, P_1 may be determined by drop-

ping a perpendicular from A_1 on OM_1 and producing it to meet O_1H_1. To determine Q it is only necessary to drop a perpendicular from P_1 on GH. Draw G_1H and note that it is parallel to OM_1, and hence perpendicular to AP_1. Thus H is the orthocenter of the triangle $A_1G_1P_1$. Finally, if P_1H is drawn meeting A_1G_1 in R_1, we see that the triangles GHR_1 and P_1HQ are similar, whence

$$GH\cdot HQ = R_1H\cdot HP_1 = H_1H\cdot HA_1,$$

which is constant (*i.e.*, the same as when P_1 is replaced by P_2, P_3, or P_4).

Also solved by the proposer.

Perfect Bridge Hands

E 508 [1942, 120]. *Proposed by R. K. Allen, Montpelier, Vermont*

How many bridge hands are there where all thirteen tricks can be taken at no trump regardless of the distribution of the cards? It is assumed that declarer will always play his highest cards and not intentionally lose any tricks. (Cf. E 448.)

Solution by H. W. Norton, University of Chicago

In order that a bridge hand may take all tricks at no trump regardless of the distribution of the remaining cards, it is necessary that that hand hold all the aces; the nine other cards may be variously distributed. For example, if the nine are all in one suit, there are three cards in that suit outstanding, and it is necessary that the nine should include the King and Queen and any seven of the remaining cards. Thus there are $4\binom{10}{7}=480$ such hands, since the nine may be in any one of the four suits. The complete list of posibilities with their respective frequencies is as follows:

9—0—0—0	480	5—2—2—0	12
8—1—0—0	1512	5—2—1—1	12
7—2—0—0	672	4—4—1—0	12
7—1—1—0	672	4—3—2—0	24
6—3—0—0	84	4—3—1—1	12
6—2—1—0	168	4—2—2—1	12
6—1—1—1	28	3—3—3—0	4
5—4—0—0	12	3—3—2—1	12
5—3—1—0	24	3—2—2—2	4

The total is 3756.

It may be noted that it is necessary to hold consecutive cards below the Ace in any suit in which fewer than seven cards (including the Ace) are held. In criticizing the frequency with which so-called "perfect" bridge hands are reported in the press (*London Times*, May 15, 1939) I called attention to this new type of hand, which is perfect in that it shows before the bidding that every trick can be taken. Though it is about three years since I called attention to this type of hand, and though the ordinary "perfect" hand is reported several times each year, I have not yet seen mention of this hand which is perfect from the

trick-taker's point of view. Yet it is nearly a thousand times as frequent, with proper shuffling; and (according to Culbertson) imperfections of shuffling should make it relatively more frequent.

Also solved by W. E. Buker and the proposer.

The Square Bedroom and the Music Room

E 509 [1942, 121]. *Proposed by the late J. E. Trevor, Cornell University*

An architect is designing a house for his client's seventy-five foot lot. The "square bedroom" is to have a square floor, and it will contain an ordinary double bed and other furniture. The owner specifies that the music room shall be two feet longer than it is wide, and that its floor-area in square feet shall be three times that of the square bedroom. It is also specified that the widths of the two rooms shall be integer numbers of feet. Find these widths.

Solution by W. E. Buker, Pittsburgh Public Schools

Let $s\times s$ and $(t-1)\times(t+1)$ be the dimensions of the bedroom and music room. Then we have to find integers s, t, satisfying $3s^2=(t-1)(t+1)$, or

$$t^2 - 3s^2 = 1. \tag{1}$$

This is an instance of the famous misnamed "Pellian Equation," which can be solved by several methods, *e.g.*, by continued fractions. We know that all solutions can be derived from the obvious solution

$$t_0 = 1, \qquad s_0 = 0$$

by means of the recurrence relations

$$t_{n+1} = 2t_n + 3s_n,$$
$$s_{n+1} = t_n + 2s_n.$$

Thus the simplest positive solutions are:

$$t = 2,\ 7,\ 26,\ 97,$$
$$s = 1,\ 4,\ 15,\ 56.$$

Clearly, the third solution alone satisfies the conditions of the problem: so the desired widths are 15 and 25.

Also solved by R. K. Allen, Paul Brock, William Douglas, Howard Eves, L. M. Kelly, C. C. Oursler, C. A. Rupp, E. P. Starke, and the proposer.

Stirling Numbers of the Second Kind

E 510 [1942, 121]. *Proposed by S. H. Gould, University of Toronto*

Given $a_{p,q}=a_{p,q-1}+qa_{p-1,q}$, $a_{p,1}=a_{1,q}=1$, $(p,\ q=1,\ 2,\ 3,\ \cdots)$, prove

$$(p-1)a_{p,q} = \sum_{k=1}^{p-1} a_{p-k,q}\left\{(q+1)^{k+1} - \binom{p+q}{k+1}\right\}.$$

Solution by the Proposer

Comparing the given recurrence relation with Bjorling's formula

$$C_n^{k+1} = nC_n^k + C_{n-1}^k$$

for the numbers $C_n^k = \Delta^n 0^k/n!$ (*Journal für die reine und angewandte Mathematik*, vol. 28, 1844, p. 284), we see that

$$a_{p,q} = C_q^{p+q-1}.$$

Thus the well-known formula

$$C_{m+1}^{n+1} = \sum_{k=m}^{n} \binom{n}{k} C_m^k$$

(Boole, *Finite Differences*, 1872 or 1926, p. 29) is equivalent to

$$a_{p+1,q+1} = \sum_{k=-1}^{p-1} \binom{p+q}{k+1} a_{p-k,q} = a_{p+1,q} + \sum_{k=0}^{p-1} \binom{p+q}{k+1} a_{p-k,q}.$$

On the other hand, by repeated application of the formula defining $a_{p,q}$ we obtain

$$a_{p+1,q+1} = a_{p+1,q} + \sum_{k=0}^{p-1} (q+1)^{k+1} a_{p-k,q}.$$

Hence, by subtraction,

$$\sum_{k=0}^{p-1} a_{p-k,q} \left\{ (q+1)^{k+1} - \binom{p+q}{k+1} \right\} = 0,$$

as desired.

The Whistling Bomb

E 511 [1942, 195 and 475]. *Proposed by W. E. Bleick, U. S. Naval Academy, Annapolis*

A whistle, which emits a sound of constant pitch, is attached to a bomb. An observer at a fixed point on the ground sees the bomb dropped from rest at an initial angular elevation B. Assume that the bomb falls with a constant acceleration g and hits the ground at a distance L from the observer. The observer hears a variable whistle pitch because of the Doppler effect. The train of sound waves of maximum apparent pitch leaves the bomb at the elevation at which the component of the bomb's velocity in the direction of the observer is a maximum. Find this angular elevation, and show that it approaches two-thirds of the initial elevation if the initial elevation is small.

Solution by Howard Eves, Syracuse University

Let A be the required angle of elevation, v the velocity of the bomb at that elevation, V the component of v in the direction of the observer, s the distance the bomb has fallen, and H the initial altitude. Then we have

$$v = gt, \qquad V = v \sin A,$$
$$s = \tfrac{1}{2}gt^2, \qquad \tan A = (H - s)/L.$$

Eliminating v, t, and s, we get

$$V^2 = 2g(H - L \tan A) \sin^2 A.$$

The condition for V^2 to be a maximum is

$$2(H - L \tan A) \sin A \cos A - L \sec^2 A \sin^2 A = 0.$$

Dividing by the non-zero factor $L \sin A \cos A$, we get

$$2 \tan B - 2 \tan A - \sec^2 A \tan A = 0,$$

or

$$(1) \qquad \tan^3 A + 3 \tan A - 2 \tan B = 0.$$

By Cardan's formula, this cubic (with discriminant $\sec^2 B > 1$) has the single real root

$$\tan A = (\sec B + \tan B)^{1/3} - (\sec B - \tan B)^{1/3},$$

which gives the sought angle of elevation A.

If B (and therefore A) is small, (1) gives approximately

$$3 \tan A - 2 \tan B = 0,$$

or (again approximately)

$$A = \tfrac{2}{3}B.$$

Also solved by Adolph Barjansky, F. A. Butter, Jr., L. M. Kelly, E. P. Starke, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers, would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4061. *Proposed by N. A. Court, University of Oklahoma*

A variable triangle has its vertices on three skew straight lines, and two of its sides meet a given plane in points lying respectively on two straight lines. Show that the points of intersection of that plane with the third side of the variable triangle are collinear.

4062. *Proposed by N. S. Mendelsohn, University of Toronto*

Show how the operation of multiplying by a real number may be expressed in terms of the operations of adding a real number and of reciprocating.

4063. *Proposed by H. S. M. Coxeter, University of Toronto*

In projective geometry the porism of triangles inscribed in one conic and self-polar for another is commonly proved by showing that if one such triangle exists, we can find another with one vertex at *any* given point on the first conic. This statement is easily seen to be valid in complex geometry. Discuss its possible failure in real geometry.

4064. *Proposed by V. Thébault, San Sebastián, Spain*

Given a tetrahedron $ABCD$: (1) Find the locus of points M such that the sum of the powers of the vertex A with respect to the spheres with diameters MB, MC, MD is constant. (2) Find the point M such that for the spheres with the diameters MA, MB, MC, MD the sum of the powers of a vertex with respect to the three spheres not passing through that vertex is the same for the four vertices. Show that the point M in this case is the symmetric of the centroid with respect to the circumcenter of the tetrahedron.

SOLUTIONS

Tetrahedron of Polar Planes

4004 [1941, 483]. *Proposed by N. A. Court, University of Oklahoma*

Given four spheres, let (E) be the tetrahedron formed by their centers, and (F) the tetrahedron formed by the four polar planes, for these spheres, of their radical center U. Prove that (i) if (E) admits U for its orthocenter, the same holds for (F); (ii) conversely, if (F) admits U for orthocenter, the same holds for (E).

I. *Solution by Howard Eves, Allen Academy, Bryan, Texas*

Let E_1, E_2, E_3, E_4 be the vertices of (E) and let F_1, F_2, F_3, F_4 be the corresponding vertices of (F).

(i) By hypothesis E_1U is perpendicular to $E_2E_3E_4$. Therefore E_1U is the radical axis of the spheres at E_2, E_3, E_4. But F_1, the common point of the polars of U for these three spheres, lies on the radical axis of the three spheres. Therefore E_1U passes through F_1. But we also have E_1U perpendicular to $F_2F_3F_4$, the polar of U for the sphere at E_1. Hence F_1U is an altitude of (F). Similar remarks hold for F_2U, F_3U, F_4U. Hence the first part of the theorem.

(ii) By hypothesis F_1U is perpendicular to $F_2F_3F_4$. Therefore F_1U passes through E_1. But F_1U is the radical axis of the spheres at E_2, E_3, E_4. Therefore F_1U is perpendicular to $E_2E_3E_4$. Hence E_1U is an altitude of (E). Similar remarks hold for E_2U, E_3U, E_4U. Hence the second part of the theorem.

Note: The corresponding theorem for the plane is similarly proved.

II. *Solution by the Proposer*

The point U is the center of a sphere (U), real, or imaginary, orthogonal to the four given spheres (A), (B), (C), (D). Since the two spheres (U), (A) are orthogonal, the polar plane of the center U of (U) for (A) is also the polar plane of the center A of (A) for (U). Similarly for the (B), (C), (D). Thus the two tetrahedrons $(E)=ABCD$, $(F)=PQRS$ are polar reciprocal with respect to the sphere (U).

Consequently the two propositions to be proved may be stated as follows. The polar reciprocal of an orthocentric tetrahedron for a sphere having the orthocenter of the tetrahedron for its center, is also an orthocentric tetrahedron having the same orthocenter as the given tetrahedron.

This proposition is fairly obvious. Indeed, using the same notations, the polar plane QRS of A for (U) is perpendicular to AU and therefore parallel to the plane BCD, and similarly for the other pairs of planes of the two tetrahedrons. Again, the pole P of the plane BCD for (U) lies on the perpendicular AU to BCD, and similarly for other pairs of corresponding vertices of the two tetrahedrons. Thus U is the homothetic center of the two tetrahedrons, hence the proposition.

Remark. Let p be the square of the radius of the sphere (U), and k, k' the squares of the radii of the polar spheres of the orthocentric tetrahedrons (E), (F). (See the proposer's *Modern Pure Solid Geometry*, art. 457.) If A', P' are the traces of the line APU in the planes BCD, QRS, we have

$$UA\cdot UA' = k, \qquad UP\cdot UP' = k', \qquad UA\cdot UP' = UA'\cdot UP = p,$$

hence $UA\cdot UA'\cdot UP\cdot UP'=kk'$, $UA\cdot UP'\cdot UA'\cdot UP=p^2$, and therefore $kk'=p^2$.

Moreover, we have

$$\frac{UA\cdot UA'}{UP\cdot UP'}=\frac{k}{k'}, \qquad \frac{UA\cdot UP'}{UA'\cdot UP}=\frac{p}{p}=1.$$

Now from the last equality we have $UA:UP=UA':UP'$, hence the homothetic ratio $UA:UP$ of the two tetrahedrons (E), (F) is equal to $\sqrt{k/k'}$.

Note. The analogue, for three circles, of the direct proposition is due to R. Tucker. *Educational Times*, Reprints, vol. 52 (1895), p. 28, Q. 12276.

Editorial Note. There is an actual tetrahedron (F) if U does not lie in a face of (E) and if the four given spheres do not have a point in common.

Perspective Tetrahedrons

4005 [1941, 483]. *Proposed by V. Thébault, San Sebastian, Spain*

An arbitrary plane (P) cuts the planes of the faces of the given tetrahedron $ABCD$ in four straight lines. The four planes through these straight lines perpendicular each to the corresponding face determine a tetrahedron $A'B'C'D'$. Show that: (1) The straight lines AA', BB', CC', DD' are concurrent. (2) The point of concurrency I is the center of one of the spheres tangent to the four

planes symmetric to (P) with respect to the faces of $ABCD$. (3) The point I remains fixed when (P) moves parallel to itself.

Editorial Note. Let α, β, γ, δ denote the planes of the faces of the tetrahedron T, α', β', γ', δ' the planes of the faces of T' where corresponding faces of T and T' intersect on a plane σ in a complete quadrilateral. Then the four straight lines each joining the corresponding vertices of T and T' meet in a point V, the center of perspectivity of T and T', and σ is called the plane of perspectivity, see Court's *Modern Pure Solid Geometry*, p. 21, and the solution of 3988 [1942, 409]. Let σ_1 be a plane parallel to σ, T_1 and T' have their corresponding faces parallel, and the corresponding faces of T_1 and T intersect on σ_1; then the vertex $(\alpha_1\beta_1\gamma_1)$ of T_1 lies on the straight line of V, $(\alpha\beta\gamma)$, $(\alpha'\beta'\gamma')$, and similarly for the other corresponding vertices. For, the figures $\alpha\beta\gamma\alpha'\beta'\gamma'\sigma$ and $\alpha\beta\gamma\alpha_1\beta_1\gamma_1\sigma_1$ have $(\alpha\beta\gamma)$ for homothetic center. This proves (1) and (3).

In this problem the corresponding faces of T and T' are perpendicular, and the planes of the faces of T'' are the symmetrics of σ with respect to the corresponding faces of T. The corresponding faces of the three trihedral angles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ intersect on σ, and σ, α, α'', α' is a harmonic axial pencil of planes, and similarly for the two other sets of faces. It follows from this harmonic property that the three vertices $(\alpha\beta\gamma)$, $(\alpha'\beta'\gamma')$, $(\alpha''\beta''\gamma'')$ are on a straight line through the center of perspectivity I for T and T'. Also the distance of $(\alpha\beta\gamma)$ from σ is its distances from the faces of $\alpha''\beta''\gamma''$, so that the straight line $(\alpha''\beta''\gamma'')I$ is an axis of the trihedral angle $\alpha''\beta''\gamma''$, and similarly for the other trihedral angles of T''. Hence I is equally distant from the faces of T'', and the proof of (2) is complete. Part of the above argument is the same as that given by N. A. Court in his solution of 3988.

A Family of Curves

4006 [1941, 561]. *Proposed by P. D. Thomas, Norman, Okla.*

Find the equation of the family of curves, each curve having the property $LR=T^2$, where L is the distance of the tangent from the origin, R is the radius of curvature, and T is the length of the tangent from the point of contact to the y-axis.

I. *Solution by J. O. Hassler, University of Oklahoma*

If we introduce for L, R, and T their values

$$\frac{y-xy'}{\sqrt{1+y'^2}}, \qquad \frac{(1+y'^2)^{3/2}}{y''}, \quad \text{and} \quad x\sqrt{1+y'^2},$$

respectively, the equation $LR=T^2$ becomes, after some reduction

$$x^2y''+xy'-y=0. \tag{1}$$

If we multiply (1) by $1/x^2$ and integrate, we get

$$y'+\frac{y}{x}=c,$$

which can be made an exact differential equation by clearing of fractions. The solution is

$$xy = c_1x^2 + c_2.$$

II. *Solution by R. K. Allen, Montpelier, Vt.*

Replace L, R, and T by their well-known and easily obtained differential expressions and the differential equation of the desired family is $x^2y''-xy'+y=0$. The general solution is $y=x(a \log x+b)$ where a and b are arbitrary constants. These curves are defined only for positive values of x, except when $a=0$, in which case the family is a pencil of straight lines through the origin, but then the expression LR degenerates into the meaningless $0\cdot\infty$, so this particular case should be excluded. For all finite values of a and b $\lim_{x\to 0}y=0$, so we state that (0, 0) lies on every curve and is indeed the endpoint of the curve. Introduction of this single point brings in no difficulty for the equality $LR=T^2$ holds for all positive values of x and will still hold in the limit.

Solved also by N. R. Amundson, E. Fleisher, T. E. Mergendahl, R. W. Wagner, A. K. Waltz, and the proposer.

Editorial Note. Half of the solvers obtained the result in I and the rest that in II by using the negative of the expression for L in I. Some of the solvers introduced a change of independent variable. The process of integration in I may be used in II by writing $d[y'-y/x]/dx=0$ and then $d[y/x]/dx=a/x$.

A later solution by S. H. Lachenbruch considers both cases.

Four Associated Quadrilaterals

4007 [1941, 561]. *Proposed by J. W. Clawson, Ursinus College*

The straight lines l_i, $(i=1, 2, 3, 4)$, determine the complete quadrilateral Q. The four triangles determined by l_j, l_k, l_l have O_i, G_i for their circumcenters and centroids. The point P_i divides O_iG_i in the ratio $r:1$. Lines through these points parallel to l_i form the quadrilaterals Q_0, Q_g, Q_p. Prove that: (1) the four quadrilaterals are congruent; (2) the homothetic center of Q and Q_0 is the orthic center of $O_1O_2O_3O_4$; (3) the homothetic center of Q and Q_g is the mean center of $G_1G_2G_3G_4$; (4) the homothetic center of Q and Q_p divides the line joining the preceding points in the ratio $r:1$; (5) these homothetic centers all lie on the common mid-diagonal line of the four quadrilaterals.

Solution by the Proposer

Since it is well known that the four points O_i are concyclic, we shall take this circle for the unit circle in the complex plane, and take O_i to be the turn t_i. The focal point, F, which lies on this circumcentric circle, will be taken as the unit point. Then circles with centers O_i, O_j and radii O_iF, O_jF intersect at A_{kl}, one of the vertices of quadrilateral Q. The point A_{kl} is $t_i+t_j-t_it_j$, and the line l_i, determined by A_{ij} and A_{ik} is

$$z - t_jt_kt_l\bar{z} = t_j + t_k + t_l - t_jt_k - t_jt_l - t_kt_l,$$

where z is any point on the line and $\bar{z}$ is the complex number conjugate to z.

G_i is $t_i + (t_j + t_k + t_l)(1 - t_i)/3$ and P_i is

$$t_i + \frac{(t_j + t_k + t_l)(1 - t_i)\cdot r}{3\cdot(r + 1)}.$$

The lines through O_i, G_i, P_i parallel to l_i are respectively

$$t_i z - s_4\bar{z} = t_i^2 - t_j t_k t_l, \qquad t_i z - s_4 \bar{z} = t_i^2 - t_j t_k t_l + s_2(1 - t_i)/3,$$

and

$$t_i z - s_4 \bar{z} = t_i^2 - t_j t_k t_l + \frac{s_2(1 - t_i)\cdot r}{3(r + 1)}.$$

In these equations, s_m is used for the sum of the products of t_1, t_2, t_3, t_4 m at a time.

Hence, O_{ij}, G_{ij}, and P_{ij}, vertices respectively of Q_0, Q_g, and Q_p are

$$t_i + t_j + t_k t_l, \qquad t_i + t_j + t_k t_l - s_2/3, \qquad t_i + t_j + t_k t_l - \frac{s_2\cdot r}{3(r + 1)}.$$

Then the mid-points of lines joining A_{ij} to O_{ij}, G_{ij} and P_{ij} are $s_1/2$, $s_1/2 - s_2/6$ and $s_1/2 - s_2\cdot r/6(r+1)$.

(1) From the symmetry of the last result, it follows that Q and Q_0, Q and Q_g, and Q and Q_j are homothetic with ratio 1:1, *i.e.*, that they are all congruent, the homothetic centers being at the points indicated. Likewise (2), (3) and (4) are easily verified. Finally, (5) follows from the fact that each of these points, as also the middle points of $A_{ij}A_{kl}$, $O_{ij}O_{kl}$, $G_{ij}G_{kl}$, $P_{ij}P_{kl}$, *viz.*, $s_1/2 - (t_it_j + t_kt_l)/2$, $s_1/2 + (t_it_j + t_kt_l)/2$, $s_1/2 + (t_it_j + t_kt_l)/2 - s_2/6$, and $s_1/2(t_it_j + t_kt_l)/2 - s_2\cdot r/6(r+1)$ all lie on the line $z - s_4\bar{z} = s_1/2 - s_3/2$.

Editorial Note. The proposer gave the following definition of the orthic center of a cyclic quadrilateral:

The four straight lines each through the midpoint of a side and perpendicular to the opposite side of a cyclic quadrilateral are concurrent; and the point of concurrency, the orthic center, bisects the segment from any vertex to the orthocenter of the triangle with the other three points as vertices, see Clawson, *Annals of Mathematics*, vol. 20, no. 4, June 1919, p. 252.

The proof is easy; let $\mathbf{a}_i$ be the vector from the center of the circle to the vertex A_i, and $\mathbf{g}$ the vector to the centroid of the four vertices. The orthocenter of triangle $A_jA_kA_l$ is given by $\mathbf{a}_j + \mathbf{a}_k + \mathbf{a}_l$, and the midpoint of the segment from this orthocenter to A_i has the vector $(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4)/2 = 2\mathbf{g}$. The vector from the midpoint of a side A_iA_j to this latter point is $(\mathbf{a}_k + \mathbf{a}_l)/2$ which is perpendicular to the vector A_lA_k, or $\mathbf{a}_k - \mathbf{a}_l$.

A solution of this problem may be obtained by use of some of the results in 3839 [1939, 604], see also 3991 [1942, 550]. We shall give the results for (2). The homothetic center of (Q) and (Q_0) has the coordinates $a(1 + \sigma_2 + \sigma_4)/4$, $a\sigma_1/2$; the orthocenter of triangle $O_iO_jO_k$ is the point $am_l(\sigma_1^l + \sigma_3^l)/2$,

$a(\sigma_1^l+\sigma_3^l+2m_l)/2$ and it lies on Δ_l, a side of (Q); and the midpoint of this orthocenter and O_l with coordinates $a(1+\sigma_2^l)/2$, $a(\sigma_1^l-\sigma_3^l)/2$ is the above homothetic center of (Q) and (Q_0). This completes the proof of (2). From these results we easily find for the center of the circle $(O_1O_2O_3O_4F)$ the point $a(3+\sigma_2-\sigma_4)/4$, $a(\sigma_1-\sigma_3)/4$. In these computations (Q) is assumed to have no two sides parallel so that there are four actual triangles.

Iterated Pedal Polygons

4008 [1941, 561]. *Proposed by V. Thébault, San Sebastián, Spain*

Given in a plane a polygon (P) of n sides having a center of symmetry S and two points M, M' symmetric with respect to S. Show that the nth pedal polygon of M and M' with respect to (P) are equal.

Note. Let Q be an arbitrary point in the plane of polygon $(A)\equiv A_1A_2\cdots A_n$. We say that the first pedal polygon of Q for the polygon (A) is the polygon $(B)\equiv B_1B_2\cdots B_n$ if the indicated consecutive vertices of (B) are the orthogonal projections of Q on the sides $A_1A_2, A_2A_3, \cdots, A_nA_1$ of (A); then the second pedal of Q for (A) is the first pedal of Q for (B), and so on.

Editorial Note. Since S is a center of symmetry of the combined figure (P), M, M', the first pedal polygons (P_1) of M and (P_1') of M' with respect to (P) form a figure with the center of symmetry S. Also the second pedal polygons (P_2) of M with respect to (P_1) and (P_2') of M' with respect to (P_1') form a figure with the center of symmetry S; and so on.

Jacobian, Alternant

4009 [1941, 638]. *Proposed by J. H. M. Wedderburn, Princeton University*

If the roots of $x^n-c_1x^{n-1}+\cdots+(-1)^nc_n$ are the variables $x_1, x_2, \cdots, x_n$, find the Jacobian of $c_n, c_{n-1}, \cdots, c_1$ with respect to $x_1, x_2, \cdots, x_n$.

I. *Solution by M. F. Smiley, Lehigh University*

It seems better to find the jacobian $\partial(c_1, \cdots, c_n)/\partial(x_1, \cdots, x_n)$ and to note that $\partial(c_n, \cdots, c_1)/\partial(x_1, \cdots, x_n)=(-1)^\mu\partial(c_1, \cdots, c_n)/\partial(x_1, \cdots, x_n)$ where $\mu=n(n-1)/2$. Direct computation reveals that $\partial(c_1, \cdots, c_n)/\partial(x_1, \cdots, x_n$ is the alternant $P(x_1, \cdots, x_n)=\pi(x_i-x_j)(j>i; i, j=1, \cdots, n)$ for $n=2, 3$. To establish this result for all n we use induction. Hence suppose that the result holds for $m=n-1$. Write $s_k(x_1, \cdots, x_n)$ for c_k and 1 for $s_0(x_1, \cdots, x_n)$ and note that

$$\frac{\partial}{\partial x_i}s_k(x_1, \cdots, x_n) = s_{k-1}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n).$$

In the jacobian $\partial(c_1, \cdots, c_n)/\partial(x_1, \cdots, x_n)$ subtract the first column from each of the remaining, noticing that

$$\begin{aligned} s_{k-1}(x_2, \cdots, x_n) &- s_{k-1}(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) \\ &= (x_j - x_1)s_{k-2}(x_2, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) \quad (j, k_1 - 2 \cdots, n). \end{aligned}$$

Expand by minors of the first row, remove the factor $\pi(x_1-x_j)(j=2, \cdots, n)$ and we find that our jacobian is

$$\partial(d_1, \cdots, d_{n-1})/\partial(x_2, \cdots, x_n)\prod_{j=2}^{n}(x_1 - x_j), \qquad d_i = s_i(x_2, \cdots, x_n).$$

Application of the hypothesis of induction shows immediately that our result holds for $m=n$, and the proof is complete.

II. *Solution by R. J. Walker, Cornell University*

Let c_r be the rth elementary symmetric function of $x_1, \cdots, x_n$. Let $c_r^{(i)}$ be the rth elementary symmetric function of $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$. Then

$$c_r^{(i)} = c_r - x_i c_{r-1}^{(i)}, \qquad r = 1, \cdots, n-1,$$

where for convenience we put $c_0^{(i)}=1$. From this we obtain

$$\frac{\partial c_r}{\partial x_i} = c_{r-1}^{(i)}, \qquad r = 1, \cdots, n.$$

Hence

$$\begin{aligned}
\left|\frac{\partial c_r}{\partial x_i}\right| &= \left| c_0^{(i)}\ c_1^{(i)}\ c_2^{(i)} \cdots c_{n-1}^{(i)}\right| \\
&= \left| 1\ \ c_1 - x_i\ \ c_2 - x_i c_1^{(i)} \cdots c_{n-1} - x_i c_{n-2}^{(i)}\right| \\
&= \left| 1\ \ -x_i\ \ -x_i c_1^{(i)} \cdots -x_i c_{n-2}^{(i)}\right| \\
&= (-1)^{n-1} x_1 \cdots x_n \left| x_i^{-1}\ 1\ c_1^{(i)} \cdots c_{n-2}^{(i)}\right| \\
&= (-1)^{n-1} x_1 \cdots x_n \left| x_i^{-1}\ 1\ c_1 - x_i \cdots c_{n-2} - x_i c_{n-3}^{(i)}\right| \\
&= (-1)^{n-1+n-2} x_1^2 \cdots x_n^2 \left| x_i^{-2}\ x_i^{-1}\ 1 \cdots c_{n-3}^{(i)}\right| \\
&= \cdots \\
&= (-1)^{n(n-1)/2} x_1^{n-1} \cdots x_n^{n-1} \left| x_i^{-n+1}\ x_i^{-n+2} \cdots 1\right| \\
&= (-1)^{n(n-1)/2} \left| 1\, x_i\, x_i^2 \cdots x_i^{n-1}\right| \\
&= \prod_{i<j}(x_i - x_j).
\end{aligned}$$

Solved also by E. Fleisher, D. T. Sigley, J. Singer, and the proposer.

Editorial Note. The remaining solvers found easily that

$$\partial c_i/\partial x_j = c_{i-1} - c_{i-2}x_j + \cdots + (-1)^{i-1} c_0 x_j^{i-1}, \qquad c_0 = 1.$$

The proposer's solution was briefly stated by using in his reduction powers of a certain nilpotent matrix, while the others used successive elementary transformations. The reductions may be put in the following form. Let M be the $n\times n$ matrix with the ith row $c_{i-1}, -c_{i-2}, \cdots, (-1)^{i-1}c_0, 0, \cdots, 0$, and let V be the matrix with the jth column $1, x_j, \cdots, x_j^{n-1}$. Then

$$(\partial c_i/\partial x_j) = MV, \qquad |\partial c_i/\partial x_j| = (-1)^{n(n-1)/2}|V|, \qquad |\partial c_{n-i+1}/\partial x_j| = |V|,$$

and the required result is the Vandermonde determinant.

Generators of Abelian Groups

4010 [1941, 638]. *Proposed by F. A. Lewis, University of Alabama*

Let G be an Abelian group of order n^m and type $(1, 1, \cdots, 1)$. Find the number of sub-groups of G of order n^r and type $(1, 1, \cdots, 1)$; and show that this number is the same as that for order n^{m-r}.

Editorial Note. This problem means that G is an Abelian group of order n^m which is generated by m independent operators each of period n. Its solution involves the function $J_k(n)$ which is the number of sets of k integers, equal or unequal, each $\leqq n$, such that the g.c.d. of each set is prime to n. It is easily shown that

$$J_k(n) = n^k \prod (1 - p_i^{-k}),$$

where the integers p_i are the distinct prime divisors of n. From this it follows that the number of elements of G of period n is $J_m(n)$. Suppose that t independent generators of period n have been chosen thus forming a sub-group of order n^t. This subgroup contains $J_t(n)$ elements of period n and there are left $J_m(n) - J_t(n)$ elements of period n. Hence there are

$$J_m(n)[J_m(n) - J_1(n)] \cdots [J_m(n) - J_{r-1}(n)]/r!$$

number of ways of choosing r independent generators each of period n. But each such generated subgroup is generated by different selections of the r generators, the number of selections being the above expression with m replaced by r. Hence the number of subgroups of order n^r generated by r elements of period n is

$$\frac{J_m(n)[J_m(n) - J_1(n)] \cdots [J_m(n) - J_{r-1}(n)]}{J_r(n)[J_r(n) - J_1(n)] \cdots [J_r(n) - J_{r-1}(n)]}.$$

If n is a prime p the above expression may be reduced to a simpler explicit form which is given in several texts, see Mathewson's *Elementary Theory of Finite Groups*, p. 93 and exs. p. 94. In this case the formula shows easily that the number of such subgroups of order p^r is that for order p^{m-r}. It appears in a recent communication from the proposer that the generalization of this special result demanded in the last lines of the problem is to be omitted.

Hyperbolic Groups of Lines

4011 [1941, 639]. *Proposed by N. A. Court, University of Oklahoma*

The pairs of straight lines a, a', b, b'; c, c'; d, d' are isogonal conjugates for the trihedral angles A, B, C, D of the tetrahedron $ABCD$. Prove that: (1) If the lines a, b, c, d are concurrent, so are also the remaining four lines (the proposer's

Modern Pure Solid Geometry, p. 242). (2) If the four lines a, b, c, d form a hyperbolic group, so also do the remaining four lines.

Solution by the Proposer

If the lines a, b, c, d passing through the vertices A, B, C, D of the tetrahedron $ABCD$ form a hyperbolic group, a line p may be drawn through A meeting the lines b, c, d. The isogonal conjugate plane of the plane (AB, p, b) for the dihedral angle AB contains both the isogonal conjugate p' of p for the trihedral angle A and the isogonal conjugate b' of b for the trihedral angle B.

It may be shown similarly that p' is coplanar with the isogonal conjugates c', d' of c, d, for the respective trihedral angles C, D; and p' is obviously coplanar with the isogonal conjugate a' of a for the trihedral angle A.

Likewise it may be shown that the lines a', b', c', d' are met by lines q', r', s', analogous to the line p' and passing respectively through B, C, D. For a given trihedral angle and a straight line through its vertex there is a one-to-one correspondence between the line and its isogonal conjugate only if the line does not lie in a face. If no one of a, b, c, d lies in a face of the tetrahedron the lines p', q', r', s' are distinct; for, if p' and q' coincide, they coincide with the straight line of AB, and then p lies in the plane of CDA and at least one of the lines c, d lies also in this plane, contrary to hypothesis. Then a', b', c', d' are skew in pairs and are cut by four distinct straight lines. In this case (2) is proved.

Solved partially by H. Eves.

TRAINING IN METEOROLOGY

Under the auspices of the Army Air Forces, the University Meteorology Committee is establishing in twenty colleges educational programs intended to prepare young men for an eight months advanced course in meteorology. Graduates of the advanced course are commissioned in the Air Forces.

The pre-meteorology courses consist of Mathematics through Advanced Calculus and Differential Equations, Vector Analysis, Mechanics with emphasis on relative motion and Coriolis force, Physics with emphasis on thermodynamics and radio, English, History, and Geography. A year's course beginning February 1st is available for students with no college background, and a six months course beginning March 1st for students with one or more years of college.

This announcement is published in the hope that teachers of mathematics and physics will call this program to the attention of their better students. There is also a limited number of vacancies on the mathematics and physics staffs of the participating colleges. Persons well qualified in the above curriculum are invited to apply.

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MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

The twenty-seventh Annual Meeting scheduled for New York, N. Y., December 30–31, 1942, was cancelled because of difficulties of transportation.

The following is a list of the Sections of the Associations, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Notre Dame, Ind., April 9–10, 1943
INDIANA, Notre Dame, April 9–10, 1943
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Brooklyn, N. Y., May 8, 1943
MICHIGAN, Notre Dame, Ind., April 9–10, 1943
MINNESOTA
MISSOURI, Kansas City
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, Jan. 30, 1943
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May 7, 1943

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